Interlacing Diffusions

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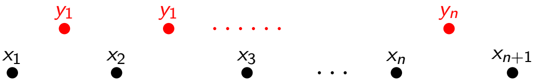
Outline

- Set up: conjugate diffusions and stochastic coalescing flow.
- Transition kernels for interlaced diffusions and SDER.
- Intertwinings and Markov functions.
- Multilevel processes in Gelfand Tsetlin patterns and examples.
- Edge particle systems.

Interlacings

- For an interval $\mathcal{I}$ with endpoints $l < r$ define the chamber $W^n = ((x) : x_i \in \mathcal{I}, x_1 \leq \cdots \leq x_n)$.
- Say $y \in W^n$ and $x \in W^{n+1}$ interlace and denote this by $y \prec x$ if $x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_{n+1}$.
- This defines the following space $W^{n,n+1} = ((x, y) : x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_{n+1})$

- Will think of as two levels:



Dual Diffusions 1

Consider a one dimensional diffusion $(X_t, t \geq 0)$ in some interval $\mathcal{I}$ with generator (with boundary conditions),

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$$

Define the scale function $s(x)$ by $s'(x) = exp\big(- \int_c^x \frac{b(y)}{a(y)}dy\big)$ where c is an arbitrary point in $\mathcal{I}^\circ$ and speed measure with density in $\mathcal{I}^\circ$ with respect to Lebesgue $m(x) = \frac{1}{s'(x)a(x)}$. The speed measure is the symmetrizing measure for $(X_t, t \geq 0)$ and,

$$L = \mathcal{D}_m \mathcal{D}_s$$

where $\mathcal{D}_m = \frac{1}{m(x)}\frac{d}{dx}$ and $\mathcal{D}_s = \frac{1}{s'(x)}\frac{d}{dx}$. Denote by P_t the semigroup associated to $(X_t, t \geq 0)$.

Dual Diffusions 2

Define conjugate/Siegmund dual diffusion $\left(\hat{X}_t, t \geq 0\right)$ with generator,

$$\hat{L} = a(x)\frac{d^2}{dx^2} + (a'(x) - b(x))\frac{d}{dx}$$

and the *dual boundary behaviour*.
Note that the ˆ operation swaps the scale functions and speed measures,

$$\hat{s}'(x) = m(x) \textit{ and } \hat{m}(x) = s'(x)$$

Key Fact

For $z, z' \in \mathcal{I}^\circ$: $\mathbb{P}_z(X(t) \leq z') = \mathbb{P}_{z'}(\hat{X}(t) \geq z)$

Coalescing Stochastic Flow

- We denote by $\mathbf{\Phi}_{s,t} : \mathcal{I} \to \mathcal{I}$ the *stochastic coalescing flow of maps* with *one point motion* $\mathbf{\Phi}_{s,t}(x)$ an L diffusion ran from time s to time t started from x.
- We can think of it as paths starting from each space time point that evolve independently until the first time they meet at which point they coalesce and move together (see Le Jan and Raymond [10]).

We have the following proposition for its finite dimensional distributions.

Proposition (A., O'Connell, Warren)

For $z, z' \in W^n$,

$$\mathbb{P}\left(\mathbf{\Phi}_{0,t}(z_i) \leq z_i' \text{ for } 1 \leq i \leq n\right) = det\left(\mathsf{P}_t \mathbf{1}_{[l,z_j']}(z_i) - \mathbf{1}(i < j)\right)_{i,j=1}^{n}$$

Transition Kernels for Interlaced Diffusions 1

We define the following kernel $q_t^{n,n+1}((x,y),(x',y'))dx'dy'$ on $W^{n,n+1}$. For $(x,y),(x',y') \in W^{n,n+1}$,

$$q_t^{n,n+1}((x,y),(x',y')) =$$
$$= \frac{\prod_{i=1}^n \hat{m}(y_i')}{\prod_{i=1}^n \hat{m}(y_i)}(-1)^n \frac{\partial^n}{\partial_{y_1}\cdots\partial_{y_n}} \frac{\partial^{n+1}}{\partial_{x_1'}\cdots\partial_{x_{n+1}'}} \mathbb{P}\big(\mathbf{\Phi}_{0,t}(x_i) \le x_i',$$
$$\mathbf{\Phi}_{0,t}(y_j) \le y_j' \;\text{ for all }\; i,j\big)$$

Note that,

$$q_t^{n,n+1}((x,y),(x',y'))dx'dy$$
$$= \frac{\prod_{i=1}^n \hat{m}(y_i')}{\prod_{i=1}^n \hat{m}(y_i)} \mathbb{P}\big(\mathbf{\Phi}_{0,t}(x_i) \in dx_i', \mathbf{\Phi}_{0,t}^{-1}(y_j') \in dy_j \;\text{ for all }\; i,j\big)$$

where $\mathbf{\Phi}_{0,t}^{-1}$ is the (generalized) inverse of $\mathbf{\Phi}_{0,t}$.

Transition Kernels for Interlaced Diffusions 2

The kernel $q_t^{n,n+1}$ can be written out explicitly,

$$q_t^{n,n+1}((x,y),(x',y')) = det \begin{pmatrix} A_t(x,x') & B_t(x,y') \\ C_t(y,x') & D_t(y,y') \end{pmatrix}$$

where,

$$A_t(x,x')_{ij} = p_t(x_i, x_j') = \partial_{x_j'} \mathsf{P}_t \mathbf{1}_{[l,x_j']}(x_i)$$

$$B_t(x,y')_{ij} = \hat{m}(y_j')(\mathsf{P}_t \mathbf{1}_{[l,y_j']}(x_i) - \mathbf{1}(j \geq i))$$

$$C_t(y,x')_{ij} = -\hat{m}^{-1}(y_i)\partial_{y_i}\partial_{x_j'} \mathsf{P}_t \mathbf{1}_{[l,x_j']}(y_i)$$

$$D_t(y,y')_{ij} = -\frac{\hat{m}(y_j')}{\hat{m}(y_i)} \partial_{y_i} \mathsf{P}_t \mathbf{1}_{[l,y_j']}(y_i) = \hat{p}_t(y_i, y_j')$$

Transition Kernels for Interlaced Diffusions 3

- Define the following family of operators acting on $f \in \mathcal{B}_b(W^{n,n+1})$
  $(Q_t^{n,n+1} f)(x,y) = \int_{W^{n,n+1}} q_t^{n,n+1}((x,y),(x',y'))f(x',y')dx'dy'$
- Then we have the following,

$$Q_t^{n,n+1} 1 \leq 1$$

$$Q_t^{n,n+1} f \geq 0 \; for \; f \geq 0$$

$$Q_{t+s}^{n,n+1} = Q_t^{n,n+1} Q_s^{n,n+1}$$

Proposition (A., O'Connell, Warren)

$Q_t^{n,n+1}$ is a sub-Markov semigroup giving rise to a Markov process $Z = (X, Y)$ (with possibly finite lifetime) with state space $W^{n,n+1}$.

Dynamics between X and Y levels and $SDER$

In fact under some further assumptions $q_t^{n,n+1}$ is the transition density of the following system of $SDER$ in $W^{n,n+1}$ up to the stopping time $T^{n,n+1} = inf\{t : \exists\ i\ Y_i(t) = Y_{i+1}(t)\ or\ Y_1(t) = l\ or\ Y_n(t) = r\}$.

$$dX_1(t) = \sqrt{2a(X_1(t))}d\beta_1(t) + b(X_1(t))dt + dK^l(t) - dK_1^+(t)$$

$$dY_1(t) = \sqrt{2a(Y_1(t))}d\gamma_1(t) + (a'(Y_1(t)) - b(Y_1(t)))dt$$

$$dX_2(t) = \sqrt{2a(X_2(t))}d\beta_2(t) + b(X_2(t))dt + dK_2^-(t) - dK_2^+(t)$$

$$\vdots$$

$$dY_n(t) = \sqrt{2a(Y_n(t))}d\gamma_n(t) + (a'(Y_n(t)) - b(Y_n(t)))dt$$

$$dX_{n+1}(t) = \sqrt{2a(X_{n+1}(t))}d\beta_{n+1}(t) + b(X_{n+1}(t))dt + dK_{n+1}^-(t) - dK^r(t)$$

where K^l increases only when $X_1 = l$, K^r increases only when $X_{n+1} = r$, K_i^+ increases only when $Y_i = X_i$ and K_i^- only when $Y_{i-1} = X_i$.

Intermediate Intertwining Relations 1

- Denote by P_t^n the Karlin McGregor semigroup associated to n L (*killed* at an absorbing boundary point) diffusions and similarly define $\hat{P}_t^n$. These are given by with $x, y \in W^n$,

$$p_t^n(x,y)dy = det(p_t(x_i, y_j))_{i,j=1}^n dy$$

  and

$$\hat{p}_t^n(x,y)dy = det(\hat{p}_t(x_i, y_j))_{i,j=1}^n dy$$

- Also define the following kernels,

$$\Lambda_{n,n+1} : \mathcal{B}_b(W^{n,n+1}) \to \mathcal{B}_b(W^{n+1})$$
$$(\Lambda_{n,n+1} f)(x) = \int_{W^{n,n+1}(x)} \prod_{i=1}^{n} \hat{m}(y_i) f(x,y) dy$$
$$\Pi_{n,n+1} : \mathcal{B}_b(W^n) \to \mathcal{B}_b(W^{n,n+1})$$
$$(\Pi_{n,n+1} f)(x,y) = f(y)$$

Intermediate Intertwining Relations 2

- Then we have the following intermediate intertwining relations,

$$P_t^{n+1}\Lambda_{n,n+1} = \Lambda_{n,n+1}Q_t^{n,n+1}$$
$$\Pi_{n,n+1}\hat{P}_t^{n} = Q_t^{n,n+1}\Pi_{n,n+1}$$

  These follow by integrating w.r.t. to y and w.r.t. to x' respectively.

- If $\hat{h}_n$ is a positive eigenfunction of $\hat{P}_t^n$ then so it is for $Q_t^{n,n+1}$. Also $h_{n+1}(x) = (\Lambda_{n,n+1}\Pi_{n,n+1}\hat{h}_n)(x)$ is a positive eigenfunction of P_t^{n+1}. Define the Markov kernel $\Lambda_{n,n+1}^{\hat{h}_n}$,

$$(\Lambda_{n,n+1}^{\hat{h}_n}f)(x) = \frac{1}{h_{n+1}(x)}\int_{W^{n,n+1}(x)}\prod_{i=1}^{n}\hat{m}(y_i)\hat{h}_n(y)f(x,y)dy$$

- Denote $P_t^{n+1,h_{n+1}}, \hat{P}_t^{n,\hat{h}_n}, Q_t^{n,n+1,\hat{h}_n}$ the h-transforms of $P_t^{n+1}, \hat{P}_t^{n}, Q_t^{n,n+1}$ by h_{n+1} and $\hat{h}_n$ respectively.

Intertwinings and Markov Functions

Denote by Ψ the operator induced by the projection on X. We have the following Theorem.

Theorem (A., O'Connell, Warren)

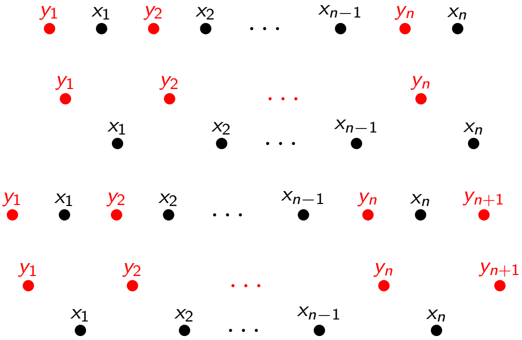
$$\Lambda_{n,n+1}^{\hat{h}_{n+1}} \Psi = Id \quad on \quad X \tag{1}$$

$$P_t^{n+1,h_{n+1}} \Lambda_{n,n+1}^{\hat{h}_{n+1}} = \Lambda_{n,n+1}^{\hat{h}_n} Q_t^{n,n+1,\hat{h}_n} \tag{2}$$

$$\Pi_{n,n+1} \hat{P}_t^{n,\hat{h}_n} = Q_t^{n,n+1,\hat{h}_n} \Pi_{n,n+1} \tag{3}$$

(1) and (2) and Rogers and Pitman Markov functions theory ([11]) gives that the X particles in their *own filtration* evolve as a Markov process with semigroup $P_t^{n+1,h_{n+1}}$ started from x if (X, Y) is started from $\Lambda_{n,n+1}^{\hat{h}_n}(x, \cdot)$.
(3) is an instance of Dynkin's criterion and implies that the Y particles are Markovian w.r.t to the joint filtration of (X, Y) (they are *autonomous*).

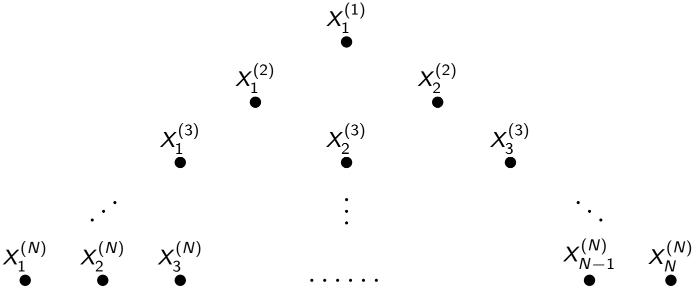
- It is also possible to define kernels on the following spaces $W^{n,n} = ((x, y) : y_1 \leq x_1 \leq y_2 \leq \cdots \leq x_n)$ and $W^{n+1,n} = ((x, y) : y_1 \leq x_1 \leq y_2 \leq \cdots \leq y_{n+1})$ shown below,



- We have analogous Markov functions theorems and description of the dynamics in terms of *SDEs* in these settings.

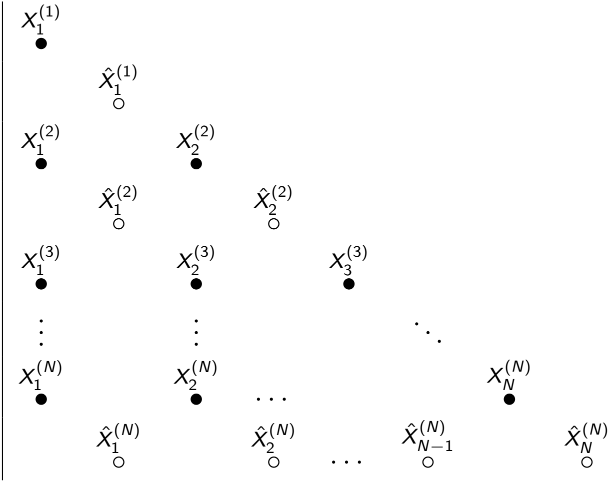
Multilevel Processes and Gelfand Tseltin Patterns 1

- Our goal will be to concatenate such 2 level processes to build a multilevel interlacing process so that each level is Markovian with explicit semigroup.
- This process will take values in the space of Gelfand Tsetlin patterns of depth N denoted by,
  $\mathbb{GT}(N) = \{(X^{(1)}, \cdots, X^{(N)}) : X^{(n)} \in W^n,\ X^{(n)} \prec X^{(n+1)}\}$



Multilevel Processes and Gelfand Tseltin Patterns 2

- ...or in the space of symplectic Gelfand Tsetlin patterns of depth N $\mathbb{GT}_{\mathbf{s}}(N) = \{(X^{(1)}, \hat{X}^{(1)} \dots, X^{(N)}, \hat{X}^{(N)}) : X^{(n)}, \hat{X}^{(n)} \in W^n,\ X^{(n)} \prec \hat{X}^{(n)} \prec X^{(n+1)}\}$



Brownian motions with drifts

- Using our main Theorem we obtain the following,

Corollary (A., O'Connell, Warren)

Consider a Markov process $(X, Y) \in W^{n,n+1}(\mathbb{R})$ started from the origin (according to an entrance law) with the Y particles evolving as n Brownian motions with drifts $(\mu_1, \cdots, \mu_n)$ conditioned to never intersect and the X particles as $n+1$ Brownian motions with drift μ_{n+1} reflected off the Y particles. Then the X particles in their own filtration evolve as $n+1$ Brownian motions with drifts $(\mu_1, \cdots, \mu_{n+1})$ conditioned to never intersect started from the origin.

- Can then concatenate to build a process in $\mathbb{GT}(N)$ with k^{th} level evolving as non intersecting *BMs* with drifts $\mu_1, \cdots, \mu_k$.
- This process was first constructed by Ferrari and Frings in [5] but studied *only at fixed times*.
- For all drifts equal to 0 this gives the process first constructed by Warren [12].

Matrix model for Brownian motions with drifts

- Let $Y_t = B_t$ be an $n \times n$ Hermitian Brownian motion.
- Let M be a diagonal $n \times n$ Hermitian matrix with distinct ordered eigenvalues $(\mu_1, \cdots, \mu_n)$.
- Then eigenvalues of the hermitian valued process $Y_t^M = B_t + tM$ evolve as BMs with drifts $\mu_1 < \mu_2 < \cdots < \mu_n$ conditioned not to collide.
- The $k \times k$ minor of Y_t^M has k eigenvalues evolving non intersecting *BMs* with drifts $\mu_1, \cdots, \mu_k$ and interlace (since Hermitian) with the eigenvalues of the $(k+1) \times (k+1)$ minor.
- So the process we constructed matches the minor process on *each level*. Also their distributions coincide at *fixed times* (see [5]).
- It is not true that the interaction between the two levels of the minor process is the same as the process we constructed via hard reflection. Involves much more complicated repulsion terms (see Adler et al. [1]).

Brownian motions in an interval 1

- Take L reflecting BM in $[0, \pi]$. $\hat{L}$ absorbing BM in $[0, \pi]$.

Corollary 1 (A., O'Connell, Warren)

Consider a process $(X, Y) \in W^{n,n+1}([0, \pi])$ with the Y particles evolving as n BMs conditioned to stay in $(0, \pi)$ and conditioned not to intersect and the X particles as $n+1$ reflecting BMs in $[0, \pi]$ reflected off the Y particles. Then (if started accordingly) the X particles in their own filtration evolve as $n+1$ non intersecting BMs reflected at the boundaries of $[0, \pi]$.

Corollary 2 (Dual of Cor. 1) (A., O'Connell, Warren)

Reflect n BMs between $n+1$ reflecting BMs in $[0, \pi]$ conditioned not to intersect then we obtain n BMs conditioned to stay in $(0, \pi)$ and conditioned not to intersect.

Brownian motions in an interval 2

- The processes studied above are related to the eigenvalue evolutions of Brownian motions on $SO(2(n+1))$ (reflecting Brownian motions in $[0,\pi]$) and $USp(2n)$ (conditioned Brownian motions in $(0,\pi)$) respectively.
- Now suppose L diffusion is a BM reflecting at π and absorbing at 0.

Corollary 3 (A., O'Connell, Warren)

Consider a process $(X,Y) \in W^{n,n}([0,\pi])$. The Y particles evolving as n BMs reflecting at 0 conditioned not to hit π and not to intersect and the X particles as n Brownian motions in $(0,\pi]$ reflecting at π being kept apart and away from 0 by the Y particles. Then the X process in its own filtration (assuming the two levels (X,Y) are started appropriately) evolves as n Brownian motions in $(0,\pi]$ reflecting at π and conditioned to stay away from 0 and not to intersect.

- These processes are related to the eigenvalues of Brownian motions on $SO(2n+1)$ and $SO^-(2n+1)$ respectively.

Squared Bessel process

- Squared Bessel Process of dimension d given by the *SDE* in $[0, \infty)$

$$dX_t = 2\sqrt{X_t}d\beta_t + ddt$$

- For positive integer d this is the modulus squared of d-dimensional Brownian motion.
- Non intersecting *BESQ* processes arise as eigenvalues of *LUE* matrix valued diffusions (see [9]).
- For $d \geq 2$ never hits the origin.
- Can also consider negative dimensions for which the origin is absorbing.
- Our main results give the following two propositions.

Intertwinings for non intersecting Squared Bessel processes

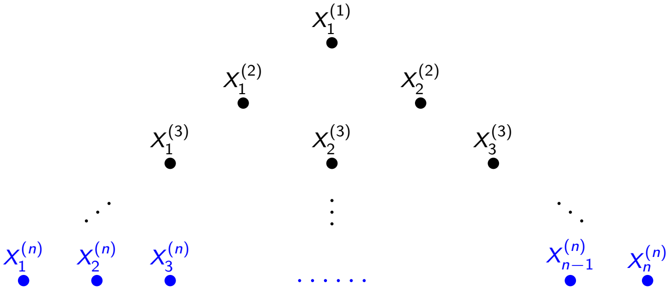
Proposition 1 (A., O'Connell, Warren)

Consider a process $(X, Y) \in W^{n,n+1}([0,\infty))$ started from the origin with the Y particles evolving as n non intersecting $BESQ(d+2)$ processes and the X particles evolving as $n+1$ $BESQ(d)$ processes being reflected off the Y particles. Then the X particles in their own filtration evolve as $n+1$ non intersecting $BESQ(d)$ issueing from the origin.

Proposition 2 (A., O'Connell, Warren)

Consider a process $(X, Y) \in W^{n,n}([0,\infty))$ started from the origin with the Y particles which evolve as n non intersecting $BESQ(d)$ processes and the X particles evolving as n $BESQ(2-d)$ processes being reflected off the Y particles. Then the X particles in their own filtration evolve as n non intersecting $BESQ(d+2)$ issueing from the origin.

- Using Proposition 1 repeatedly we obtain a process in $\mathbb{GT}(n)$.



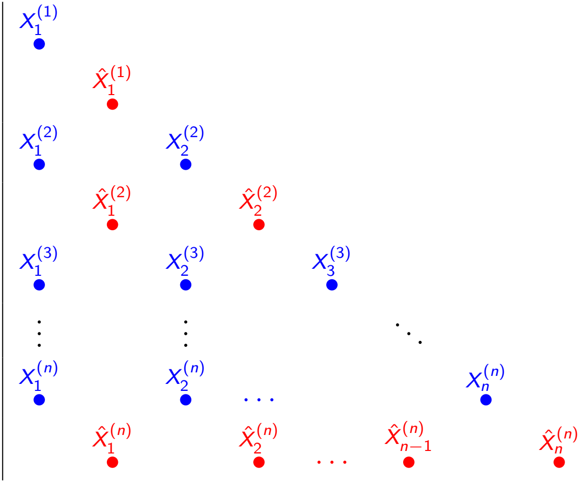
- Fixing the bottom level *blue* particles to be n non intersecting $BESQ(d)$ started from the origin.
- Then the k^{th} level particles $(X_1^{(k)}, \cdots, X_k^{(k)})$ evolve as k non intersecting $BESQ(d + 2(n-k))$ started from the origin.

Matrix model for non intersecting $BESQ$ in $\mathbb{GT}(n)$

- We explain the connection to a minor process, with $d = 2$ for simplicity.
- Let $A(t)$ to be an $n \times n$ complex Brownian matrix and consider $H(t) = A(t)A(t)^*$.
- Then $\left(\lambda^{(k)}(t), t \geq 0\right)$ the eigenvalues of the $k \times k$ minor of $H(t)$ evolve as k non colliding $BESQ(2(n-k+1))$ (see [9]).
- These eigenvalues then interlace with $\left(\lambda^{(k-1)}(t), t \geq 0\right)$ which evolve as $k-1$ non colliding $BESQ(2(n-k+1)+1)$ with the fixed time conditional distribution of $\lambda^{(k-1)}(t)$ given $\lambda^{(k)}(t)$ being proportional to $\Delta(\lambda^{(k-1)}(t))$ on $W^{k-1,k}\left(\lambda^{(k)}(t)\right)$.
- Again we expect (not studied in detail yet) that the interaction between two consecutive levels of this minor process involves some more complicated repulsion and not hard reflection.

Non intersecting *BESQ* in $\mathbb{GT}_{\mathbf{s}}(n)$

Using Proposition 2 and 1 interchangeably with an increasing number of particles we construct a process in $\mathbb{GT}_{\mathbf{s}}(n)$. Blue particles evolve as non intersecting $BESQ(d)$ and Red particles as non intersecting $BESQ(d+2)$.

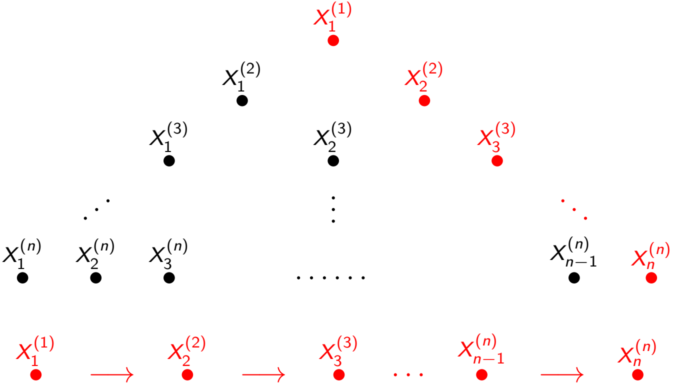


Matrix model for non intersecting $BESQ$ in $\mathbb{GT}_{\mathsf{s}}(n)$

- Let d be even.
- Start with a row vector $A^{(d)}(t)$ of independent $d/2$ standard complex Brownian motions. Then (by definition) $X^{(d)}(t) = A^{(d)}(t)A^{(d)}(t)^*$ evolves as a $BESQ(d)$.
- Now add another independent complex Brownian motion to make $A^{(d)}(t)$ a row vector of length $d/2+1$.
- $X^{(d)}(t) = A^{(d)}(t)A^{(d)}(t)^*$ now evolves as a $BESQ(d+2)$ interlacing with the $BESQ(d)$ described above.
- At fixed times the conditional distribution of the $BESQ(d)$ process given the position x of the $BESQ(d+2)$ process is proportional to $y^{\frac{d}{2}-1}$ in $[0, x]$.
- Now add a row of $d/2+1$ independent complex Brownian motions, the eigenvalues of $X^{(d)}(t) = A^{(d)}(t)A^{(d)}(t)^*$ evolve as 2 $BESQ(d)$ which interlace with the $BESQ(d+2)$.
- Can continue this construction *ad infinitum* by adding columns and rows successively of independent complex Brownian motions.

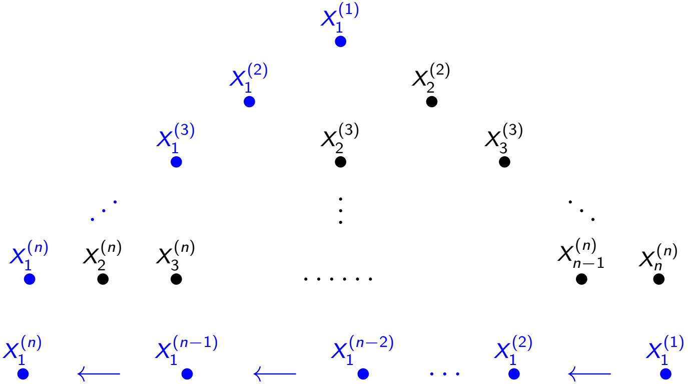
Edge Particle System 1

System 1



Edge Particle System 2

System 2



Edge Particle Systems Generators

We will be concerned with generators,

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$$

with,

$$a(x) = a_0 + a_1 x + a_2 x^2 \quad b(x) = b_0 + b_1 x$$

and boundary points are either *natural* or *entrance* (i.e. the diffusion can never reach them). Define,

$$\begin{aligned} b^{(k)}(x) &= b(x) + (n-k)a'(x) \\ &= b_0 + (n-k)a_1 + (b_1 + 2(n-k)a_2)x \end{aligned}$$

Edge Particle System 1 SDEs

System 1 is given in terms of the following *SDEs* with oblique reflection.

$$X_1^{(1)}(t) = x_1^1 + \int_0^t \sqrt{2a(X_1^{(1)}(s))}d\gamma_1^1(s) + \int_0^t b^{(1)}(X_1^{(1)}(s))ds$$

$$\vdots$$

$$X_m^{(m)}(t) = x_m^m + \int_0^t \sqrt{2a(X_m^{(m)}(s))}d\gamma_m^m(s) + \int_0^t b^{(m)}(X_m^{(m)}(s))ds + K_m^{m,-}(t)$$

$$\vdots$$

$$X_n^{(n)}(t) = x_n^n + \int_0^t \sqrt{2a(X_n^{(n)}(s))}d\gamma_n^n(s) + \int_0^t b^{(n)}(X_n^{(n)}(s))ds + K_n^{n,-}(t)$$

where γ_i^i are independent standard Brownian motions and $K_i^{i,-}$ are positive finite variation processes supported on $X_i^{(i)} = X_{i-1}^{(i-1)}$.

Edge Particle System 2 SDEs

Similarly System 2 is given in terms of the following *SDEs* with oblique reflection.

$$X_1^{(1)}(t) = x_1^1 + \int_0^t \sqrt{2a(X_1^{(1)}(s))}d\gamma_1^1(s) + \int_0^t b^{(1)}(X_1^{(1)}(s))ds$$

$$\vdots$$

$$X_1^{(m)}(t) = x_1^m + \int_0^t \sqrt{2a(X_1^{(m)}(s))}d\gamma_1^m(s) + \int_0^t b^{(m)}(X_1^{(m)}(s))ds - K_1^{m,+}(t)$$

$$\vdots$$

$$X_1^{(n)}(t) = x_1^n + \int_0^t \sqrt{2a(X_1^{(n)}(s))}d\gamma_1^n(s) + \int_0^t b^{(n)}(X_1^{(n)}(s))ds - K_1^{n,+}(t)$$

where γ_1^i are independent standard Brownian motions and $K_1^{i,+}$ are positive finite variation processes supported on $X_1^{(i)} = X_1^{(i-1)}$.

Edge Particle System Transition Densities 1

Denote by $p_t^{(k)}(x,y)$ the transition kernel associated to the diffusion with generator,

$$L^{(k)} = a(x)\frac{d^2}{dx^2} + b^{(k)}(x)\frac{d}{dx}$$

Define,

$$\mathcal{S}_t^{(k),j}(x,x') = \begin{cases} \int_l^{x'} \frac{(x'-z)^{j-1}}{(j-1)!} p_t^{(k)}(x,z)dz & j \geq 1 \\ \partial_{x'}^{-j} p_t^{(k)}(x,x') & j \leq 0 \end{cases}$$

and,

$$s_t(x,x') = det(\mathcal{S}_t^{(i),i-j}(x_i,x_j'))_{i,j\leq n}$$

Proposition (A., O'Connell, Warren)

The process $(X_1^{(1)}(t),\cdots,X_n^{(n)}(t); t\geq 0)$ has transition densities $s_t(x,x')$.

Edge Particle System Transition Densities 2

Define

$$\bar{\mathcal{S}}_t^{(k),j}(x,x') = \begin{cases} -\int_{x'}^{r} \frac{(x'-z)^{j-1}}{(j-1)!} p_t^{(k)}(x,z)dz & j \geq 1 \\ \partial_{x'}^{-j} p_t^{(k)}(x,x') & j \leq 0 \end{cases}$$

and,

$$\bar{s}_t(x,x') = det(\bar{\mathcal{S}}_t^{(i),i-j}(x_i,x_j'))_{i,j\leq n}$$

Proposition (A., O'Connell, Warren)

The process $(X_1^{(1)}(t),\cdots,X_1^{(n)}(t); t\geq 0)$ has transition densities $s_t(x,x')$.

Formulas for largest and smallest eigenvalues

Corollary (A., O'Connell, Warren)

$$\mathbb{P}_{x^{(0)}}\left(X_n^{(n)}(t) \le z\right) = det\left(\mathcal{S}_t^{(i),i-j+1}(x_i^{(0)}, z)\right)_{i,j=1}^{n}$$

$$\mathbb{P}_{\bar{x}^{(0)}}\left(X_1^{(n)}(t) \ge z\right) = det\left(-\bar{\mathcal{S}}_t^{(i),i-j+1}(\bar{x}_i^{(0)}, z)\right)_{i,j=1}^{n}$$

where $x^{(0)} = (x_1^{(0)} \le \cdots \le x_n^{(0)})$ and $\bar{x}^{(0)} = (\bar{x}_1^{(0)} \ge \cdots \ge \bar{x}_n^{(0)})$.

- With $p_t^{(k)}$ the heat kernel and $x^{(0)} = (0, \cdots, 0)$ this recovers a formula for the *GUE* from [12].
- In the $BESQ(d)$ case and $t = 1$ the above give expressions for the largest and smallest eigenvalues for the *LUE* ensemble. Analogous expressions in the Jacobi case as $t \to \infty$.

Further Connections and Developments

- Connection to the notion of *Strong Stationary Duality (SSD)* (see [6]). The framework outlined above gives a *coupling* between a diffusion and its SSD.
- Brownian motions in an interval and diffusions with orthogonal polynomial eigenfunctions fit into framework of diffusions with discrete spectrum which can be analyzed uniformly. Related to the classical theory of Chebyshev systems (see Karlin [8]).
- Similar theory developed in the setting of *Birth and Death* chains (*in preparation*). In particular provides a new (computation free) proof of the *key commutativity relation* used by Borodin and Olshanski in order to construct an infinite dimensional Markov process on the boundary of the Gelfand Tsetlin graph in [3].

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Thank You

Thank you for your attention