Duality, Statistical Mechanics and Random Matrices

Bielefeld Lectures

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Overview

Statistical mechanics motivated by Random Matrix theory

Topics: Mean Field models, Phase transitions, Symmetry breaking, Goldstone modes, Infra-red bounds, Mermin-Wagner, Lee-Yang Zeros, SUSY Hyperbolic Sigma model

Duality : SUSY Statistical mechanics provides equivalent way to study spectral properties of such disordered quantum systems.

Especially useful for analysis of **Random Band Matrices**.

Advantage of Statistical Mechanics Perspective

• It enables one to get estimates and insights not available by direct analysis. Role of Symmetry and Geometry - transparent.

• We will see that ordered and disordered phases of lattice spin systems correspond to different spectral types and quantum time evolutions.

• **Wigner Dyson Universality** of level spacing can be *understood* via the study of the saddle manifold.

Disadvantages

SUSY statistical mechanics is complicated:

Spins are 4 by 4 matrices with both commuting and Grassmann variables.

Complex measure with oscillations and determinants.

For this reason SUSY is often difficult to control analytically.

Focus on simple models to illustrate the main ideas

Duality and Phase transitions in $D \geq 2$

- **Ising model**: Disordered $T > T_c$, ordered $< T_c$. Symmetry Z_2. In 2D dual to Grassmann Free Field - Determinant of "Dirac" .
- **2D XY dual to 2D Coulomb Gas** : plasma phase and dipole phase. Vortices $\sim$ charges. U(1) symmetry.
- **Reinforced Random Walks** in 3D strong reinforcement walk is recurrent; weak reinforcement it is transient Dual to SUSY hyperbolic sigma model.
- **Quantum Heisenberg** Dual to Random adjacent permutations macroscopic loops - BE condensate, $SU(2)$ symmetry.
- **Anderson Transition** in 3D: Localization to delocalization. Dual to SUSY mechanics with $U(1,1|2)$ Symmetry.

Background and History

E. Wigner (1955) studied the *energy level statistics* of a large nucleus.

Wigner's brilliant insight: Statistics Energy levels are given by the eigenvalues of a large random matrix - Mean Field model - each matrix element has equal variance.

Conjecture: After rescaling, statistical properties of local energy level spacings depend only on symmetry .

Universality of Wigner-Dyson Statistics

Proof of Universality for Mean Field Models

F. J. Dyson, M. Gaudin and M. L. Mehta
P. Deift et al; L. Pastur and M. Shcherbina: Invariant ensembles
K. Johansson for perturbation of GUE ensembles

More Recent work:
L. Erdős, B. Schlein and H-T Yau;
T. Tao and V. Vu
L. Erdős, A. Knowles, H-T Yau, J.Yin (Random Graphs.)

Universality of eigenvalue statistics at spectral edges for **RBM** ,
$W \gg N^{5/6}$, Tracy-Widom (A. Sodin)

Gaussian Random Band Matrices:

A. GUE: Gaussian Unitary Ensemble, $H = H^*$,

$$\langle H_{ij} \rangle = 0 \quad \langle H_{ij} \, H_{i'j'} \rangle = \frac{1}{N} \delta_{ij'} \, \delta_{ji'} \quad i, j \in [1, 2, \ldots N]$$

B. RBM: Random Band matrices, $H = H^*$, $\Lambda =$ Large Box

$$\langle H_{ij} \rangle = 0, \quad \langle H_{ij} \, H_{i'j'} \rangle = J_{ij} \, \delta_{ij'} \, \delta_{ji'} \quad i, j \in \Lambda \cap \mathbb{Z}^d$$

where $J_{ij} \approx 0$ *for* $|i - j| > \mathbf{W}$, $\mathbf{W} =$ *band width*. Example:

$$J_{jk} = (-W^2 \Delta + 1)^{-1}(j, k) \approx (2W)^{-1} e^{-|j-k|/W} .$$

Spectrum $H \approx [-2, 2]$.

Conjectures Beyond Mean Field Theory

Fix $W \gg 1$ and let $\Lambda \uparrow \mathbb{Z}^d$, Energy in the Bulk

Conjecture 1: For $\mathbf{D} \geq \mathbf{3}$ local eigenvalue statistics of Random Band Matrix in large box match those of GUE after scaling.

Random Schrödinger version: $-\Delta + \lambda v$, v real, random potential, $|\lambda| \ll 1$, local statistics **GOE**.

ID version: If $W^2 \gg N = |\Lambda|$ local statistics is GUE.

Conjecture 2: In **2D**, Localization length $\approx e^{W^2}$. If $N \gg e^{W^2}$. Poisson statistics.

Simple Examples of Duality

$$N! = \int_0^\infty e^{-t} t^N dt = N^{N+1} \int_0^\infty e^{-N[s - \ln s]}\, ds, \quad (t = Ns)$$

$$\approx N^N e^{-N} \sqrt{2\pi N} \quad \text{expand about } s = 1$$

P(N) $\equiv$ Number of Partitions of N , Hardy - Ramanujan (1918),

$$P(N) = \frac{1}{2\pi i} \oint \frac{1}{\prod_m (1 - z^m)} z^{-N-1} dz \approx \frac{1}{4\sqrt{3}N} e^{\pi\sqrt{2N/3}}$$

Circle Method in Number Theory.

Exercises:

Work out corrections to Stirling's formula

Why can P(N) be represented as above contour integral?

Stationary Phase: Evaluate for large real $k > 0$:

$$\frac{1}{2\pi}\int e^{i\,k\cos(\theta)}d\theta \approx (2\pi k)^{-1/2}\,\cos(k-\pi/4)$$

Key Gaussian Integral Identity

Let $z = (z_1, z_2, ..., z_N), \quad z_j = x_j + iy_j \quad x_j, y_j \in \mathbb{R}$

If $H = H^*$ is an $N \times N$ matrix.

$$\det(H - E - i\epsilon)^{-1} = \int e^{-i\, z^*(H - E_\epsilon) z}\, D_N z$$

$$E_\epsilon = E + i\,\epsilon, \quad \epsilon > 0, \qquad D_N z \equiv \prod_j^N \frac{dx_j dy_j}{i\pi}$$

If H is Gaussian of mean 0 then we have

$$\langle e^{-i\, z^* H z} \rangle_H = e^{-1/2 \langle (z^* H z)^2 \rangle_H} = e^{-(z^* z)^2/2N}\ GUE$$

GUE Average of $\det(H - E_\epsilon)^{-1}$

$$\langle \det(H - E_\epsilon)^{-1} \rangle_{GUE} = \int e^{-(z^* z)^2/2N - iE_\epsilon z^* z} D_N z$$

$$e^{-(z^* z)^2/2N} = \frac{1}{2\pi N} \int e^{-N s^2/2} e^{-i s z^* z} ds$$

Now we can integrate quadratic expression in $z^* z$ to get

$$2\pi N \langle \det(H - E_\epsilon)^{-1} \rangle_{GUE} = \int_{-\infty}^{\infty} e^{-Ns^2/2} (s - E_\epsilon)^{-N} ds$$

Saddle Point: $s = E/2 - i\sqrt{1 - (E/2)^2}$

Exercise: Find the asymptotics for $|E| \leq 1.8$

Mean field Heisenberg Model

Heisenberg spins: $s_j \in \mathbb{S}^2,\ j = 1, 2, ...N.\quad \beta \sim Temp^{-1}$

Magnetic field $h \in \mathbb{R}^3$, Dual variable $x \in \mathbb{R}^3$,

$$Z_N(\beta, h) \equiv \int_{s_j \in \mathbb{S}^2} exp\ \{\frac{\beta}{2N}(\sum_j^N s_j)^2 + \ h \cdot \sum s_j\} \prod_j^N d\mu(s_j)$$

Decoupling trick (Hubbard-Stratonovich):

$$exp\ \frac{\beta}{2N}\{(\sum_j^N s_j)^2\} = (\frac{N}{2\pi\beta})^{3/2} \int e^{x\cdot(\sum^N s_j)}\, e^{-\frac{N}{2\beta}x^2}\, d^3x$$

$$Let\ \ F(|x|) = \int e^{x\cdot s_j} d\mu(s_j) = |x|^{-1} \sinh |x|$$

Phase transition

$$Z_N(\beta, h) = (\frac{N}{2\pi\beta})^{3/2} \int e^{N\{\ln F(|x+h|-|x|^2/2\beta\}} d^3x.$$

When $h = 0$, Minima of $\{\ln F(|x|) - |x|^2/2\beta\}$:

If $\beta \leq 3$, $x^*(\beta) = 0$, $M(\beta) = 0$

If $\beta > 3$, $x^*(\beta) = r(\beta)U$, $\quad r(\beta) > 0$, $\quad U \in \mathbb{S}^2$, Saddle manifold

Second order transition: $M(\beta) = r(\beta)/\beta$ and vanishes continuously as $\beta \downarrow \beta_c = 3$.

$$M^2(\beta) \equiv \frac{1}{N^2} \langle\, [\sum_j^N s_j\,]^2\,\rangle(\beta, 0)$$

Exercises:

1) Prove that for $\beta < 3$, $\{\ln F(|x|) - |x|^2/2\beta\}$ is a concave function of $r = |x|$, $F(r) = \frac{\sinh r}{r}$, thus r=0 only saddle point.

2) For GUE

$$2\pi N \langle \det(H - E_\epsilon)^{-1} \rangle_{GUE} = \int_{-\infty}^{\infty} e^{-Ns^2/2} (s - E_\epsilon)^{-N} ds$$

Saddle Point: $s = E/2 - i\sqrt{1 - (E/2)^2}$

Show that this saddle is correct and analyze the integral by contour deformation. for $|E| \leq 1.8$

To see the dependence on h expand about the minimum:

$$\ln F(|x + h/N|) - \ln F(r) = \frac{1}{N} M(\beta) U \cdot h$$

Thus as $N \to \infty$,

$$\frac{Z_N(\beta, h/N)}{Z_N(\beta, 0)} = \int_{\mathbb{S}^2} e^{M(\beta) h \cdot U} d\mu(U) = |Mh|^{-1} \sinh(M|h|)$$

Formula is valid if we replace $s_j \in \mathbb{S}^2$ by a potential distribution $V(s_j) = ((s_j)^2 - 1)^2, \; s \in \mathbb{R}^3$.

The large N limit depends only on the symmetry group and on $M(\beta)$ - Analogous, to Wigner-Dyson universality!

Lattice Spin systems and Lee-Yang

Let $\Lambda \subset \mathbb{Z}^d$. For $j \in \Lambda$ we consider the spin $S_j = \pm 1$ Ising; $S_j = (\cos(\theta_j), \sin(\theta_j))$ XY model; $S_j \in \mathbb{S}^2$ Heisenberg.

The partition function

$$Z_\Lambda(\beta, h) = \int \exp[\sum_{j,k} \beta J_{j,k} S_j \cdot S_k + \sum_j h \cdot S_j] \prod_{j \in \Lambda} d\mu(S_j)$$

Lee-Yang Theorem If $\beta J_{j,k} \geq 0$, then all Zeros of $Z_\Lambda(\beta, h)$ are on the imaginary h axis.

Corollary: $f(\beta, h) = \lim_{\Lambda \to Z^d} |\Lambda|^{-1} \log Z_\Lambda(\beta, h)$ is analytic if $Re\, h > 0$. No Phase transition for $Re\, h > 0$.

$$\frac{d^2}{dh^2} f(\beta, h) = \sum_j [\langle S_0 S_j \rangle(\beta, h) - \langle S_0 \rangle(\beta, h) \langle S_j \rangle(\beta, h)] < \infty$$

Lattice Spin systems and Mean Field

Let

$$J_{jk} = (-W^2\Delta + 1)^{-1}(j,k) \approx e^{-|j-k|/W} ,$$

For Heisenberg spins:

If $\beta < 3$, $h = 0$. The spin correlations decay exponentially fast for all W.

$$\langle S_0 S_j \rangle(\beta, h = 0) \leq e^{-|j|/\ell(\beta)}$$

After duality the Gibbs weight is log concave.

In 3D , if $\beta > 3$ and W is large then there is a magnetization $M(\beta) > 0$ and order

$$\langle S_0 S_j \rangle(\beta, h = 0) \to M^2(\beta) > 0 \ as \ |j| \to \infty$$

Mean Field Conjecture for $d \geq 3$

Conjecture: Let Λ be a periodic cube of side L

$$\frac{Z_\Lambda(\beta, h/|\Lambda|)}{Z_\Lambda(\beta, 0)} = \int_{|u|=1} e^{Mu\cdot h} d\mu(u)\,(1 + c\beta^{-1}\,L^{-(d-2)})$$

Theorem (Fröhlich-Sp) Holds for almost all β for X-Y model.

Conjecture Same result for 3D SUSY models: Saddle manifold is $U(1,1|2)/U(1|1) \times U(1|1)$. Gives Wigner-Dyson Universality with corrections for 3D RBM.

Product of Characteristic Polynomials - GUE

Let H denote and $N \times N$ GUE matrix Define

$$D_N(E, \delta) = \det(H - E + \delta/N)\ \det(H - E - \delta/N)$$

If $|E| < 2$ then as $N \to \infty$

$$\frac{\langle D_N(E, \delta) \rangle_{GUE}}{\langle D_N(E, 0) \rangle_{GUE}} = \sin(\bar{\rho}(E)\delta)/(\bar{\rho}(E)\delta)$$

Here $\bar{\rho}(E)$ is proportional to the density of states.

Characteristic Polynomial for Gaussian Band Matrices

Let H be an $N \times N$ Gaussian Band matrix of width W.

Theorem (T. Shcherbina) In **1D** if $|E| < 2$, *and* $W^2 \gg N$ then

$$\frac{\langle D_N(E,\delta)\rangle_{RBM}}{\langle D_N(E,0)\rangle_{RBM}} = \sin(\bar{\rho}(E)\delta)/(\bar{\rho}(E)\delta)$$

Theorem (M. and T. Shcherbina) If $W^2 \ll N$ then the limit is 1.

We shall see that 1D band is similar to a 1D Heisenberg spin chain of length N at with $\beta \sim W^2$.

The spins in the chain are aligned when $W^2 \sim \beta \gg N$.

Grassmann Integral Identity

Let ψ_j, ψ_j^* anti-commuting:

$$\psi_k \psi_j = -\psi_j \psi_k, \;\; \psi_k^* \psi_j = -\psi_j \psi_k^*, \;\; (\psi_k^*)^2 = (\psi_j)^2 = 0$$

$$\Psi = (\psi_1, \psi_2, ..., \psi_N) \quad and \quad \Psi^* = (\psi_1^*, \psi_2^*, ..., \psi_N^*)$$

$$\det(H - E_\epsilon) = \int e^{-\,\Psi^*(H-E_\epsilon)\Psi}\, D_N\Psi$$

where $D_N\Psi = \prod_j^n d\psi_j^* \, d\psi_j$

Integration Rule: $\int \prod_j^N \psi_j \psi_j^* \;\, D_N\Psi \equiv 1$

Integrals of lower order polynomials vanish.

Sketch of proof for GUE

If $\psi_j,\ \bar{\psi}_j\,,\chi_j,\ \bar{\chi}_j,\ \ 1 \le j \le N,$ are Grassmann variables then

$$D_N(E,\delta) = \int e^{-\{\bar{\psi}\cdot(H-E)\psi + \bar{\chi}\cdot(H-E)\chi + \frac{\delta}{N}(\bar{\psi}\cdot\psi - \bar{\chi}\cdot\chi)\}}\, D\psi\, D\chi$$

$$\langle D_N(E,\delta)\rangle_{GUE} = \int e^{-\{tr\ Q^2/2N + tr\ QE + \frac{\delta}{N}(\bar{\psi}\cdot\psi - \bar{\chi}\cdot\chi)\}}\, D\psi\, D\chi$$

$$where\ \ Q = \begin{pmatrix} \bar{\psi}\cdot\psi & \bar{\psi}\cdot\chi \\ \bar{\chi}\cdot\psi & \bar{\chi}\cdot\chi \end{pmatrix}$$

Exercise: Prove this Formula

Let X be a 2×2 Hermitian matrix and let DX be the flat measure.

$$e^{-tr\, Q^2/2N} = C_N \int e^{-i\, tr\, XQ} e^{-N\, tr\, X^2/2}\, DX$$

the Grassmann's are now quadratic and we can trace them out:

$$\langle D_N(E,\delta)\rangle_{GUE} = C_N \int \det(i\, X - \tilde{E})^N\, e^{-N\, tr\, X^2/2}\, DX$$

where $\tilde{E} = diag(E + \delta/N, E - \delta/N) = E\, I_2 + \delta/N\, \sigma_3$

By diagonalizing X we find the saddle X_s (at $\delta = 0$) has the following form :

$$\check{X}_s = \{iE/2 \pm \bar{\rho}(E)\}\, I_2 \quad or \quad \hat{X}_s = U^* \{iE/2 \ I_2 + \bar{\rho}(E)\, \sigma_3\}\, U$$

where $\sigma_3 = diag(1, -1)$, $U \in SU(2)$ and

$$\bar{\rho}(E) = 2\pi\rho(E) = \sqrt{1 - (E/2)^2}.$$

Since

$$\frac{det(iX - E + \delta/N\sigma_3)}{det(iX - E)} = e^{tr(iX-E)^{-1}\delta\sigma_3/N}$$

Integration over the saddle manifold $\hat{X}_c$ is

$$\int e^{i\delta\, tr\,(X-iE)^{-1}\sigma_3} d\mu(U) = \int e^{i\delta\, \bar{\rho}\, tr\, U^*\sigma_3 U\sigma_3} d\mu(U)$$

$$= \sin(\bar{\rho}(E)\delta)/(\bar{\rho}(E)\delta)$$

$\hat{X}_s$ is Dominant Saddle Manifold

The contribution of $\check{X}_s$ *and* $\hat{X}_s$ has the same modulus.

However at $\check{X}_s$, the Jacobian in

$$DX = (\lambda_1 - \lambda_2)^2 d\lambda_1\, d\lambda_2\, d\mu(U)$$

vanishes since the eigenvalues λ_1, λ_2 coincide.

Spectral information via Green's Function

The Green's function for H provides spectral information near E:
Let N or box Λ be fixed. Let $\varepsilon > 0$

$$tr\, Im\,(H - E - i\varepsilon)^{-1} = tr\frac{\varepsilon}{(H-E)^2 + \varepsilon^2} \equiv \pi\, tr\, \delta_\epsilon(H - E)$$

Counts eigenvalues in ϵ - neighborhood of E.

$$tr\,(H - E - i\varepsilon)^{-1} = \frac{d}{dE'}\,\frac{\det(H - E - i\epsilon)}{\det(H - E' - i\epsilon)}\,|_{E=E'}$$

GUE Average Green's Function

$$\rho_N(E,\varepsilon) \equiv Im \frac{1}{N} \langle tr(H - E - i\epsilon)^{-1} \rangle_{GUE} = Im \langle s_1 \rangle_{SUSY}$$

$$\equiv \frac{N}{2\pi} \int s_1 \, e^{-N(s_1^2+s_2^2)/2} \frac{(i\, s_2 - E_\epsilon)^N}{(s_1 - E_\epsilon)^N} \cdot R(s_1, s_2) ds_1 ds_2$$

where $R \equiv 1 - (s_1 - E_\epsilon)^{-1}(i\, s_2 - E_\epsilon)^{-1}$

Note $\langle 1 \rangle_{SUSY} = 1$. Deform the contour of integration.

Analysis about saddle point: $s_1 = E/2 - i\sqrt{1 - (E/2)^2}$
There is another sub-dominant saddle point at s_1^*.

Block GUE Matrix

Let H_1 and H_2 be independent $N \times N$ GUE matrices.

$$H \equiv \begin{pmatrix} H_1 & c\, I_N \\ c\, I_N & H_2 \end{pmatrix}$$

where $c > 0$.

$$\rho_\Lambda(E, \varepsilon) = \frac{1}{2\pi N} Im \, \langle tr(H - E - i\epsilon)^{-1} \rangle_{GUE}$$

Integral in 4 variables. Saddle points solve **cubic** equation.

Universality: *local eigenvalue statistics* and correlations should be the same as for GUE.

Density of States ρ for **RBM**

Let $S = (S_1(j), S_2(j)) \in \mathbb{R}^2, \;\; j \in \Lambda \cap \mathbb{Z}^d,$

$$\rho_\Lambda(E,\varepsilon) \equiv \frac{1}{|\Lambda|}\langle tr(H - E - i\epsilon)^{-1}\rangle_{RBM} = \langle S_1(0)\rangle_{SUSY}$$

$$= C_N \int S_1(0) e^{-\sum_j [\mathbf{W}^2(\nabla S)(j)^2 + S(j)^2]/2} \cdot \mathbf{R} \cdot \prod_j \frac{(i\, S_2(j) - E_\epsilon)}{(S_1(j) - E_\epsilon)} dS_j$$

$\mathbf{W}$ = Band Width fixed but $\Lambda \uparrow \mathbb{Z}^d$ - Statistical Mechanics

$$\mathbf{R} = \det\{-W^2\Delta + 1 - \delta_{ij}(S_1(j) - E_\epsilon)^{-1}(i\, S_2(j) - E_\epsilon)^{-1}\}$$

Theorem (Disertori, Pinson, Sp)

For **RBM** with W large and fixed, and $|E| \leq 1.8$,
$\rho_\Lambda(E, \varepsilon)$ is smooth and uniformly bounded as $\varepsilon \downarrow 0$, $\Lambda \uparrow \mathbb{Z}^3$.

Remarks:

Constantinescu, Felder, Gawedzki and Kupiainen
Earlier work for N-orbital model.

Theorem gives sharp estimates on density of states for narrow energy windows: $\varepsilon \approx 1/|\Lambda|$.

Perturbative methods restricted to $\varepsilon \geq 1/W$.