Spectral information via Green's Function

The Green's function for H provides spectral information near E:
Let N or box Λ be fixed. Let $\varepsilon > 0$

$$tr\, Im\, (H - E - i\varepsilon)^{-1} = tr \frac{\varepsilon}{(H-E)^2 + \varepsilon^2} \equiv \pi\, tr\, \delta_\epsilon(H - E)$$

Counts eigenvalues in ϵ - neighborhood of E.

$$tr\, (H - E - i\varepsilon)^{-1} = \frac{d}{dE'} \frac{\det(H - E - i\epsilon)}{\det(H - E' - i\epsilon)} |_{E=E'}$$

GUE Average Green's Function

$$\rho_N(E,\varepsilon) \equiv Im\frac{1}{N}\langle tr(H-E-i\epsilon)^{-1}\rangle_{GUE} = Im\langle s_1\rangle_{SUSY}$$

$$\equiv \frac{N}{2\pi}\int s_1\, e^{-N(s_1^2+s_2^2)/2}\frac{(i\,s_2-E_\epsilon)^N}{(s_1-E_\epsilon)^N}\cdot R(s_1,s_2)ds_1ds_2$$

where $R \equiv 1-(s_1-E_\epsilon)^{-1}(i\,s_2-E_\epsilon)^{-1}$

Note $\langle 1\rangle_{SUSY} = 1$. Deform the contour of integration.

Analysis about saddle point: $s_1 = E/2 - i\sqrt{1-(E/2)^2}$
There is another sub-dominant saddle point at s_1^*.

Block GUE Matrix

Let H_1 and H_2 be independent $N \times N$ GUE matrices.

$$H \equiv \begin{pmatrix} H_1 & c\, I_N \\ c\, I_N & H_2 \end{pmatrix}$$

where $c > 0$.

$$\rho_\Lambda(E, \varepsilon) = \frac{1}{2\pi N} Im \, \langle tr(H - E - i\epsilon)^{-1} \rangle_{GUE}$$

Integral in 4 variables. Saddle points solve **cubic** equation.

Universality: *local eigenvalue statistics* and correlations should be the same as for GUE.

Density of States ρ for **RBM**

Let $S = (S_1(j), S_2(j)) \in \mathbb{R}^2, \;\; j \in \Lambda \cap \mathbb{Z}^d$,

$$\rho_\Lambda(E,\varepsilon) \equiv \frac{1}{|\Lambda|}\langle tr(H - E - i\epsilon)^{-1}\rangle_{RBM} = \langle S_1(0)\rangle_{SUSY}$$

$$= C_N \int S_1(0) e^{-\sum_j [\mathbf{W}^2 (\nabla S)(j)^2 + S(j)^2]/2} \cdot \mathbf{R} \cdot \prod_j \frac{(i\,S_2(j) - E_\epsilon)}{(S_1(j) - E_\epsilon)} dS_j$$

$\mathbf{W}$ = Band Width fixed but $\Lambda \uparrow \mathbb{Z}^d$ - Statistical Mechanics

$$\mathbf{R} = \det\{-W^2\Delta + 1 - \delta_{ij}(S_1(j) - E_\epsilon)^{-1}(i\,S_2(j) - E_\epsilon)^{-1}\}$$

For $0 < \delta \leq |E| \leq 1.8$ analysis RBM completed in 1,2,3 dimensions for fixed large W

Brascamp-Lieb Inequality

Let F(x) be a real function of $x \in \mathbb{R}^N$.
Suppose $F(x) \geq c_o + c_1|x|^2,\ c_1 > 0$. For smooth g define

$$\langle g \rangle_F = \frac{\int e^{-F(x)} g(x) d^N x}{\int e^{-F(x)} d^N x}, \ \ and \ \partial g = (\frac{\partial g}{\partial x_j})$$

Theorem (Brascamp-Lieb) If Hessian, $F''(x) > 0$ is a positive matrix and g is differentiable then

$$Var_F(g) = \langle\, (g - \langle g \rangle_F)^2 \rangle_F \leq \langle \partial g \ \cdot [F''(x)]^{-1} \partial g \rangle_F$$

Exercise: If $g = x \cdot v$ and F is quadratic, then equality.

Helffer-Sjöstrand-Witten Formula

Let F(x), be smooth with suitable growth for large $|x|$. Let

$$\partial_j = \frac{\partial}{\partial x_j} \text{ and } \partial_j^* = -\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} F(x).$$

Define $\partial^* \partial = \sum_j \partial_j^* \partial_j$, note $[\partial_k, \partial_j^*] = \partial_j \partial_k F$

$$\langle g_1 g_2 \rangle_F - \langle g_1 \rangle_F \langle g_2 \rangle_F = \langle \partial g_1, \, [\partial^* \partial + F(x)"]^{-1} \, \partial g_2 \rangle_F$$

Since $\sum_j \partial_j^* \partial_j \geq 0$, Brascamp-Lieb follows when $g_1 = g_2 = g$.

$$[\partial^* \partial + F(x)"]^{-1} \text{ acts on } g_j(x) \in L_2(R^N, e^{-F}), \; j \in \Lambda$$

Proof of HSW formula

Let U(x) be the solution to the elliptic equation:

$$\sum_j \partial_j^* \partial_j \, U(x) = \partial^* \partial \, U = g_1 - \langle g_1 \rangle_F$$

Then

$$\langle [g_1 - \langle g_1 \rangle_F] \, g_2 \rangle_F = \langle \partial^* \partial \, U \, g_2 \rangle_F = \langle \partial \, U \, \partial \, g_2 \rangle_F$$

Equation for $\partial \, U$:

$$[\partial^* \partial + F(x)''] \partial \, U = \partial \, g_1$$

To complete the proof invert left side to solve for ∂U.

Application of Helffer-Sjöstrand-Witten Formula

Let $x_j \in \mathbb{R}^3$, and $J_{jk} = (-W^2\Delta + 1)^{-1}(j,k) \approx e^{-|j-k|/W}$,

$$Z_\Lambda(\beta) = \int \exp[\sum_{j,k} \beta J_{j,k} S_j \cdot S_k] \prod_{j \in \Lambda} d\mu(S_j)$$

$$= Z_G^{-1} \int e^{\sum x_j s_j} e^{-\frac{1}{2\beta}[x \cdot (-W^2\Delta_\Lambda + 1)x]} \prod_{j \in \Lambda} dx_j d\mu(S_j)$$

$$= Z_G^{-1} \int e^{\sum_j \ln f(|x_j|) - \frac{1}{2\beta}[x \cdot (-W^2\Delta_\Lambda + 1)x]} \prod_{j \in \Lambda} dx_j$$

$$\equiv \int e^{-F_\beta(x)} \prod_j dx_j \,, \qquad \textit{and} \;\; f(r) = r^{-1}\sinh r.$$

If $\beta < 3$, then Hessian $F''_\beta(x)$ is positive:

$$F'' = \beta^{-1}(-W^2\Delta+1)-[\ln f(|x_j|)]''\delta_{ij} \geq -\beta^{-1}W^2\Delta+\delta, \quad \delta > 0$$

By HSW the spin correlations decay exponentially fast.

$$\langle S_0 S_j \rangle(\beta) \leq C\, e^{-|j|\sqrt{\delta\beta/W^2}}$$

Infrared bounds and Reflection Positivity

Infrared bounds and Phase Transitions: Fröhlich-Simon-Sp (1976)

Let Λ be a periodic box side L, x_j, $h_j \in R^n$, $j \in \Lambda$. Define

$$F_\Lambda(x) \equiv \frac{1}{2} \sum_{|j-j'|=1} (x_j - x_{j'})^2 + \sum_j V(x_j)$$

$$F_\Lambda(x,h) \equiv \frac{1}{2} \sum_{|j-j'|=1} (x_j - x_{j'})^2 + \sum_j V(x_j - h_j)$$

Example: $V(x) = \lambda(x^2-1)^2$, $x \in \mathbb{R}^3$, large $\lambda \to$ Heisenberg.

$$Z_\Lambda(\beta,h) \equiv \int e^{-\beta F_\Lambda(x,h)} \prod_\Lambda d^n x_j$$

Infrared bound:

$$Z_\Lambda(\beta, h) \leq Z_\Lambda(\beta, 0)$$

$$\langle x_j x_0 \rangle_{\beta F_\Lambda} \equiv Z_\Lambda^{-1} \int x_j x_0 \, e^{-\beta F_\Lambda(x)} \prod_\Lambda d^n x_j$$

$$0 \leq \sum_j e^{ip \cdot j} \langle x_j \, x_0 \rangle_{\beta F_\Lambda} \leq \frac{n}{-\beta \Delta(p)}, \quad p \neq 0$$

where $-\pi \leq p_i \leq \pi$, $i = 1, 2, .., d$ and $p_i = 2\pi j_i / L$, $j_i \in \mathbb{Z}_L$

$$-\Delta(p) = 2 \sum_i (1 - cos \, p_i) \approx p^2 \, .$$

$$
\begin{aligned}
\langle x_0^2 \rangle_{\beta F_\Lambda} &= L^{-d} \sum_p \sum_j e^{ip\cdot j} \langle x_j \, x_0 \rangle_{\beta F_\Lambda} \\
&= L^{-d} \sum_j \langle x_j \, x_0 \rangle_{\beta F_\Lambda} + L^{-d} \sum_{p \neq 0} \sum_j e^{ip\cdot j} \langle x_j \, x_0 \rangle_{\beta F_\Lambda} \\
&\leq L^{-d} \sum_j \langle x_j \, x_0 \rangle_{\beta F_\Lambda} + \int \frac{n dp}{-\beta \Delta(p)}
\end{aligned}
$$

If

$$
\langle x_0^2 \rangle_{\beta F_\Lambda} - n \int_{|p_i| \leq \pi} \frac{dp}{\beta p^2} > 0, \quad d > 2
$$

then

$$
L^{-d} \sum_j \langle x_j \, x_0 \rangle_{\beta F_\Lambda} = M^2 > 0 \, .
$$

In dimension 3 Long Range Order is established for nearest neighbor XY, Heisenberg for $\beta \geq 1$.

There are similar results for the Antiferromagnetic Quantum Heisenberg (Dyson-Lieb-Simon).

However, RP does not apply to Ferromagnetic Quantum Heisenberg. LRO open.

Needed new techniques to understand continuous symmetry breaking especially for SUSY systems.

O(2) symmetric X-Y model

$$s_j = (cos(\theta_j), sin(\theta_j)), \quad j \in \Lambda \subset \mathbb{Z}^d, \quad |j - j'| = 1$$

$$Z_\Lambda(\beta, \varepsilon) = \int e^{\beta\{\sum_{j\sim j'} cos(\theta_j - \theta_{j'}) + \varepsilon \sum_j cos(\theta_j)\}} \prod_\Lambda d\theta_j$$

$$< \cos(\theta_0 - \theta_x) >_\Lambda (\beta, \varepsilon) =$$

$$Z_\Lambda^{-1} \int cos(\theta_0 - \theta_x) e^{\beta\{\sum_{j\sim j'} cos(\theta_j - \theta_{j'}) + \varepsilon \sum_j cos(\theta_j)\}} \prod_\Lambda d\theta_j$$

$$M(\beta) \equiv lim_{\varepsilon \downarrow 0} \langle cos(\theta_0) \rangle (\beta, \varepsilon)$$

$M(\beta) \neq 0 \Rightarrow$ continuous symmetry breaking.

Results for XY model

- For $\varepsilon > 0$, exponential decay of correlations (Lee-Yang, Penrose, Lebowitz, Gu-Ro-Si, Fröhlich)

- In **2D** $0 \leq \langle \cos(\theta_0 - \theta_x) \rangle(\beta) \leq C|x|^{-(2\pi\beta)^{-1}}$ (Mc-Sp)

- In **2D** For $\beta \gg 1$, $\langle \cos(\theta_0 - \theta_x) \rangle(\beta) \geq C|x|^{-(2\pi\bar{\beta})^{-1}}$, $\bar{\beta} \approx \beta$

Duality - map to Coulomb gas - charges-vortices need RNG. (Kosterlitz-Thouless, Fröhlich-Sp)

- In **3D** $\beta > 1/2$, $M(\beta) > 0$ (Fröhlich-Simon-Sp)

Conjecture: **2D** XY models at β_c (Kosterlitz-Thouless)

$$0 \leq \langle \cos(\theta_0 - \theta_x) \rangle (\beta_c) \approx |x|^{-1/4}$$

Recent mathematical work of Falco.

Conjecture: **2D** correlations of the Heisenberg model decay exponentially for all $\boldsymbol{\beta}$. (Polyakov)

Closely related conjectures in **2D**:

Exponential Localization for the Random Schrödinger for **weak disorder** .

Recurrence and localization of Edge Reinforced random walk for **weak reinforcement** SUSY Hyperbolic model.

Mermin-Wagner Theorem and Goldstone mode

The XY model obeys the Ward identity:

$$0 \leq \sum_j \langle \sin(\theta_0) \sin(\theta_j) \rangle(\beta, \epsilon) = \epsilon^{-1} \langle \cos(\theta_0) \rangle(\beta, \epsilon)$$

Proof of Ward Identity: $\theta_j \to \theta_j + \phi$ leaves measure invariant, hence

$$0 = \frac{d}{d\phi} \int \sin(\theta_0 + \phi) e^{\beta\{\sum_{j\sim j'} \cos(\theta_j - \theta_{j'}) + \varepsilon \sum_j \cos(\theta_j + \phi)\}} \prod_\Lambda d\theta_j$$

Now set $\phi = 0$ to get the Ward Identity.

Mermin-Wagner Theorem:

$$\sum_j e^{i\,j\cdot p} \langle \sin(\theta_0)\ \sin(\theta_j) \rangle(\beta, \varepsilon) \geq M^2(\beta p^2 + \varepsilon M)^{-1}$$

Hence if there is symmetry breaking $M > 0$ as $\varepsilon \downarrow 0$ we have **Goldstone mode** $1/p^2$.

Analog for RBM or random Schrödinger: Quantum Diffusion.

$$\sum_{j \in Z^3} e^{ij\cdot p} \langle\, |(H - E - i\varepsilon)^{-1}(0, j)|^2\, \rangle \approx (D(E)p^2 + \varepsilon)^{-1}$$

$D(E)$ is the diffusion at energy E.

Quantum Dynamics

$$H = -\Delta + \lambda V(j), \quad V(j), \text{ random, iid.}$$

$$\sum_j \langle |e^{-itH}(0,j)|^2 \rangle_V |j|^2 \equiv R^2(t)$$

$R^2(t) \approx t^2 \leq C$ Localization.
$R^2(t) \approx Ct^2$ Ballistic,
$R^2(t) \approx D|t|$ Diffusion.

Exercise For any V, show that $\sum_j |e^{-itH}(0,j)|^2 |j|^2 \leq Ct^2$

Manhattan Pinball

Quantum Network Model with random scatterers

Motivation: Chalker's network model Integer Quantum Hall.

Particle moves on $\mathbb{Z}^2$ along streets with alternating orientations scattered by independent random obstructions.

Equivalent to Unitary evolution with SU(2) bond disorder.

Beamond, Cardy, Owczarek, and Gruzberg, Ludwig, Read

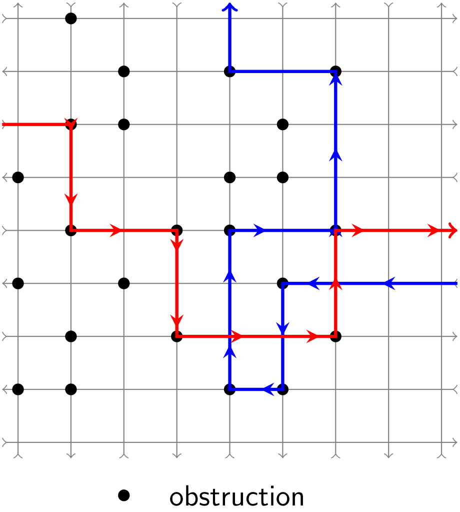

Figure: Manhattan Lattice

Theorem If $p > 1/2$ then all trajectories are closed with probability 1. Localization. Proof (Chalker) by percolation.

Conjecture: All trajectories are closed for any $p > 0$

$$\text{Average loop diameter} \approx e^{c\,p^{-2}} \gg 1.$$

Thus in 2D *any randomness* produces Localization

What is the distribution of the length of **long** cycles?

Mirror model: mirrors at vertices randomly placed at ± 45.
Fully packed mirrors equivalent to critical percolation.
There is **always** an Extended channel. (Kozma-Sidovaricius)