Why Hyperbolic Symmetry?

Consider the very trivial case when $N = 1$ and $H = h$ is a real Gaussian variable of unit variance. To simplify notation set $E_\epsilon = 0 - i\epsilon$ and $z, w \in \mathbb{C}$

$$\langle |G(E_\epsilon)|^2 \rangle_h = \langle [(h - i\epsilon)(h + i\epsilon)]^{-1} \rangle_h$$

$$= \int < exp[ih(z^*z - w^*w) - \epsilon(z^*z + w^*w)] >_h \, dz\, dw$$

$$= \exp[-\frac{1}{2}(z^*z - w^*w)^2 - \epsilon(z^*z + w^*w)]\, dz\, dw.$$

Note that ϵ breaks the hyperbolic symmetry to make the integral well defined.

Hyperbolic sigma models and Reinforced walk

• Quantum Scattering with random impurities - Manhattan Pinball (Beamond, Cardy, Owczarek and Gruzberg, Ludwig, Read)

• Linearly Edge Reinforced Random Walk - Persi Diaconis

• Zirnbauer's SUSY sigma model with a hyperbolic target. Anderson like phase transition in 3D.

• Cycle structure created by Random Adjacent Transpositions on $\Lambda \subset \mathbb{Z}^d$. Universality of Mean Field for macroscopic cycle lengths? (Ueltschi, Chalker, Nahum, Schramm, Toth)

Manhattan Pinball

Quantum Network Model with random scatterers

Motivation: Chalker's network model Integer Quantum Hall.

Particle moves on $\mathbb{Z}^2$ along streets with alternating orientations scattered by random obstructions.

Equivalent to Unitary evolution with SU(2) bond disorder.

Beamond, Cardy, Owczarek, and Gruzberg, Ludwig, Read

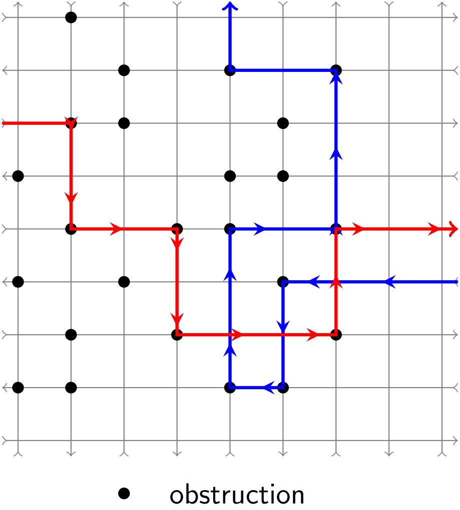

Figure: Manhattan Lattice

Theorem If $p > 1/2$ then all trajectories are closed with probability 1. **Localization**. Proof (Chalker) by percolation.

Conjecture: All trajectories are closed for any $p > 0$

$$\text{Average loop diameter} \approx e^{c\,p^{-2}} \gg 1.$$

Thus in 2D *any randomness* produces **Localization**

Mirror model: mirrors at vertices randomly placed at ± 45.
Fully packed mirrors equivalent to critical percolation.
There is **always** an **Extended channel**. (Kozma-Sidovaricius)

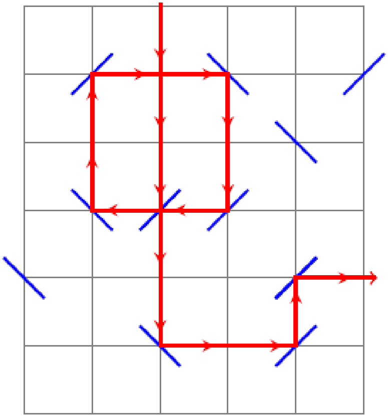

Figure: Cohen's Mirror Model, Mirrors ±45°

Linearly Edge Reinforced Random Walk

History dependent walk $W_n \in \mathbb{Z}^d$, $n \in \mathbb{Z}^+$:

Walk takes nearest neighbor steps and favors edges $j, k, \in \mathbb{Z}^d$, $|j-k| = 1$, it has visited in the past.

Introduced by P. Diaconis while wandering the streets of Paris. He liked to return to streets he had visited in the past.

Remarks:

Not Markovian but is a superposition of Markov Processes

Equivalent to a random walk in a correlated random environment given by statistical mechanics.

Definition of Reinforced Random Walk

Let $C_{jk}(n) =$ number of times the walk has crossed edge jk up to time n and let $\beta > 0$.

$$Prob\{W_{n+1} = k | W_n = j\} = \frac{1 + C_{jk}(n)/\beta}{\mathcal{N}_\beta}, \quad |j - k| = 1.$$

$\mathcal{N}$ is the normalization:
$\mathcal{N}_\beta = \sum_{k'}(1 + C_{jk'}(n)/\beta), \quad |j - k'| = 1$

$0 < \beta \ll 1$, **strong** reinforcement, (high temperature)
$\beta \gg 1$, **weak** reinforcement, (low temperature)

Questions: Is ERRW localized? recurrent? transient?

P. Diaconis and D. Coppersmith (1986):

ERRW $\approx$ random walk in a random environment.

Environment: The rate at which an edge j, j' is crossed $w_{j,j'} > 0$ are correlated random variables. (conductances)

Distribution of $w_{j,j'} > 0$ is correlated, unbounded and given by an **explicit statistical mechanics model**.

The generator for the RW is a weighted Laplacian D_w

$$s^t \cdot D_w\, s = \sum_{|j-j'|=1} w_{j,j'}(s_j - s_{j'})^2, \quad s_j \in \mathbb{R}$$

Local conductance: $\mathrm{w}_{j,j'}$

Quantum of particle on $\mathbb{Z}^3$ scattered by impurities

Quantum Time Evolution is analyzed via the Green's function:

$$G(E + i\epsilon; j, k) = [H(\omega) - E - i\epsilon]^{-1}(j, k) \ .$$

Average of $|G(j,k)|^2$ is the **spin-spin correlation** in SUSY Statistical mechanics.

Spins s_j, $j \in \mathbb{Z}^d \cap \Lambda$ are 4×4 super matrices in box Λ.

Efetov sigma model target: $U(1,1|2)/[U(1|1) \times U(1|1)]$

Study a simpler vector version - Zirnbauer's model.

Zirnbauer's SUSY Hyperbolic Model - $H^{(2|2)}$

Sigma constraint : $z_j^2 - x_j^2 - y_j^2 - 2\eta_j\xi_j = 1$

Horospherical parametrization: $s_j, t_j \in \mathbb{R}$:

$$x = \sinh t - \mathrm{e}^t \left(\tfrac{1}{2}s^2 + \bar\psi\psi\right) \ , \quad y = \mathrm{e}^t s \ , \quad \xi = \mathrm{e}^t \bar\psi \ , \quad \eta = \mathrm{e}^t \psi \ ,$$

$$S_\Lambda = \sum_{i \sim j \in \Lambda} S_{ij} + \epsilon \sum_{k \in \Lambda} z_k$$

$$S_{ij} = \cosh(t_i - t_j) + \tfrac{1}{2}(s_i - s_j)^2 \, \mathrm{e}^{t_i + t_j} + (\bar\psi_i - \bar\psi_j)(\psi_i - \psi_j) \, \mathrm{e}^{t_i + t_j} \ ,$$

$$z_k = \cosh t_k + \left(\tfrac{1}{2}s_k^2 + \bar\psi_k\psi_k\right) \mathrm{e}^{t_k} \ .$$

Local conductance: $w_{j,j'} = e^{t_j + t_{j'}}$.

Partition function and Ward identity: $H^{(2|2)}$

$$Z_\Lambda(\beta,\epsilon) = C \int exp\left[-\beta S_\Lambda\right] \prod_\Lambda e^{-t_j}\, dt_j ds_j d\psi_j d\bar{\psi}_j$$

By supersymmetry $Z_\Lambda(\beta,\epsilon) \equiv 1$ all β,ϵ.

Ward identity:

$$\left\langle e^{\alpha\, t_0}\right\rangle = \left\langle e^{(1-\alpha)\, t_0}\right\rangle, \qquad \left\langle e^{t_0}\right\rangle = 1.$$

$$\beta \approx \rho(E)^2\lambda^{-2}, \quad \rho(E) = density\ of\ states,\ \lambda = disorder$$

Effective Action: Integrate out Grassmann $\bar{\psi}, \psi$ and s variables.

$$E_{SUSY}(\{t_j\}) = \beta \sum_{j \sim j'} cosh(t_j - t_{j'}) - 1/2 \log \det D_{\beta,\epsilon}(t)$$

$D_{\beta,\epsilon}(t)$ a finite difference elliptic operator:

$$[s\,; D_{\beta,\epsilon}(t)\, s]_\Lambda = \beta \sum\nolimits_{(j' \sim j)} \mathrm{e}^{t_j + t_{j'}} (s_j - s_{j'})^2 + \epsilon \sum\nolimits_{k \in \Lambda} \mathrm{e}^{t_k} s_k^2$$

Local conductance $= \; e^{t_j + t_{j'}}$.

Saddle Point t_s of Effective Action:

If $t_s = 0$, $as \, \epsilon \downarrow 0$ - Conduction, **3D**, when β large

If $\epsilon e^{-t_s} \geq e^{-\beta}$ $as \, \epsilon \downarrow 0$ localization, **2D**

Spin-Spin correlation:

$\langle y_0 y_k \rangle =< s_0 e^{t_0} s_k e^{t_k} > (\beta, \epsilon) =< e^{t_0+t_k} D_{\beta,\epsilon}(t)^{-1}(0,k) >_{SUSY} (\beta, \epsilon)$

random walk in a random environment:

In 3D if the local conductance $e^{t_j+t_{j'}}$ does not fluctuate or $t_j \approx 0$
Then correlation

$$< e^{t_0+t_k} D_{\beta,\epsilon}(t)^{-1}(0,k) >_{SUSY} (\beta, \epsilon) \approx (-\beta\Delta + \epsilon)^{-1}(0,k)$$

is diffusive.

Phase Transition in 3D

Theorem (Disertori, S., Zirnbauer '10)

A) For $\beta \gg 1$ and $d \geq 3$ **Q Diffusion:**

$$\langle y_0 y_k \rangle \approx (-\beta\Delta + \epsilon)^{-1}(0, k), \quad k \in \mathbb{Z}^3$$

B) For $0 < \beta \ll 1$, **localization:**

$$\langle y_0 y_k \rangle(\beta, \epsilon) \approx \epsilon^{-1}\, e^{-|k|/\ell(\beta)}$$

Remark:

The Hyperbolic sigma model (No Fermions) has
No phase transition in 3D - always delocalized.

Effective action is convex for all β. (Sp, Zirnbauer, Brydges)

Gruzberg-Mirlin (96) Replica analysis of $H^{(2|2)}$ on Bethe Lattice

Conjectures:

A)There is a multi-fractal transition in 3D

B) In 2D exponential localization holds for all β.

SUSY Hyperbolic sigma model is essentially equivalent to ERRW

Relation made precise by Sabot and Tarres following earlier observations of Kozma, Sznitman, Gawedzki

Phase Transition for **ERRW** in 3D

Theorem (Sabot-Tarres) For **strong** reinforcement,
ERRW is recurrent and we have Localization:

$$Prob\{|W(t) - W(0)| \geq R\} \leq Ce^{-R/\ell(\beta)}, \quad \mathcal{E}\, W^2(t) \leq Const$$

and $\ell(\beta)$ is the localization length.

Theorem (Disertori-Sabot-Tarres) For **weak** reinforcement, $\beta \gg 1$, and $d \geq 3$, $W(t)$ is Transient, quasi-diffusion.

Ideas of Proof in 3D: Ward identities

To prove diffusion must show the t field has very small fluctuation for large β:

$$\boxed{\langle cosh^p(t_j - t_k)\rangle(\beta) \leq Const \quad all\, j, k, \quad \beta \gg 1}$$

If $F(t, s, \overline{\psi}, \psi)$ is $\mathbf{0Sp(2|2)}$ invariant, then

$$< F >_{SUSY} (\beta, \epsilon) = F(0)$$

.

Example: $< F^m >= 1$ all $m \geq 1$.

$$F = cosh(t_j - t_k) + \; e^{t_j + t_k}\left[\frac{1}{2}(s_j - s_k)^2 + (\overline{\psi}_j - \overline{\psi}_k)(\psi_j - \psi_k)\right]$$

Products of Random Transpositions on $\mathbb{Z}^d$.

Let $\Lambda = \Lambda_L \subset \mathbb{Z}^d$ be a box of side L,

$\mathcal{P}_\Lambda(0)$ be the Identity permutation of its vertices at time 0.

Each adjacent edge $(j, j') \in \Lambda,$ has an independent Poisson clock which rings at rate 1.

When the clock rings, we make a transposition $j \leftrightarrow j'$.

$\mathcal{P}_\Lambda(T)$ is the product of these transpositions up to time T.

Question: After time T, what is the cycle structure of $\mathcal{P}_\Lambda(T)$ for large L?

In 2D are these cycles like those of the Manhattan Pinball
$T \approx p^{-2}$??

Theorem (Caputo, Liggett, Richthammer) In Λ_L, if $T \gg L^2$ then $\mathcal{P}_\Lambda(T) \Rightarrow$ uniform weight on all permutations of Λ_L.

Conjecture: For $d \geq 3$ there is a T^* independent of L, such that if $T > T^*$ there are **macroscopic cycles** of length

$$\ell_1 > \ell_2 > \ell_3 \ldots \textit{ where } \ell_j \geq c_j L^d \, .$$

Moreover for $T > T^*$ as $L \to \infty$ the length distribution is **Mean Field** - Poisson-Dirichlet (Schramm, Ueltschi) after scaling.

Thus ℓ_1/ℓ_2 reaches its Mean Field distribution long before $T = L^2$!

Theorem The partition function $Z_\Lambda(\beta)$ for the spin 1/2 Quantum Heisenberg Ferromagnet is

$$Z_\Lambda(\beta) = tr\, e^{\beta \sum_{j\sim j'}^{\Lambda} S_j \cdot S_{j'}} = \mathbb{E}_\Lambda(\beta)\,[\,2^{\# cycles}]$$

where $\mathbb{E}_\Lambda(\beta)$ denotes the expectation of the independent Poisson clocks up to time $T = \beta$.

Macroscopic cycles correspond to Bose-Einstein condensation.

Theorem In 2D there are no macroscopic cycles - Mermin-Wagner

Remark: PD(2) implies that

$$Z(\beta)^{-1} tr\, e^{\beta \sum_{j\sim j'}^{\Lambda} S_j \cdot S_{j'} + \sum_j^{\Lambda} |\Lambda|^{-1} h \cdot s_j} = \int_{S^2} e^{M(\beta) h \cdot S_0} d\mu$$

END