Hole Probabilities for finite and infinite Ginibre ensembles

Randomness 2

ZiF - Bielefeld University

Nanda Kishore Reddy
(IISc Bangalore)

Joint work with
Kartick Adhikari
(IISc Bangalore)

Finite Ginibre ensemble:-

$n \times n$ random matrix G_n with i.i.d Standard Complex Gaussian entries is called n-th Complex Ginibre ensemble.

jpdf of eigenvalues $z_1, z_2, z_3, \ldots, z_n$ of G_n is

$$\text{jpdf} \propto e^{-\sum_{k=1}^{n} |z_k|^2} \prod_{i<j} |z_i - z_j|^2$$

& $$\frac{1}{n!} \det\left(K_n(z_i, z_j) \right)_{1 \leq i,j \leq n}$$

where

$$K_n(z,w) = \sum_{k=1}^{n} \frac{1}{\pi} \frac{z^{k-1} \bar{w}^{k-1}}{(k-1)!} e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2}.$$

$$\mathcal{X}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\frac{z_i}{\sqrt{n}}}$$

$z_1, z_2 \ldots z_n$ are eigenvalues of G_n

As $n \to \infty$ $\quad \mathcal{X}_n \xrightarrow{d} \text{Unif}(\mathbb{D})$

<u>Heuristics:-</u>

Set $\quad \omega_i = \dfrac{z_i}{\sqrt{n}} \quad i = 1, 2 \ldots n$

density of $\omega_1, \omega_2 \ldots \omega_n$ is prop. to

$$e^{-n \sum_{k=1}^{n} |\omega_k|^2} \prod_{i<j} |\omega_i - \omega_j|^2$$

$$= e^{n^2 \left[\frac{1}{n^2} \sum_{i \neq j} \log|\omega_i - \omega_j| - \frac{1}{n} \sum_{k=1}^{n} |\omega_k|^2 \right]}$$

$$\mathcal{X}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\omega_i} \Longrightarrow \mu_e$$

Energy of μ

$$R_\mu = \iint \log|z-w|\, d\mu(z)\, d\mu(w) - \int |z|^2 d\mu(z)$$

μ_e maximizes R_μ.

log. Potential $P_\mu(z) = \int \log|z-w|\, d\mu(w)$

Conditions on μ_e:-

$$P_{\mu_e}(z) - |z|^2 = C \quad \text{for q.e } z \in \operatorname{Supp}(\mu_e)$$

$$P_{\mu_e}(z) - |z|^2 \le C$$

Jensen's formula:-

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z-re^{i\theta}|}\, d\theta = \begin{cases} \log 1/r & \text{if } |z|\le r \\ \log\frac{1}{|z|} & \text{if } |z|>r \end{cases}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|z-re^{i\theta}|\, d\theta = \log(\max\{|z|, r\}).$$

$$\mathcal{X}_n \xRightarrow{d} \mathrm{Unif}(\mathbb{D}).$$

Hole probabilities for finite Ginibre ensembles

$$P(\mathcal{X}_n(U) = 0) \longrightarrow 0 \quad \text{as } n \to \infty \qquad \underset{\text{open set}}{U \subseteq \mathbb{D}}$$

$$P(\mathcal{X}_n(U)=0) = \frac{\int_{U^c} \cdots \int_{U^c} e^{-n\sum_{k=1}^{n}|w_i|^2} \prod_{i<j}|w_i - w_j|^2 \prod_{k=1}^{n} dm(w_k)}{\int_{\mathbb{C}} \cdots \int_{\mathbb{C}} e^{-n\sum_{k=1}^{n}|w_i|^2} \prod_{i<j}|w_i - w_j|^2 \prod_{k=1}^{n} dm(w_k)}.$$

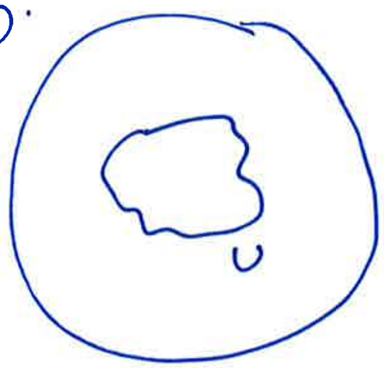
for very large n,

$$P(\mathcal{X}_n(U)=0) \approx \frac{e^{n^2\left[\max\{R_\mu : \mu(U)=0\}\right]}}{e^{n^2 R_{\mathrm{Unif}(\mathbb{D})}}}$$

$$\approx e^{-n^2\left(R_{\mathrm{Unif}(\mathbb{D})} - R_{\mu_{e,U^c}}\right)}$$

μ_{e,U^c} is the measure which maximizes R_μ under the condition that $\mu(U) = 0$

$\mathbb{D}$.



$$\mu_e = \mu_1 + \mu_2.$$

μ_1: restriction of μ_e to U^c

μ_2: restriction of μ_e to U

$$P_{\mu_{e,U^c}}(z) - |z|^2 = C \quad \text{for a.e } z \in \mathrm{Supp}(\mu_{e,U^c})$$

$$\le C \quad \text{outside} \quad "$$

$\mu_{e,U^c} = \mu_1 + \mu_3$ where μ_3 is a measure on ∂U such that

$$P_{\mu_3}(z) = P_{\mu_2}(z) \quad \forall z \in (\overline{U})^c$$

To find μ_3:-

$$\int_{\partial U} \log|z-w|\, d\mu_3(w) = \int_U \log|z-w|\, d\mu_2(w)$$
$$\forall\, |z| \geq 1$$

$$\int_{\partial U} \log(z-w)\, d\mu_3(w) = \int_U \log(z-w)\, d\mu_2(w) + c$$
$$\forall\, |z| \geq 1$$

$$\int_{\partial U} \left[\log z + \sum_{n=1}^{\infty} \frac{w^n}{n z^n}\right] d\mu_3(w) = \int_U \left[\qquad\right] d\mu_2(w) + c$$

$$\Leftrightarrow \quad \int_{\partial U} w^n\, d\mu_3(w) = \int_U w^n\, d\mu_2(w), \quad \forall\, n \geq 0$$

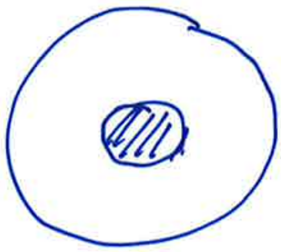
$$\boxed{\int_{\partial U} w^n\, d\mu_3(w) = \frac{1}{\pi} \int_U w^n\, dm(w)}$$

$dm(w)$ denotes Lebesgue measure.

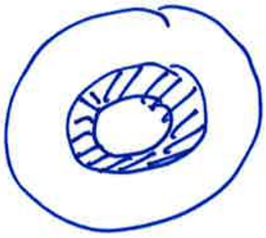
μ_3 is called balayage (sweeping) measure corresponding to μ_2.

Examples with exact computations of R_{μ_e, U^c}

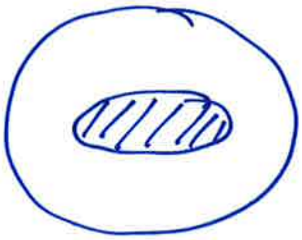
Disk



Annulus



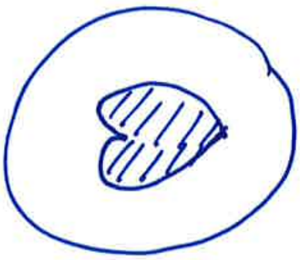
Ellipse



$x = a\cos\theta, \quad y = b\sin\theta$

$$d\mu_3(\theta) = \frac{ab}{2\pi}\left[1 - \frac{a^2-b^2}{a^2+b^2}\cos 2\theta\right] d\theta$$

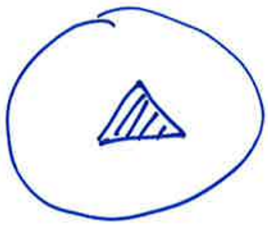
Cardioid



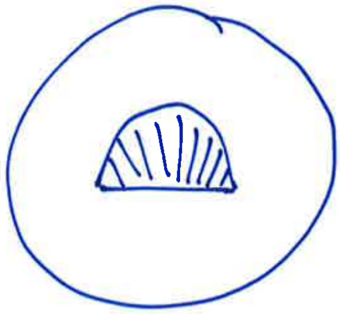
$r = b(1+2a\cos\theta)$

$$d\mu_3(\theta) = \frac{b^2}{2\pi}(1+a^2+2a\cos\theta)\, d\theta$$

Triangle (equilateral)



Semidisk



$$R_{\mu_{e,U^c}} = \frac{3}{4} + \frac{1}{2}\left[\int_{\partial U} |z|^2 d\mu_3(z) - \frac{1}{\pi}\int_U |z|^2 dm(z)\right]$$

Mathematical Steps:-

U open $\quad U \subseteq \mathbb{D}$

$$\lim_{n\to\infty} \frac{1}{n^2} \log P(\mathcal{X}_n(U)=0) = ?$$

Upper bound:-

$$\limsup_{n\to\infty} \frac{1}{n^2} \log P(\mathcal{X}_n(U)=0) \le \sup_{\mu(U)=0} R_\mu - R_{\mu_e}$$

Valid for any open U.

Proof involves Fekete points.

Points $\omega_1^*, \omega_2^*, \ldots \omega_n^*$ where

$$\prod_{i<j}\left(|\omega_i - \omega_j|^2 e^{-|\omega_i|^2} e^{-|\omega_j|^2}\right) \text{ is maximum.}$$

Lower bound :-

(i) $$\liminf_{n\to\infty} \frac{1}{n^2} \log P(\mathcal{X}_n(U)=0) \geq \sup_{\text{dist}(\mu,\bar{U})>0} R_\mu - R_{\mu_e}$$

for nice open sets U

(for ex. sets U which can contain scaled copies of themselves)

we have

$$\lim_{n\to\infty} \frac{1}{n^2} \log P(\mathcal{X}_n(U)=0) = \sup_{\mu(U)=0} R_\mu - R_{\mu_e}$$

(ii)

For sets U which satisfy exterior ball condition. i.e

$\exists\ \varepsilon>0$ such that for every $z \in \partial U$ there exists $\eta \in U^c$ such that

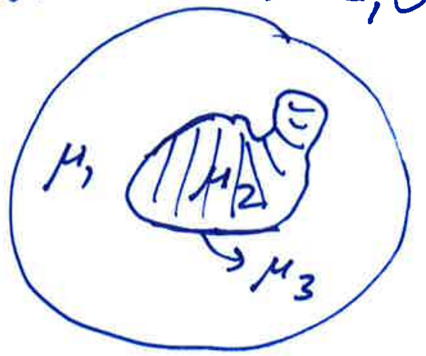
$B(\eta,\varepsilon) \subset U^c$ and $|z-\eta| = \varepsilon$.

Key idea :- Seperation of Fekete points.

$$\min\{|w_i^* - w_j^*| : 1 \leq i \neq j \leq n\} \geq \frac{c}{n^3}$$

Limit :-

$\sup\limits_{\mu(U)=0} R_\mu$ is attained. Let's call ~~that~~ by μ_{e,U^c}



$$\mu_e = \mu_1 + \mu_2$$

$$\mu_{e,U^c} = \mu_1 + \mu_3$$

μ_3 is balayage measure on ∂U, corresponding to μ_2.

$P_{\mu_3} = P_{\mu_2}$ outside support of μ_2.

Reference :-

Edward B. Saff, Vilmos Totik

Logarithmic potentials with external fields

Thomas Ransford

Potential theory in complex plane

n-th Ginibre ensemble is a determinantal point process on Complex plane.

i.e $\rho_k(z_1, z_2 \ldots z_k) = \det\left(K_n(z_i, z_j)\right)_{1 \le i,j \le k}$ (ρ_k: k-th correlation function)

Infinite Ginibre ensemble:-

determinantal point process with kernel.

$$K(z,w) = \frac{1}{\pi} e^{z\bar{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2}$$

$\mathcal{X}_\infty$ denotes the empirical measure of points in the point process.

$$\lim_{r \to \infty} \frac{1}{r^4} \log P\left[\mathcal{X}_\infty(rU) = 0\right] = R_{\mu_{e,U^c}} - R_{\mu_e}$$

$$P\left[\mathcal{X}_{2r^2}\left(\frac{U}{\sqrt{2}}\right) = 0\right] \ge \underbrace{P\left[\mathcal{X}_\infty(rU) = 0\right] \ge}_{\text{hard step}} c\, P\left[\mathcal{X}_{2r^2}\left(\frac{U}{\sqrt{2}}\right) = 0\right]$$

Further directions :-

(i)
$$\prod_{i<j} |z_i - z_j|^2 \prod_{i=1}^{n} e^{-|z_i|^{2\alpha}}$$

also, infinite determinantal point process. (done by Kartick).
arising from it.

open:-

(ii)
$$\prod_{i<j} |z_i - z_j|^2 \prod_{i=1}^{n} e^{-V(z_i)}$$

and the infinite determinantal point process from it

(iii) Real and Complex:-

$$\prod_{i<j} |z_i - z_j|^{2\beta} \prod_{i=1}^{n} e^{-V(z_i)}$$

Key ideas :-

equilibrium measure, Fekete points.