Moscow State University and IITP RAS

On the Local Semicircular Law for Wigner Ensembles

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based on joint work with F. Götze and A. Tikhomirov

Bielefeld 17.08.2016

Papers

1. F. Götze, A. Tikhomirov, Rate of convergence to the semi-circular law. PRTF, 127(2):228–276, 2003.
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4. T. Tao and V. Vu. Random matrices: The universality phenomenon for Wigner ensembles, arXiv:1202.0068, 2012
5. F. Götze and A. Tikhomirov. Optimal bounds for convergence of expected spectral distributions to the semi-circular law. PTRF, 2015.
6. F. Götze, A. Naumov, A. Tikhomirov, Local semicircle law under moment conditions. Part I: The Stieltjes transform, arXiv:1510.07350, 2015
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8. F. Götze, A. Naumov, A. Tikhomirov, On the Local Semicircular Law for Wigner Ensembles, arXiv:1602.03073, 2016

Random matrices

- We consider a random symmetric matrix $\mathbf{X} = [X_{jk}]_{j,k=1}^{n}$ such that:

  $X_{jk}, 1 \leq j \leq k \leq n,$ are independent random variables and $X_{kj} = X_{jk}$.

  We shall assume for $1 \leq j \leq k \leq n$ that

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  $$\mathbb{E}\, X_{jk} = 0, \quad \mathbb{E}\, X_{jk}^2 = 1.$$

- For symmetric matrix $\mathbf{W} := \frac{1}{\sqrt{n}}\mathbf{X}$ we denote its n eigenvalues in the increasing order as

  $$\lambda_1(\mathbf{W}) \le \ldots \le \lambda_n(\mathbf{W})$$

  and introduce the eigenvalue counting function

  $$N_I(\mathbf{W}) := |\{1 \le k \le n : \lambda_k(\mathbf{W}) \in I\}|$$

  for any interval $I \subset \mathbb{R}$. We also define $F_n(\lambda) := \frac{1}{n} N_{(-\infty,\lambda]}(\mathbf{W})$.

Semicircle law, Classical result

- Denote by $g_{sc}(x)$ the density function of the semicircle law

$$g_{sc}(\lambda) := \frac{1}{2\pi}\sqrt{4-\lambda^2}\,\mathbb{I}[|\lambda| \leq 2].$$

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- **T h e o r e m (E. Wigner, 1955)**. Let $X_{jk}, 1 \leq j \leq k \leq n$, be i.i.d. r.v. taking values $+1$ and -1 with probability $1/2$ (in this case $\mathbb{E}\, X_{jk} = 0, \mathbb{E}\, X_{jk}^2 = 1$). Then

$$\lim_{n\to\infty} \frac{1}{n}\,\mathbb{E}\, N_I(\mathbf{W}) = \int_I g_{cs}(\lambda)\, d\lambda.$$

E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Ann. of Math. (2)*, 62:548–564, 1955.

This result has been extended in various aspects. See Wigner, 1958; Grenader, 1963; Arnold, 1967; Pastur, 1973; Girko, 1985, ...

Local semicircle law

- We may rewrite the result of Wigner in the following form

$$\frac{1}{n|I|} N_I(\mathbf{W}) = \frac{1}{|I|} \int\limits_I g_{sc}(\lambda)\, d\lambda + \mathcal{O}\left(\frac{\Delta_n^*}{|I|}\right), \tag{1}$$

where

$$\Delta_n^* := \sup_{x \in \mathbb{R}} |F_n(x) - G_{sc}(x)|, \quad G_{sc}(x) := \int\limits_{-\infty}^{x} g_{sc}(\lambda)\, d\lambda.$$

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- It is of the great interest to investigate the case of smaller intervals where the number of eigenvalues cease to be macroscopically large. Here an appropriate analytical tool for asymptotic approximations is the Stieltjes transform.

Local semicircle law

- Define the Stieltjes transform of F_n by

$$m_n(z) := \int_{-\infty}^{\infty} \frac{dF_n(\lambda)}{\lambda - z} = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{\lambda_j - z} = \frac{1}{n}\,\mathrm{Tr}(\mathbf{W} - z\mathbf{I})^{-1},$$

where $z = u + iv, v \geq 0$ (i.e. $z \in \mathbb{C}^+$).

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- $m_n(z)$ is Nevanlinna function (complex function which is an analytic function on the open upper half-plane $\mathbb{C}^+$ and has non-negative imaginary part. A Nevanlinna function maps the upper half-plane to itself).

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- $m_n(z)$ is Nevanlinna function (complex function which is an analytic function on the open upper half-plane $\mathbb{C}^+$ and has non-negative imaginary part. A Nevanlinna function maps the upper half-plane to itself).
- The Stieltjes transform of Wigner's semicircle law is given by

$$s(z) = \int_{-\infty}^{\infty} \frac{g_{sc}(\lambda)\, d\lambda}{\lambda - z} = -\frac{z}{2} + \sqrt{\frac{z^2}{4} - 1}.$$

Local semicircle law

- Remember that

$$\operatorname{Im} m_n(u+iv) = \int_{-\infty}^{\infty} \frac{v}{(\lambda-u)^2+v^2}\, dF_n(\lambda) = \frac{1}{v}\int_{-\infty}^{\infty} K\left(\frac{u-\lambda}{v}\right) dF_n(\lambda)$$

is the kernel density estimator with Poisson's kernel K and bandwidth v. For a meaningful estimator of the spectral density we cannot allow v to be smaller than the typical $\frac{1}{n}$ -distance between eigenvalues.

Hence, we shall be mostly interested in the situations when

$$v \geq \frac{c}{n}, c > 0,$$

where in some situations c may depend on n and grow to infinity, for example, like $\log n$.

Density recovery

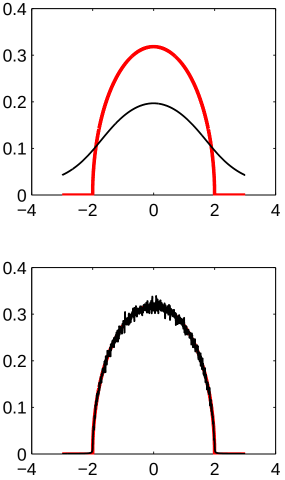

Figure : Let $n = 5000$. In the top row $v = 1$ (on the left) and $v = 0.1$ (on the right) In the bottom row $v = 0.005$ (on the left) and $v = 0.0007$ (on the right)

Local semicircle law

- Significant progress was recently made in a series of results by L. Erdös, B. Schlein, H.-T. Yau and et al., showing that with high probability uniformly in $u \in \mathbb{R}$
$$|m_n(u+iv) - s(u+iv)| \leq \frac{\log^\beta n}{nv}, \quad \beta > 0, \tag{2}$$
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- It means that the fluctuations of $m_n(z)$ around $s(z)$ are of order $(nv)^{-1}$ (up to a logarithmic factor). The value of β depends on n, to be exact $\beta := \beta_n = c \log\log n$.
- To prove (2) it was assumed that the distribution of X_{jk} for all $1 \le j, k \le n$ has sub-exponential tails. Later on this assumption had been relaxed to requiring $\mathbb{E}\,|X_{jk}|^p \le \mu_p$ for all $p \ge 1$, where μ_p are some constants. Finally in the series of papers by Erdös, Knowles, Lee, Yau, Yin and Götze, N., Tikhomirov it was proved under condition $\mathbb{E}\,|X_{jk}|^{4+\delta} < \infty, \delta > 0$.

- **T h e o r e m** (Götze, N., Tikhomirov). Assume that $X_{jk}, 1 \le j \le k \le n$ are i.i.d. r.v. such that $\mathbb{E}\, X_{jk} = 0, \mathbb{E}\, X_{jk}^2 = 1$ and $\mathbb{E}\,|X_{jk}|^{4+\delta} < \infty$. Let $V > 0$ be some constant.

  (i) There exist positive constants A_0, A_1 and C depending on V and δ such that

  $$\mathbb{E}\,|m_n(z) - s(z)|^p \le \left(\frac{Cp}{nv}\right)^p,$$

  for all $1 \le p \le A_1 \log n$, $V \ge v \ge A_0 n^{-1} \log n$ and $|u| \le 2 + v$.

  (ii) For any $u_0 > 0$ there exist positive constants A_0, A_1 and C depending on u_0, V and δ such that

  $$\mathbb{E}\,|\operatorname{Im} m_n(z) - \operatorname{Im} s(z)|^p \le \left(\frac{Cp}{nv}\right)^p,$$

  for all $1 \le p \le A_1 \log n$, $V \ge v \ge A_0 n^{-1} \log n$ and $|u| \le u_0$.

Local semicircle law

- As a consequence of this result we may show that for all $K > 0$

$$\mathbb{P}\left(|m_n(z) - s(z)| \geq \frac{K}{nv}\right) \leq \left(\frac{Cp}{K}\right)^p, \tag{3}$$

valid for all $1 \leq p \leq A_1 \log n$, $V \geq v \geq A_0 n^{-1} \log n$ and $|u| \leq 2 + v$. Taking p and K of order $\log n$ we may guarantee that (3) is less then, for example, n^{-2}. Thus, one has $\beta = 1$.

Rate of convergence to the semicircle law

- Recall that

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Rate of convergence to the semicircle law

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- Assuming $\max_{1\le j,k\le n} \mathbb{E}\,|X_{jk}|^{12} =: \mu_{12} < \infty$, one may obtain the following estimate
$$\mathbb{E}\,\Delta_n^* \le \mu_{12}^{\frac{1}{6}} n^{-\frac{1}{2}}.$$
See Götze and Tikhomirov, 2003. In particular this estimate implies by Markov's inequality that
$$\mathbb{P}\,(\Delta_n^* \ge K) \le \frac{\mu_{12}^{\frac{1}{6}}}{K n^{\frac{1}{2}}}.$$

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- It is easy to see from the previous bound that one may take $K \gg n^{-\frac{1}{2}}$. This result has been extended by Bai and et al. 2011, showing that instead of 12 finite moments it suffices to require finiteness of 6 moments.

Rate of convergence to the semicircle law

- **T h e o r e m** (Götze, N., Tikhomirov) Assume that $X_{jk}, 1 \le j \le k \le n$ are i.i.d. r.v. such that $\mathbb{E}\, X_{jk} = 0, \mathbb{E}\, X_{jk}^2 = 1$ and $\mathbb{E}\, |X_{jk}|^{4+\delta} < \infty, \delta > 0$. There exist positive constants c and C depending on δ such that for all $1 \le p \le c \log n$

$$\mathbb{P}\left(\Delta_n^* \ge K\right) \le \frac{C^p \log^{2p} n}{K^p n^p}$$

for all $K > 0$.

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$$\mathbb{P}\left(\Delta_n^* \geq \frac{C' \log^2 n}{n}\right) \leq n^{-C''}.$$

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- Lower bound for (GUE Gustavsson, 2005)
$$\Delta_n^* = \Omega\left(\frac{\sqrt{\log n}}{n}\right).$$
Similar estimate for GOE (O'Rourke).

Rate of convergence to the semicircle law

- Instead of Δ_n^* one may study the following distance of the mean spectral distribution to its limit

$$\Delta_n := \sup_{x\in\mathbb{R}} |\mathbb{E}\, F_n(x) - G_{sc}(x)|.$$

T h e o r e m (Götze, Tikhomirov (2015); Götze ; N., Tikhomirov (2016)) Assume that $X_{jk}, 1 \leq j \leq k \leq n$ are i.i.d. r.v. such that $\mathbb{E}\, X_{jk} = 0, \mathbb{E}\, X_{jk}^2 = 1$ and $\mathbb{E}\, |X_{jk}|^{4+\delta} < \infty, \delta > 0$. There exists a positive constant $C(\delta)$ depending on δ such that

$$\Delta_n \leq \frac{C(\delta)}{n}.$$

Rigidity

- Let us define the quantile position of the j-th eigenvalue by

$$\gamma_j : \quad \int_{-\infty}^{\gamma_j} g_{sc}(\lambda)\, d\lambda = \frac{j}{n}, \quad 1 \le j \le n.$$

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(i). For all $j \in [K, n-K+1]$ there exist constants c and C, C_1 depending on δ such that such that for all $1 \le p \le c \log n$ we have

$$\mathbb{P}(|\lambda_j - \gamma_j| \ge C_1 K[\min(j, n-j+1)]^{-\frac{1}{3}} n^{-\frac{2}{3}}) \le \frac{C^p \log^{2p} n}{K^p}.$$

(ii). Assume that $\delta = 4$. For any $0 < \phi < 2$ and all $j \le K$ or $j \ge n-K+1$ there exist constants c and C, C_1 depending on ϕ and μ_8 such that for $5 \le p \le c \log n$

$$\mathbb{P}(|\lambda_j - \gamma_j| \ge C_1 K[\min(j, n-j+1)]^{-\frac{1}{3}} n^{-\frac{2}{3}}) \le \frac{C}{n^{2-\phi}} + \frac{C^p \log^{12p} n}{K^p}.$$

Rigidity, "Bulk" case

- Let $j \in [n\Delta_n^*, n - n\Delta_n^*]$. Without loss of generality we may assume that in this case $\lambda_j \in [-2, 2]$.
- Obviously, the maximum in Δ_n^* is reached at the jump points of F_n, i.e.

$$\Delta_n^* = \max_{1 \le k \le n} |F_n(\lambda_k) - G_{sc}(\lambda_k)| = \max_{1 \le k \le n} \left| \frac{k}{n} - G_{sc}(\lambda_k) \right| .$$

- This fact implies that for every j there exists $\theta, |\theta| \le 1$ such that

$$\lambda_j = G_{sc}^{-1} \left(\frac{j}{n} + \theta \Delta_n^* \right) .$$

By Taylor's formula we get

$$\lambda_j = G_{sc}^{-1} \left(\frac{j}{n} \right) + \mathbb{E}_\theta \frac{2\pi\theta\Delta_n^*}{\sqrt{4 - \left(G_{sc}^{-1}\left(\frac{j}{n} + \theta\Delta_n^*\right)\right)^2}} . \qquad (4)$$

Optimal delocalization of eigenvectors

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$$\mathbb{P}\left(\max_{1 \leq j,k \leq n} |u_{jk}| \geq C_1 \sqrt{\frac{\log n}{n}}\right) \leq \frac{C}{n^{2-\phi}}.$$

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- We remark here that it is possible to relax the moment conditions to the case $0 < \delta < 4$ as well. But here we may only conclude that there exists some constant $c(\delta) > 0$ depending on δ such that

$$\mathbb{P}\left(\max_{1 \le j,k \le n} |u_{jk}| \ge C_1 \sqrt{\frac{\log n}{n}} \right) \le \frac{C}{n^{c(\delta)}}.$$

A comparison with a similar result for the GOE ensemble and the delocalization of eigenvectors of the unit sphere shows that this result is optimal with respect to the power of logarithm. It is not clear though whether it is still possible to strengthen the probability bounds above.

Eigenvectors, GOE case

- The collections $(\mathbf{u}_1, ..., \mathbf{u}_n)$ is independent of the eigenvalues $(\lambda_1(\mathbf{W}), ..., \lambda_n(\mathbf{W}))$. Each of the eigenvectors is distributed uniformly on

$$S_+^{n-1} := \{\mathbf{x} = (x_1, ..., x_n) : x_j \in \mathbb{R}, \|x\| = 1, x_1 > 0\}.$$

  Further, $(\mathbf{u}_1, ..., \mathbf{u}_n)$ is distributed like a sample of Haar measure on $O(n)$, with each column multiplied by a scalar so that the columns all belong to S_+^{n-1}.

- With exponentially high probability

$$\max_{j,k} |u_{jk}| \leq C\sqrt{\frac{\log n}{n}},$$

  where C is some positive constant

Delocalization of eigenvectors

- Let us introduce the following distribution function

$$F_{nj}(x) := \sum_{k=1}^{n} |u_{jk}|^2 \mathbb{I}[\lambda_k(\mathbf{W}) \leq x].$$

Using the eigenvalue decomposition of $\mathbf{W}$ it is easy to see that

$$\mathbf{R}_{jj}(z) := [(\mathbf{W} - z\mathbf{I})^{-1}]_{jj} = \sum_{k=1}^{n} \frac{|u_{jk}|^2}{\lambda_k(\mathbf{W}) - z} = \int_{-\infty}^{\infty} \frac{1}{x - z}\, dF_{nj}(x),$$

which means that $\mathbf{R}_{jj}(z)$ is the Stieltjes transform of $F_{nj}(x)$.

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- For any $\lambda > 0$ we have

$$\max_{1 \leq k \leq n} |u_{jk}|^2 \leq \sup_x (F_{nj}(x + \lambda) - F_{nj}(x)) \leq 2 \sup_u \lambda \operatorname{Im} \mathbf{R}_{jj}(u + i\lambda). \quad (5)$$

To finish the proof we need to show that with high probability the r.h.s. of (5) is bounded by $n^{-1} \log n$.

Moment bound for diagonal entries of the resolvent

- Define the following region in the complex plain

$$\mathbb{D} := \{z = u + iv \in \mathbb{C} : |u| \le u_0, V \ge v \ge v_0 := A_0 n^{-1} \log n\},$$

- There exists positive constant C_0 depending on u_0, V and positive constants A_0, A_1 depending on C_0, H_0, δ such that for all $z \in \mathbb{D}$ and $1 \le p \le A_1 \log n$ we have

$$\max_{j=1,\dots,n} \mathbb{E}\,|\mathbf{R}_{jj}(z)|^p \le C_0^p. \tag{6}$$

Self-contained equations

- We know that $s(z)$ is a solution of

$$1 + zs(z) + s^2(z) = 0 \quad \text{or} \quad s(z) = -\frac{1}{z + s(z)}.$$

Self-contained equations

- We know that $s(z)$ is a solution of

$$1 + zs(z) + s^2(z) = 0 \quad \text{or} \quad s(z) = -\frac{1}{z + s(z)}.$$

- We may express $\mathbf{R}_{jj}$ in the following way

$$\mathbf{R}_{jj} = \frac{1}{-z + \frac{X_{jj}}{\sqrt{n}} - \frac{1}{n}\sum_{l,k\in T_j} X_{jk}X_{jl}\mathbf{R}^{(j)}_{kl}}. \tag{7}$$

Let $\varepsilon_j := \varepsilon_{1j} + \varepsilon_{2j} + \varepsilon_{3j} + \varepsilon_{4j}$, where

$$\varepsilon_{1j} = \frac{1}{\sqrt{n}}X_{jj}, \varepsilon_{2j} = -\frac{1}{n}\sum_{l\neq k\in T_j} X_{jk}X_{jl}\mathbf{R}^{(j)}_{kl}, \varepsilon_{3j} = -\frac{1}{n}\sum_{k\in T_j}(X^2_{jk} - 1)\mathbf{R}^{(j)}_{kk},$$

$$\varepsilon_{4j} = \frac{1}{n}(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)}).$$

- Using these notations we may rewrite (7) as follows

$$\mathbf{R}_{jj} = -\frac{1}{z + m_n(z) - \varepsilon_j} = -\frac{1}{z + m_n(z)} + \frac{1}{z + m_n(z)}\varepsilon_j\mathbf{R}_{jj}. \tag{8}$$

Self-contained equations

- After summation over all $j = 1, ..., n$ we get that $m_n(z)$ satisfies

$$1 + zm_n(z) + m_n^2(z) = T_n, \tag{9}$$

where

$$T_n := \frac{1}{n}\sum_{j=1}^{n} \varepsilon_j \mathbf{R}_{jj}, \tag{10}$$

We arrive at the following representation for $\Lambda_n := m_n(z) - s(z)$ in terms of T_n

$$\Lambda_n = \frac{T_n}{z + m_n(z) + s(z)} = \frac{T_n}{b_n(z)}.$$

Stein's approach

- Use Stein's method to estimate $\mathbb{E}\,|T_n|^p$. Let $\varphi(z) := \overline{z}|z|^{p-2}$. Then

$$\mathbb{E}\,|T_n|^p = \mathbb{E}\,T_n\varphi(T_n) = \frac{1}{n}\sum_{\nu=1}^{4}\sum_{j=1}^{n}\varepsilon_{\nu j}\mathbf{R}_{jj}\varphi(T_n)$$

We get

$$\mathbb{E}\,|T_n|^p \le \frac{Cp}{nv}\,\mathbb{E}^{\frac{p-1}{p}}\,|T_n|^p + \frac{Cp^2}{(nv)^2}\,\mathbb{E}^{\frac{p-2}{p}}\,|T_n|^p + \frac{C^p p^p}{(nv)^p}.$$

It follows that

$$\mathbb{E}\,|T_n|^p \le \frac{C^p p^p}{(nv)^p}.$$

This idea goes back to F. Götze, A. Tikhomirov, Rate of convergence to the semi-circular law. PTRF, 2003.

Thank you!