The Energy Density in the Toric Ising Model

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Introduction to Ising model

- Who and When: Ernst Ising 1924 (Wilhelm Lenz 1920).
- What: Model of ferromagnet to study the magnetic phase change at the Curie temperature. Iron: Curie 770 °C, melting 1538 °C.
- A very much studied and archetypal model, a "test laboratory", of statistical physics. The simplest model with interesting behaviour. Baxter, Enting: 399th Solution of the Ising Model, 1978.
- Lots of articles (12 000 1969-1997) published over the last 90-odd years in physics, mathematics, computer science and even in biology and other sciences. Applied to (for example) study of ferromagnetism, lattice gases, chemical absorption, ecology and image processing.
- Due to vastness of the subject, in this talk we shall only focus on some specific aspects (mainly CFT and conformal covariance).
- Integrable, exactly solvable model in 2D with order-disorder phase transition in $D \geq 2$.
- 1D solved by Ising, 2D solved by Onsager 1944, 3D unsolved.
- Mathematical study focuses on the 2D case.

Introduction to Ising model

- $G = (V(G), E(G))$ finite graph. A spin configuration on G is a map $\sigma : V(G) \to \{\pm 1\}$.
- The Ising model on graph G with inverse temperature $\beta > 0$, interaction strength $J \in \mathbb{R}$ and magnetic field $h \in \mathbb{R}$ is a model of random spin configurations

$$H(\sigma) := -J \sum_{\{x,y\} \in E(G)} \sigma(x)\sigma(y) - h \sum_{x \in V(G)} \sigma(x)$$

$$\mathrm{P}[\sigma] = \frac{1}{Z} e^{-\beta H(\sigma)} \qquad Z = \sum_{\sigma \in \Sigma(G)} e^{-\beta H(\sigma)}$$

Usually one considers $h = 0$ case.

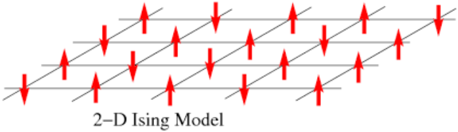


Figure: Picture by Navinder Singh.

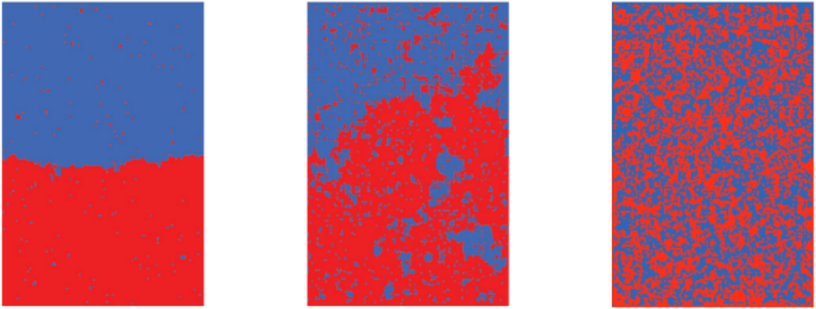
Introduction to Ising model

The quantities of main interest are:

- Partition function Z.
- Magnetization $\mathrm{E}[\sigma(z)]$.
- Energy density $\mathrm{E}[\sigma(x)\sigma(y)]$ for $\{x, y\} \in E(G)$.
- General spin correlation $E[\sigma(z_1)\ldots\sigma(z_n)]$.
- General energy correlation $E[\sigma(x_1)\sigma(y_1)\ldots\sigma(x_n)\sigma(y_n)]$ where $\{x_i, y_i\} \in E(G)$.
- Mixed correlation $E[\sigma(x_1)\sigma(y_1)\ldots\sigma(x_n)\sigma(y_n)\sigma(z_1)\ldots\sigma(z_m)]$ where $\{x_i, y_i\} \in E(G)$.
- Traditionally physicist are interested in the temperature-dependence if these quantities. Lately, after emergence of CFT, interest has move to the conformal properties of these quantities at critical temperature. That is where our interest also lies.

Introduction to Ising model

Phase transition and critical behaviour:



Ising model by physicists

- Algebraic approach by transfer matrix: Onsager 1944; Onsager, Kaufman (spinor analysis) 1949.
- Combinatorial approach: Van der Waerden (high temperature expansion) 1941; Kac and Ward (Kac-Ward determinant) 1952; Hurst, Green, Kasteleyn, Fisher (Pfaffian method) 1960; McCoy and Wu (The Two-Dimensional Ising Model, 1973).
- Renormalization group arguments: The Ising model has a continuum limit described by a quantum field theory (QFT).
- Belavin, Polyakov, Zamolodchikov 1984: The limiting QFT for the critical case has conformal symmetry (CFT).
- Physicist believe that the limiting theory is the free fermion field theory. The theory is massless for the critical case (massive for non-critical).
- Di Francesco, Mathieu, Sénéchal: Yellow book.
- Prediction: Universality, conformal covariance.
- Prediction: Feynman path integrals → determinants of Laplacians.

Physicist have made predictions; mathematicians try to prove them.

Ising model by mathematicians, from our perspective

- The algebraic and combinatorial approaches to Ising model are (mostly) rigorous. The approach by QFT/CFT is non-rigorous.

For us, the main combinatorial method is the Kac-Ward determinant method:

- Proved 1999 by Dolbilin, Zinov'ev, Mishchenko, Shtan'ko, Shtogrin (Kac, Ward 1952).
- Generalized by David Cimasoni (2010, 2012).

Ising model by mathematicians, from our perspective

Let us now consider only the critical case, and the calculation of correlations:

- Discrete complex analysis method: Stanislav Smirnov, Dmitry Chelkak 2006, 2010. Result: interfaces converge to SLE(3).
- Full-plane: energy correlations (Boutillier, de Tiliere 2010, 2011); spin correlations (Palmer 2007).
- Clement Hongler thesis: energy correlations in simply connected planar domains 2010.
- Hongler, Smirnov: energy density in simply connected planar domains 2013.
- Chelkak, Konstantin Izyurov: discrete holomorphic spinors 2013.
- Chelkak, Hongler, Izyurov: spin correlations in simply connected planar domains 2015.

Critical square lattice Ising model on a torus

- $G = (V(G), E(G))$ finite square grid made into torus (periodic boundary). Set parameters $\beta = \frac{1}{2}\log\left(\sqrt{2}+1\right)$, $J = 1$, $h = 0$. Let grid mesh be $\delta > 0$.
- The critical Ising model on the square lattice on torus (without magnetic field) is a model of random spin configurations

$$\mathrm{P}[\sigma] = \frac{1}{Z} e^{\beta \sum_{\{x,y\}\in E(G)} \sigma(x)\sigma(y)} \qquad Z = \sum_{\sigma\in\Sigma(G)} e^{\beta \sum_{\{x,y\}\in E(G)} \sigma(x)\sigma(y)}$$

The quantity $\mathrm{E}[\sigma(x)\sigma(y)]$, where $\{x, y\} \in E(G)$ is called the energy density of the edge $\{x, y\}$. We calculate the asymptotics of this quantity in the scaling limit $\delta \to 0$.

$$\mathrm{E}[\sigma(x)\sigma(y)] = \frac{1}{Z}\sum_{\sigma\in\Sigma(G)} \sigma(x)\sigma(y)\, e^{\beta \sum_{\{a,b\}\in E(G)} \sigma(a)\sigma(b)}$$

Note that the scaling limit should be a number independent of the sites x and y in the graph.

Our (aimed for) result

Proposition (Izyurov, Kemppainen, Tuisku (ongoing work))

$$\mathrm{E}\left[\sigma(x)\,\sigma(y)\right] = \frac{1}{\sqrt{2}} + \frac{|\eta(\tau)|^2}{\frac{1}{2}\left(\left|\frac{\theta_2(\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_3(\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|\right)}\delta + \mathfrak{o}(\delta) \qquad \delta \downarrow 0$$

(Not proved for general τ yet).

- Asymptotics only depends on modular parameter τ, the shape of the torus.
- Arthur Ferdinand and Michael Fisher 1969 proved a similar result by brute-force calculations extending the results of Onsager and Kaufman.
- Our result is obtained arguably more natural way from point of view of CFT, namely by relating the quantities to determinants of Laplacians, thus providing a connection to CFT not present in the previous calculation.

How do we do it?

1. Partition function.
   - Linear combination of determinants of 4 Kac-Ward matrixes.
   - Determinants of discrete Laplacian operators.
2. The sum in the energy density: discrete complex analysis.
   - Affine combination of 4 specifically constructed spinor observables.
   - Use discrete complex analysis to find a formula for the observables and prove that three of the observables do not contribute.
   - Analyse the fourth observable using similar techniques as for the partition function (determinants of Kac-Ward matrixes, determinants of discrete Laplacians).
3. Determinants of Laplacian operators.
   - Relate determinant of discrete Laplacian to the determinant of continuum Laplacian.
   - Calculate determinants of continuum Laplacians.

The result is obtained by combining the three steps above.

Partition function

- Van der Waerden high-temperature expansion + Euler's theorem (graphs) = loop configuration.
- Parity of the number of non-trivial loops of given parity is invariant of resolvation.
- Twisted partition functions = partition function weighed by $(-1)^{\text{nontrivial loops of given type}}$.
- Twisted partition functions = square root of determinant of twisted Kac-Ward matrix (Cimasoni 2010).
- Determinants of twisted discrete Laplacian operators and twisted Kac-Ward matrixes related (Cimasoni 2012).

The sum in the energy density: discrete complex analysis

$$F_{(i,j)}(a,z) := (-1)\,\text{const.} \sum_{\omega \in C(a,z)} \alpha^{|\omega|} e^{-\mathrm{i}\frac{W(\gamma)}{2}} (-1)^{\mathbb{1}_{\gamma : a \rightsquigarrow z}} (-1)^{\text{nontrivial loops of given type}}$$

- The sum in the energy density can be expressed as an affine combination of special values of 4 discrete holomorphic spinor observables.
- Using discrete complex analysis, three of the four observables can be shown to equal (linear combinations of) discrete Cauchy kernels (periodic). This allows asymptotic analysis; the result is that these observables do not contribute.
- So all boils down to one special value of one discrete holomorphic observables; we use discrete arguments to relate this value to the determinant of Laplacian with zero mode removed.

Determinants of Laplacian operators

- Using recent results by Gautam Chinta, Jay Jorgenson, Anders Karlsson (arXiv:1110.6841) we find out the asymptotics of the determinants of the various discrete Laplacians.
- After the cancellations between the numerator and denominator, we are left with relations of the determinants of (zeta function regularized) continuum Laplacians. These are classical in mathematics, and we are able to calculate them.

Summary

Proposition (Izyurov, Kemppainen, Tuisku (ongoing work))

$$\mathrm{E}\left[\sigma(x)\,\sigma(y)\right] = \frac{1}{\sqrt{2}} + \frac{\left|\eta(\tau)\right|^2}{\frac{1}{2}\left(\left|\frac{\theta_2(\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_3(\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|\right)}\delta + \mathfrak{o}(\delta) \qquad \delta \downarrow 0$$

Note the correct behaviour under conformal maps between tori.
For the case of the straight $N \times N$-torus (here $\delta = \frac{1}{N}$) as $N \to \infty$ we get ($\tau = \mathrm{i}$):

$$\begin{aligned}\mathrm{E}\left[\sigma(x)\,\sigma(y)\right] &= \frac{1}{\sqrt{2}} + \frac{\Gamma\left(\frac{1}{4}\right)^2}{\left(2^{\frac{9}{4}} + 2\sqrt{2}\right)\pi^{\frac{3}{2}}}\frac{1}{N} + \mathfrak{o}\left(\frac{1}{N}\right) \\ &\approx \frac{1}{\sqrt{2}} + 0.31122\ldots\frac{1}{N} + \mathfrak{o}\left(\frac{1}{N}\right)\end{aligned}$$

Future: Universality (isoradial graphs), general correlations, higher genus Riemann surfaces...

Thank you