Geometric Dyson Brownian motion and May–Wigner stability

Jesper R. Ipsen
University of Melbourne

Summer school: Randomness in Physics and Mathematics 2
Bielefeld 2016

joint work with
Henning Schomerus (Lancaster)

arXiv:1602.06364

AUSTRALIAN RESEARCH COUNCIL CENTRE OF EXCELLENCE FOR
MATHEMATICAL AND STATISTICAL FRONTIERS

Outline:

- The problem in a nutshell
  May's model & a matrix-valued geometric Brownian motion
- Geometric Dyson Brownian motion
  Stochastic and Fokker–Planck eq. for the Lyapunov exponents
- Solving the Fokker–Planck equation
  Solution for $\beta = 2$ as a biorthogonal ensemble
- Some open problems
  What to do next?

The problem in a nutshell

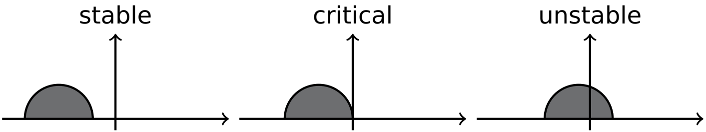
May’s model & a matrix-valued geometric Brownian motion

May's model ('72)

Consider the stability problem

$$\frac{d\mathbf{x}}{dt} = -\mu\mathbf{x} + \sigma\mathbf{J}\,\mathbf{x} \qquad \left[\ \mathbf{x}(t) = e^{(-\mu+\sigma\mathbf{J})t}\mathbf{x}(0)\ \right]$$

- $\mathbf{x}$: N-dimensional vector
- $\mathbf{J}$: $N \times N$ random matrix
- μ, σ: constants



We are interested in a dynamical generalisation

See review by Majumdar & Schehr ('14)

Matrix-valued geometric Brownian motion:

$$d\mathbf{x}(t) = -\mu dt\, \mathbf{x}(t) + \sigma d\mathbf{B}(t) \circ \mathbf{x}(t),$$

- $\mathbf{x}(t)$: N-dimensional vector
- $\mathbf{B}(t)$: $\mathfrak{gl}_N$-valued Brownian motion
- μ, σ: constants

Question:
Behaviour of the finite-time Lyapunov exponents
(i.e. the logarithm of the singular values)
The largest Lyapunov exponent is of special interest for stability.

Matrix-valued geometric Brownian motion:

$$d\mathbf{Y}(t) = -\mu dt\,\mathbf{Y}(t) + \sigma d\mathbf{B}(t) \circ \mathbf{Y}(t), \qquad \mathbf{Y}(0) = \mathbf{I}$$

- $\mathbf{Y}(t)$: GL_N-valued stochastic process
- $\mathbf{B}(t)$: $\mathfrak{gl}_N$-valued Brownian motion
- μ, σ: constants
- $\mathbf{I}$: $N \times N$ identity matrix

Question:
Behaviour of the finite-time Lyapunov exponents
(i.e. the logarithm of the singular values)
The largest Lyapunov exponent is of special interest for stability.

Symmetries:

$$B(t) \stackrel{d}{=} UB(t)U^\dagger$$

- U is orthogonal or unitary for real and complex fields, respectively
- This determines $B(t)$ up to a parameter $\tau \in (-1, 1)$ which interpolate between Hermitian and anti-Hermitian noise

Side remarks:

- Can be interpreted as advection in an N-dimensional homogeneous, isotropic, white-in-time stochastic fluid
  Sometimes referred to as the Kraichnan ensemble
- A different symmetry analysis with certain similarities appear in the study of disordered wires
  Here the symmetries are flux conservation plus chiral symmetry

Geometric Dyson Brownian motion

Stochastic and Fokker–Planck equation
for the Lyapunov exponents

Lyapunov exponents

We are interested in the finite-time Lyapunov exponents

$$\lambda_1(t), \ldots, \lambda_N(t),$$

i.e. the eigenvalues of the Hermitian matrix

$$\mathbf{W}(t) := \frac{1}{2} \log \mathbf{Y}^\dagger(t)\mathbf{Y}(t)$$

Question:
Can we find the stochastic and Fokker–Planck equation for the Lyapunov exponents (i.e. integrate out unitary degrees of freedom) and what do this tells us about stability?

Geometric Dyson Brownian Motion

Stochastic equation:

$$d\lambda_i(t) = \kappa dB_i(t) + \frac{\beta}{2}\sum_{j\neq i}\frac{\kappa^2 dt}{\tanh(\lambda_i - \lambda_j)} - \mu dt$$

Fokker–Planck equation:

$$\frac{\partial \rho_t(\lambda)}{\partial t} = \sum_i \left(\frac{\kappa^2}{2}\frac{\partial^2}{\partial \lambda_i^2} + \frac{\partial}{\partial \lambda_i}\left(\mu - \frac{\beta}{2}\sum_{j\neq i}\frac{\kappa^2}{\tanh(\lambda_i - \lambda_j)}\right)\right)\rho_t(\lambda)$$

Diffusion constant: $\kappa^2 = (1+\tau)\sigma^2/\beta$

Idea of derivation:

- Perturbation theory for $\mathbf{Y}^\dagger(t)\mathbf{Y}(t)$
- Change variables to the Lyapunov exponents

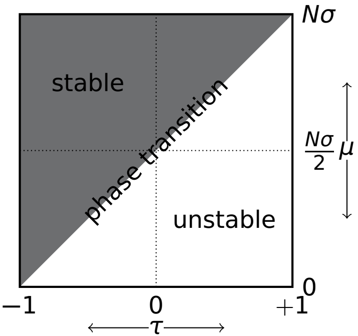
Observations:

- Dependence on κ can't be removed by rescaling
- $\tau = -1$ implies $\kappa = 0$: Here the "noise"-term is anti-Hermitian, thus the singular values do not diffuse. The eigenvalues still diffuse
- A naïve expansion in small κ reproduces ordinary Dyson Brownian motion
- The Lyapunov exponents becomes equidistant in the long-time limit

$$\lambda_k(t)/t \xrightarrow{t\to\infty} -\mu + (1+\tau)(2k-1-N)\sigma^2/2$$

and their fluctuations become Gaussian if N is kept fixed. For $\beta = 1$ this dates back to Le Jan ('85), Newman ('86)

Phase diagram



Question: What happens at the phase transition?

- $\kappa t \gg N$: Fluctuations are known to be Gaussian.
- $\kappa t \ll N$: Fluctuations are expected to be Tracy–Widom
- Intermediate: Unknown

Solving the Fokker–Planck equation

Solution for $\beta = 2$ as a biorthogonal ensemble

How to solve the Fokker–Planck eq.

- Use ansatz

$$\rho_t(\lambda|\nu) = \prod_{i<j}\left(\frac{\sinh(\lambda_j - \lambda_i)}{\sinh(\nu_j - \nu_i)}\right)^{\beta/2} \psi_t(\lambda|\nu)$$

  in the Fokker–Planck equation
- This gives a (Calogero–Sutherland) Schrödinger eq.:

$$-\frac{\partial \psi_t(\lambda|\nu)}{\partial t} = -\frac{\kappa}{2}\nabla^2 \psi_t(\lambda|\nu) + V(\lambda)\psi_t(\lambda|\nu)$$

$$V(\lambda) = \frac{\kappa\beta(\beta-2)}{8}\sum_{i<j}\frac{1}{\tanh(\lambda_j - \lambda_i)^2} + \frac{\kappa\beta^2(N+1)N(N-1)}{12}$$

- The "particle" interaction vanishes for $\beta = 2$ and the equation is easily solved.

Similar idea as Beenakker & Rajaei ('93) for the DMPK

Solution as a biorthogonal ensemble

$$\rho_t^{\beta=2}(\lambda) = \prod_{\ell=1}^{N} \frac{1}{\ell!} \det_{1\le i,j\le N}\left[\left(\frac{\lambda_i}{2\kappa t}\right)^{j-1}\right] \det_{1\le i,j\le N}\left[\frac{e^{-(\lambda_i-\mu_j t)^2/2\kappa t}}{\sqrt{2\pi\kappa t}}\right]$$

$$\mu_k = -\mu + (1+\tau)(2k-1-N)\sigma^2/2$$

Observations:

- The JPDF is identical to a GUE matrix coupled to an equispaced external source
- The JPDF is the same as for a disordered wire with chiral symmetry and broken time-reversal symmetry
  Here one would be interested in linear statistics

Some open problems

What to do next?

Open problems

- Study the fluctuations of the largest Lyapunov exponent and the consequences for the phase transition
  The biorthogonal ensembles for $\beta = 2$ seems to be the best starting point
- Study the evolution of the global spectrum
  - The long-time limit (properly rescaled) is a uniform dist.
  - Heuristics and numerics suggest semi-circle-like behaviour on short time scales

  Free SDE techniques might be applicable, Kargin ('08)
- Number of stable points in dynamical random landscapes
  An extension of Fyodorov & Khoruzhenko ('16)
- The product ensembles of Akemann et al. ('13) may be seen as discrete-time analogues of what considered here. Is there a scaling which relates them?
  The answer to this question is: yes!

Thanks for your attention!

Open problems

- Study the fluctuations of the largest Lyapunov exponent and the consequences for the phase transition
  The biorthogonal ensembles for $\beta = 2$ seems to be the best starting point
- Study the evolution of the global spectrum
  - The long-time limit (properly rescaled) is a uniform dist.
  - Heuristics and numerics suggest semi-circle-like behaviour on short time scales

  Free SDE techniques might be applicable, Kargin ('08)
- Number of stable points in dynamical random landscapes
  An extension of Fyodorov & Khoruzhenko ('16)
- The product ensembles of Akemann et al. ('13) may be seen as discrete-time analogues of what considered here. Is there a scaling which relates them?
  The answer to this question is: yes!

Thanks for your attention!