Density of States for Random Band Matrices in $d = 2$

via the supersymmetric approach

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The model: Random Band matrices

- model for conducting properties of disordered materials (similar to Random Schrödinger operator)
- $d \geq 1$, $\Lambda \subset \mathbb{Z}^d$ finite, $H : \Lambda \times \Lambda \to \mathbb{C}$ hermitian random matrix
- matrix elements: Gaussian random variables

$$H_{ii} \sim \mathcal{N}_{\mathbb{R}}(0, J_{ii}), \qquad H_{ij} \sim \mathcal{N}_{\mathbb{C}}(0, J_{ij}), \quad \text{for } i < j,$$

with $<$ order relation on $\mathbb{Z}^d$

- covariance

$$J_{ij} := (-W^2\Delta + \mathbb{1})^{-1}_{ij} \lesssim \mathrm{e}^{-|i-j|/W} \quad \text{for } |i-j| > W,$$

  - $-\Delta \in \mathbb{R}^{\Lambda \times \Lambda}$ discrete Laplacian on Λ with periodic b.c.
  - $W \gg 1$ band width

Special cases

Gaussian unitary ensemble (GUE):

- $W \gg |\Lambda|$, $J_{ij} = \frac{1}{N}$
- $\Lambda \to \mathbb{Z}^d$: extended eigenvectors, a.c. spectrum

Diagonal case:

- $W = 0$, $J_{ij} = \delta_{ij}$
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$\Rightarrow$ RBM interpolates between GUE and diagonal case

Density of States

quantity of interest: **averaged density of states**
$\equiv$ probability density of eigenvalues

$$\bar{\rho}_\Lambda(E) := \frac{1}{|\Lambda|}\mathbb{E}\left[\sum_j \delta_{\lambda_j}(E)\right] = -\frac{1}{\pi|\Lambda|}\lim_{\varepsilon\to 0}\mathbb{E}[\mathrm{Im}\,\mathrm{Tr}\,G^+]$$

with $E \in \mathbb{R}$, λ_j (random) eigenvalues of H and Green's function

$$G^+ := ((E + i\varepsilon)\cdot \mathbb{1} - H)^{-1}$$

Result in $d = 2$

Theorem (Disertori, L. 2016)

Let $d = 2$ and $\mathcal{I} = \{E : \eta < |E| \leq 1.8\}$. For each fixed $\alpha \in (0, 1)$, there exist $W_0(\alpha)$ such that for all $W \geq W_0(\alpha)$ and $E \in \mathcal{I}$

$$|\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| \leq W^{-2}\,\mathrm{e}^{K(\ln W)^\alpha},$$
$$|\partial_E^n \bar{\rho}_\Lambda(E)| \leq C_n \qquad \forall n \leq n_0(W),$$

where C_n and K are independent of Λ and W and $\lim_{W\to\infty} n_0(W) = \infty$. ρ_{SC} is Wigner's semicircle law

$$\rho_{SC}(E) = \begin{cases} \frac{1}{\pi}\sqrt{1 - \frac{E^2}{4}} & \text{if } |E| \leq 2, \\ 0 & \text{if } |E| > 2. \end{cases}$$

Both estimates are uniform in Λ and hold also for $\Lambda \to \mathbb{Z}^2$.

Other known results

- $d = 3$ [Disertori, Pinson, Spencer 2002]

$$|\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| \leq W^{-2}$$

- $d = 1$ [M. Shcherbina, T. Shcherbina 2016]

$$|\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| \leq W^{-1}$$

Rough idea of the proof in $d = 2$

- dual representation via the supersymmetric approach as integral over $2|\Lambda|$ real variables

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- estimates in a finite cube of volume W^2

region	correction	
dominant saddle	$\mathcal{O}(W^{-2})$	Brascamp-Lieb inequality
second saddle	$\mathcal{O}(\mathrm{e}^{-c \ln W})$	decay from Fermionic integral
$\rightarrow$ need exp. decay in W		
large field region	$\mathcal{O}(\mathrm{e}^{-cW})$	small probability argument

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- infinite volume $\Rightarrow$ cluster expansion with complex covariance

Dual representation via supersymmetry

$$\frac{1}{|\Lambda|}\mathbb{E}[\mathrm{Tr}\, G^+] = \int \mathrm{d}M \, \mathrm{e}^{-\frac{1}{2}\sum_{i,j\in\Lambda} J_{ij}^{-1}\,\mathrm{Str}[M_i M_j]} \prod_{j\in\Lambda} \mathrm{Sdet}[E_\varepsilon - M_j]^{-1} a_0$$

$$= \int \mathrm{d}a\,\mathrm{d}b\,\mathrm{e}^{-\frac{1}{2}((a,J^{-1}a)+(b,J^{-1}b))} \prod_{j\in\Lambda} \frac{E_\varepsilon - ib_j}{E_\varepsilon - a_j} \; \det\left[\frac{J^{-1}-F}{2\pi}\right] \; a_0,$$

- $M_j = \begin{pmatrix} a_j & \bar{\rho}_j \\ \rho_j & ib_j \end{pmatrix}$ supermatrix, $\begin{cases} a_j, b_j \in \mathbb{R} \\ \bar{\rho}_j, \rho_j \text{ Grassman variables} \end{cases}$
- $E_\varepsilon = E + i\varepsilon$
- $\mathrm{Str}\, M_j = a_j - ib_j$, $\mathrm{Sdet}\, M_j = \det[a_j - \bar{\rho}_j b_j^{-1} \rho_j] \det b_j^{-1}$
- $F(a,b)_{ij} = \delta_{ij} \frac{1}{(E_\varepsilon - a_j)(E_\varepsilon - ib_j)}$

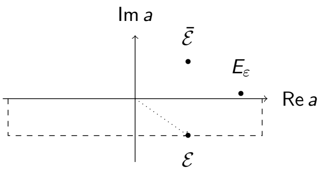
saddle point analysis

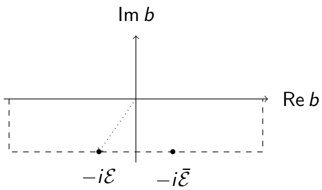
$$(a, J^{-1}a) = \sum_j a_j^2 + W^2 \sum_{i\sim j}(a_i - a_j)^2 \quad \Rightarrow \quad a_j \approx a \text{ constant}$$

$$\mathrm{e}^{-\frac{|\Lambda|}{2}(a^2+b^2)} \left(\frac{E-ib}{E-a}\right)^{|\Lambda|} = \mathrm{e}^{-|\Lambda|\left(\frac{a^2}{2}+\ln(E-a)+\frac{b^2}{2}-\ln(E-ib)\right)},$$

critical points $a_s^{\pm} = \mathcal{E}_r \pm i\mathcal{E}_i$ and $b_s^{\pm} = -i\mathcal{E}_r \pm \mathcal{E}_i$, where

$$\mathcal{E} = \mathcal{E}_r - i\mathcal{E}_i = \frac{E}{2} - i\sqrt{1 - \frac{E^2}{4}}$$





obtaining the semicircle law

Lemma

By a complex deformation $a_j \mapsto a_j + a_s^+$ *and* $b_j \mapsto b_j + b_s^+$

$$\lim_{\varepsilon \to 0} \frac{1}{|\Lambda|} \mathbb{E}[\mathrm{Tr}\, G^+] = a_s^+ + \int \mathrm{d}\mu_B(a,b) \mathcal{R}(a,b) a_0,$$

where $\mathrm{d}\mu_B(a,b)$ *is Gaussian with* $B := (-W^2\Delta + (1 - \mathcal{E}^2))^{-1}$ *and*

$$\mathcal{R}(a,b) := \det[1 + DB]\ \mathrm{e}^{\mathcal{V}(a,b)},$$

where $D_{ij} = D_{ij}(a,b) = \delta_{ij} D_j(a,b)$ *is diagonal and*

$$\mathcal{V}(a,b) = \sum_{j \in \Lambda} V(a_j) - V(ib_j), \quad V(x) = \int_0^1 \frac{x^3(1-t)^2}{(\bar{\mathcal{E}} - tx)^3} \mathrm{d}t.$$

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Remember:

- $\bar{\rho}_\Lambda(E) = -\frac{1}{\pi|\Lambda|} \lim_{\varepsilon \to 0} \mathbb{E}[\mathrm{Im}\,\mathrm{Tr}\, G^+]$
- $a_s^+ = \frac{E}{2} - i\sqrt{1 - \frac{E^2}{4}}$
- $\rho_{SC}(E) = \frac{1}{\pi}\sqrt{1 - \frac{E^2}{4}} \quad \text{if } |E| \leq 2,$

$\Rightarrow$ estimate remainder $\int \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b)a_0$ (cluster expansion)

Thank you for your attention!
Any questions?