Density of States of Gaussian Random Band Matrices

Martin Lohmann, Bonn

August 17, 2016

Outline

- Same problem as in Mareike's talk, but $d = 1$
- Semicircle law proven by T.+M. Shcherbina [Sh16]
- This talk: Sketch of a simpler proof
- Supersymmetry: Grassmann variables "beyond" determinants

  $\rightarrow$ Algebra: unexpected computational simplifications

  $\rightarrow$ Analysis: Simplified proof of spectral gap

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Model

Definition

Average Green's function for $d = 1$ Gaussian random band matrix:

$$\langle G_{00}(E) \rangle = \lim_{\varepsilon \to 0} \int \mathrm{d}M e^{-\frac{1}{2}\sum_{i,j\in\Lambda} J_{ij}^{-1} \mathrm{Str}\ M_i M_j} \prod_{j\in\Lambda} \mathrm{Sdet}\,(E_\varepsilon - M_j)^{-1} a_0,$$

where $\Lambda = \{-N, \dots, N\}$, $J_{ij}^{-1} = (-W^2\Delta + 1)$, Δ n.n. Laplacian with Neumann b.c.

- Universality: $-\frac{1}{\pi}\Im\langle G_{00}(E)\rangle \sim \frac{1}{2\pi}\sqrt{(4-E^2)_+}$ for $W, N \gg 1$
- Semicircle law explicit after change of contour
- Overall strategy: Get rid of Grassmann variables, bound remaining high - dim. integral
- $(\Lambda, \Delta_{\mathrm{Neumann}})$ tree: local recursive structure

$$\langle G_{00}(E) \rangle = a_s^+ + \int \mathrm{d}M_0\ [\mathfrak{T}^N e^{-V}](M_0)^2 e^{V(M_0)}\ a_0$$

$$(\mathfrak{T}F)(M) = e^{-V(M)} \int \mathrm{d}M' e^{-\frac{W^2}{2}\mathrm{Str}\,(M-M')^2} F(M') \qquad e^{-V(M)} = \frac{e^{-\frac{1}{2}\mathrm{Str}\,(M+a_s^+)^2}}{\mathrm{Sdet}\,(\overline{a}_s^+ - M)}$$

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  $\Rightarrow \mathfrak{T}$ becomes integral operator $FCT(\mathbb{R}^2) \otimes \mathbb{R}^4 \to FCT(\mathbb{R}^2) \otimes \mathbb{R}^4$
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  $\to$ Speed of convergence determined by $|\frac{\lambda_2}{\lambda_1}| < 1$, $P_{v_1} = |v_1\rangle\langle v_1|$ if $\mathfrak{T}$ s.a.
- Our $\mathfrak{T}$ is not s.a. $\Rightarrow$ Complicated pert. theory for $\lambda_{1,2}$ / construction of P_{v_1}

  $\to$ [Sh16]: Matrix pert. theory (Schur complements) in Hermite basis

SUSY restricts to subspace $FCT(\mathbb{R}^2) \otimes \mathbb{R}^4 \to FCT(\mathbb{R}^2)$

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- Consider superrotations $M \to \exp\begin{pmatrix} 0 & \xi \\ \eta & 0 \end{pmatrix} M \exp\begin{pmatrix} 0 & -\xi \\ -\eta & 0 \end{pmatrix}$, η, ξ odd

  $\Rightarrow$ Sdet , Str , $\mathrm{d}M$ invariant

  $\Rightarrow$ $\mathfrak{T}$ commutes with superrotations

- Use superrotations to change to supermatrix polar coordinates:

$$M = \begin{pmatrix} a & \bar\rho \\ \rho & ib \end{pmatrix} = \left(\exp\begin{pmatrix} 0 & -\eta \\ -\xi & 0 \end{pmatrix}\right)\begin{pmatrix} \lambda_1 & 0 \\ 0 & i\lambda_2 \end{pmatrix}\left(\exp\begin{pmatrix} 0 & \eta \\ \xi & 0 \end{pmatrix}\right)$$

$\lambda_1 = a + \frac{\bar\rho\rho}{a-ib}$, $\lambda_2 = b - i\frac{\bar\rho\rho}{a-ib}$, $\eta = \frac{\bar\rho}{a-ib}$, $\xi = -\frac{\rho}{a-ib}$.

Proposition (Constantinescu (1988))

If $F(M) = f(\lambda_1, \lambda_2)$, then $(\mathfrak{T}f)(M) = (\mathcal{T}f)(\lambda_1, \lambda_2)$ with ($\lambda := \lambda_1 - i\lambda_2$)

$$(\mathcal{T}f)(\lambda_1,\lambda_2) = e^{-V-\frac{W^2}{2}|\lambda|^2} f(0) + \lambda e^{-V}\int \frac{W^2}{2\pi} e^{-\frac{W^2}{2}|\lambda-\lambda'|^2} f(\lambda_1',\lambda_2')\frac{\mathrm{d}\lambda_1'\,\mathrm{d}\lambda_2'}{\lambda'}$$

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SUSY

- Consider superrotations $M \to \exp\begin{pmatrix} 0 & \xi \\ \eta & 0 \end{pmatrix} M \exp\begin{pmatrix} 0 & -\xi \\ -\eta & 0 \end{pmatrix}$, η, ξ odd

  $\Rightarrow$ Sdet , Str , $\mathrm{d}M$ invariant

  $\Rightarrow \mathfrak{T}$ commutes with superrotations

- Use superrotations to change to supermatrix polar coordinates:

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Proof: Single application of Cauchy's theorem.

General framework: Berezin change of variables $\quad\to\quad$ non-linear σ model

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Corollary

$$\langle G_{00}(E) \rangle = a_s^+ + \int [\mathcal{T}^N e^{-V}](\lambda_1, \lambda_2)^2 e^{V(\lambda_1,\lambda_2)} \frac{\mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2}{\lambda}$$

Analysis of SUSY reduced transfer operator

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  $\Rightarrow$ Eigenvalue gap $|\lambda_2|$: Replaced by bound $\|\mathcal{T}|_E\|_{\tilde{L}^2\to\tilde{L}^2}$, $\tilde{L}^2 = L^2(\mathbb{R}^2, \frac{\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2}{|\lambda|})$.

Result

$$\langle G_{00}(E)\rangle = a_s^+ + \int [\mathcal{T}^N e^{-V}](\lambda_1,\lambda_2)^2 e^{V(\lambda_1,\lambda_2)} \frac{\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2}{\lambda}$$

- Decompose $e^{-V} = P_{v_1} e^{-V} + (1-P_{v_1})e^{-V} = v_1 + (e^{-V} - v_1)$
- By construction, $e^{-V} - v_1 \in E \quad \Rightarrow \quad \mathcal{T}^N(e^{-V} - v_1) = \mathcal{O}((1-\frac{c}{W})^N) \to 0.$

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- Write $v_1 = v_1^0 + \text{error}$, and compute explicitly $\langle G_{00}(E)\rangle = a_s^+ + \frac{c}{W^2} + \mathcal{O}(W^{-\frac{5}{2}})$

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If $N \geq CW \log W$ *then* $|\rho_N(E) - \frac{1}{2\pi}\sqrt{(4-E^2)_+}| \leq \frac{C'}{W \log W}$.

Proof: rather unique interplay between beautiful algebraic manipulations and hard analysis

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Thank you for your attention!