

Fourth ZIF Summer School

Randomness in Physics and Mathematics

1 - 13 August 2022

From Integrable Probability to Disordered Systems

Lecture Series

Riemann - Hilbert Techniques for Determinantal Point Processes

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Program of the lecture series

Lecture 1: Introduction to Determinantal Point Processes
(Monday 14-15h)

Lecture 2: Transformations of Determinantal Point Processes
(Wednesday 9h30-10h30)

Lecture 3: The LKS method and variants
(Friday 11h-12h)

Lecture 4: Asymptotics and Integrable Differential Equations
(Saturday 11h - 12h)

General Motivation

- Determinantal Point Processes arise naturally in
 - Random Matrix Theory
 - Eigenvalue Distributions for finite size
 - (Universal) limiting processes when size $\rightarrow \infty$
 - Combinatorics and Statistical Mechanics
 - Tiling Models
 - Vertex Models
 - Machine Learning
 - Search for Diversity in Samples
- Toolbox with powerful analytic techniques, often with elegant probabilistic interpretation, to study properties and asymptotics.

Lecture 1: Introduction to Determinantal Point Processes

① What is a Point Process ?

- Factorial Moment Measures and Correlation Functions
- Janossy Measures and Janossy Densities
- Laplace Functional

② What is a Determinantal Point Process ?

- Correlation functions
- Laplace Functional and Fredholm Determinants

③ Subclasses of Determinantal Point processes

- N-point DPPs: (Orthogonal) Polynomial Ensembles, Bi-Orthogonal Ensembles
- (Orthogonal) Projection DPPs
- Integrable Kernel DPPs

① What is a Point Process ?

Informal: A Random Configuration of Points

Examples: ① N independent Gaussians on \mathbb{R}


Symmetric probability distribution on \mathbb{R}^N :

$$\frac{1}{Z_N} \prod_{j=1}^N e^{-x_j^2} dx_j$$

② Eigenvalues of an $N \times N$ GUE Random Matrix

Symmetric probability distribution on \mathbb{R}^N :

$$\frac{1}{Z_N} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2 \cdot \prod_{j=1}^N e^{-x_j^2} dx_j$$

In general :- Configurations can be infinite

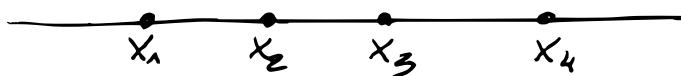
- Number of points in a configuration can be random

To define Point Processes properly, we need:

- A complete separable metric space \mathcal{L} ,
with Borel σ -algebra $\mathcal{B}_{\mathcal{L}}$.

For us: $\mathcal{L} = \mathbb{R}$ with the Lebesgue Borel σ -algebra $\mathcal{B}_{\mathbb{R}}$.

- The space $\text{Conf}(\mathcal{L})$ of locally finite point configurations!



→ We represent ξ as a Borel counting measure

$$\xi = \sum_j \delta_{x_j}, \quad x_j \in \mathcal{L}, \quad \xi: \mathcal{B}_{\mathcal{L}} \rightarrow \{\text{NU}\} \cup \{0\}$$

→ $\xi(B)$ denotes the number of points in B

- There is a natural σ -algebra C_L on $\text{Conf}(L)$,
 namely the smallest σ -algebra containing all
 sets of the form $\{\xi \in \text{Conf}(L) : \xi(B) = k\}$
 with $B \in \mathcal{B}_L$, $k \in \text{INU} \cup \{0\}\}$.

→ This is the minimal σ -algebra such that
"counting points" is measurable:

$$\#_B : \text{Conf}(L) \rightarrow \text{INU} \cup \{0, \infty\} : \xi \mapsto \xi(B)$$

is measurable $\forall B \in \mathcal{B}_L$.

Definition: A (random) Point Process

is a probability distribution P on

the space $(\text{Conf}(\mathcal{L}), \mathcal{C}_{\mathcal{L}})$

Example 1: $\mathcal{L} = \mathbb{R}$, probability density $f: \mathbb{R} \rightarrow [0, \infty)$

$$P(\{\xi \in \text{Conf}(\mathcal{L}) : \xi(\mathbb{R}) \neq 1\}) = 0$$

$$P(\{\xi \in \text{Conf}(\mathcal{L}) : \xi(B) = 1\}) = \int_B f(x) dx, B \in \mathcal{B}_{\mathcal{L}}$$

defines a probability measure on $(\text{Conf}(\mathbb{R}), \mathcal{C}_{\mathbb{R}})$

→ 1 random variable with density f .

Example 2: A probability density f on \mathbb{R}^N

which is symmetric under permutation

of the variables,

$$\frac{1}{Z_N} f(x_1, \dots, x_N) dx_1 \dots dx_N,$$

naturally induces a point process

by identifying (x_1, \dots, x_N) with $\xi = \sum_{i=1}^N \delta_{x_i}$.

Note that it is often not practical to define

$P: C_R \rightarrow [0, 1]$ explicitly.

- Definition: The Factorial Moment Measure of order 1 of IP is defined as

$$M_1(B) = \mathbb{E} \xi(B), \quad \text{for } B \in \mathcal{B}_L.$$

If M_1 is absolutely continuous with density ρ_1 wrt μ , then

$$dM_1(x) = \rho_1(x) dx, \quad \mathbb{E} \xi(B) = \int_B \rho_1(x) d\mu(x),$$

ρ_1 is called the 1-point correlation function of IP wrt μ .

- Factorial Moment Measures and Correlation Functions of order m are more complicated to define, but

$\rho_m(x_1, \dots, x_m) dx_1 \dots dx_m$ will represent the infinitesimal probability to have points at x_1, \dots, x_m .

- Definition: The Factorial Moment Measure M_m
of order m is the symmetric measure on \mathcal{L}^m
such that

$$M_m(B_1 \times \dots \times B_m) = \mathbb{E} \xi(B_1) \dots \xi(B_m)$$

$$= \mathbb{E} \# \{m\text{-tuples } (x_1, \dots, x_m) : x_j \in \text{Supp } \xi \cap B_j\}$$

for $B_1, \dots, B_m \in \mathcal{B}_L$ disjoint, and

$$M_m(B_1^{k_1} \times \dots \times B_l^{k_l}) = \mathbb{E} \xi(B_1)^{\{k_1\}} \dots \xi(B_l)^{\{k_l\}}, \quad N^{\{k_i\}} = N(N-1)\dots(N-k_i)$$

$$= \mathbb{E} \# \{m\text{-tuples } (x_1^{(1)}, \dots, x_1^{(k_1)}, \dots, x_l^{(1)}, \dots, x_l^{(k_l)}) : x_j^{(k)} \in \text{Supp } \xi \cap B_j\}$$

for $B_1, \dots, B_l \in \mathcal{B}_L$ disjoint, $\sum_{j=1}^l k_j = m$.

- In particular, $M_m(B^m) = \mathbb{E} \xi(B)!$

$$= \mathbb{E} \# \{m\text{-tuples } (x_1, \dots, x_m) : x_j \in \text{Supp } \xi \cap B\}$$

- Definition: If there exists a reference measure μ on Λ such that

$$dM_m(x_1, \dots, x_m) = p_m(x_1, \dots, x_m) d\mu(x_1) \dots d\mu(x_m),$$

we say that $p_m(x_1, \dots, x_m)$ is the m -point correlation function of P wrt μ .

- " $p(x_1, \dots, x_m) d\mu(x_1) \dots d\mu(x_n)$ "

$$= \mathbb{P} \left(\xi([x_1, x_1 + dx_1]), \dots, \xi([x_m, x_m + dx_m]) \neq \emptyset \right)^{**}$$

Further Assumptions:

A. The Point Process is Simple,

$$\text{i.e. } \mathbb{P}(\exists x \in L : \xi(\{x\}) > 1) = 0$$

B. The Factorial Moment Measure of any order m exists, and there exists a reference measure μ on L such that

$$dM_m(x_1, \dots, x_m) = \rho_m(x_1, \dots, x_m) d\mu(x_1) \dots d\mu(x_m)$$

C. For any bounded $B \in \mathcal{B}_L$, $\exists \varepsilon_B > 0$:

$$\sum_{m=1}^{\infty} \frac{(1+\varepsilon_B)^m}{m!} M_m(B^m) < \infty.$$

- Under these assumptions, it is a classical result [Lenard 1975] that the correlation functions uniquely characterize the point process.

Definition: The Janossy Measure of order m associated to $B \in \mathcal{B}_L$ is the symmetric measure on L^m given by

$$J_m^B(B_1^{k_1} \times \dots \times B_m^{k_m}) = k_1! \dots k_m! P(\xi(B_i) = k_i, i=1, \dots, n)$$

with $B = \bigcup_{i=1}^n B_i$, $\sum_{i=1}^n k_i = m$

If $dJ_m^B(x_1, \dots, x_m) = \gamma_m^B(x_1, \dots, x_m) dx_1(x_1) \dots dx_m(x_m)$,

we call $\gamma_m^B(x_1, \dots, x_m)$ the Janossy density.

Heuristically: $\gamma_m^B(x_1, \dots, x_m)$ is the infinitesimal probability that ξ has exactly m points x_1, \dots, x_m in B .

- Under Assumptions A-C, the Janossy densities of any order m exist for bounded $B \in \mathcal{B}_L$,

and $\boxed{\gamma_m^B(x_1, \dots, x_m) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} \dots \int p_{m+n}(x_1, \dots, x_m, y_1, \dots, y_n) d\mu(y_1) \dots d\mu(y_n)}$

In particular, $\gamma_0^B(\phi) = \mathbb{P}(\xi(B) = 0)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} \dots \int p_n(y_1, \dots, y_n) d\mu(y_1) \dots d\mu(y_n)$$

- The Global Janossy density γ_m^L does not necessarily exist.

- If the Global Janossy density γ_m^L exists,

$\boxed{p_m(x_1, \dots, x_m) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{B^n} \dots \int \gamma_m^L(x_1, \dots, x_m, y_1, \dots, y_n) d\mu(y_1) \dots d\mu(y_n)}$

Definition: A linear statistic of Π_P is a random variable of the form

$$\sum_{x \in \text{Supp } \xi} f(x) = \int f d\xi, \quad f: \mathcal{L} \rightarrow \mathbb{C}.$$

Definition: The Laplace functional of Π_P is

$$\mathcal{L}[f]: B_+(\mathcal{L}) \rightarrow \mathbb{R}^+: f \mapsto \mathcal{L}[f] := \mathbb{E} e^{-\int f d\xi},$$

where $B_+(\mathcal{L})$ is the space of bounded measurable $f: \mathcal{L} \rightarrow [0, +\infty)$ with bounded support.

→ The Laplace functional characterizes Π_P uniquely
(cf. Characteristic Function of a Random Variable)

A close cousin of the Laplace functional

is the probability-generating functional

or multiplicative functional:

- $L[g] := \mathbb{E} \prod_{x \in \text{Supp } g} g(x), \quad L[f] = L[e^{-f}]$

- It follows from Fubini's theorem and the definition of ρ_n that

$$L[1-\phi] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int \dots \int \rho_n(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n) d\mu(x_1) \dots d\mu(x_n)$$

for $\phi \in B_+(\mathcal{L})$.

Exercise 1: Consider N independent Gaussians with distribution

$$\frac{1}{Z} e^{-x^2} dx \text{ on } \mathbb{R}, Z = \sqrt{\pi}.$$

- (a) Compute first the symmetric joint probability density on \mathbb{R}^N , and then the probability generating functional.
- (b) Compute the correlation functions
- (c) Compute the Global Joinossy density.

Exercise 2: The Poisson Point Process

→ Definition: Let $\rho: \mathbb{R} \rightarrow [0, \infty)$ be bounded and locally integrable (i.e. integrable over any bounded Borel set).
The Poisson Point Process with Intensity ρ is the unique Point Process on \mathbb{R} such that $P_m(x_1, \dots, x_m) = \prod_{i=1}^m \rho(x_i)$ for any m .

(a) Prove that the prob. gen. functional is given by

$$L[g] = e^{-\int (\rho - g(x)) \rho(dx)}$$

(b) Compute the Janossy densities.

(c) Prove that

$$P(\xi(B)=n) = \left(\int_B \rho dx \right)^n e^{-\int_B \rho dx}$$

$\frac{n!}{n^n}$ is Poisson distributed.

② What is a Determinantal Point Process?

Definition (Macchi 1975): A Determinantal Point Process (DPP) on \mathcal{L} is a Point Process on \mathcal{L} with correlation functions given by

$$p_m(x_1, \dots, x_m) = \det \left[K(x_i, x_j) \right]_{i,j=1}^m,$$

for some function $K: \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{C}$ which is called the (correlation) kernel of the DPP.

- ! The kernel of a DPP is not unique
 - $K(x, y) \mapsto \frac{g(x)}{g(y)} K(x, y)$ preserves the DPP
- ! K has to be such that $p_m(x_1, \dots, x_m) \geq 0$.

→ Also called Fermionic Point Process.

- We associate an integral operator to the kernel K :

$$\mathcal{K}: L^2(\mathcal{L}, \mu) \rightarrow L^2(\mathcal{L}, \mu)$$

$$[\mathcal{K}f](x) = \int K(x, y) f(y) d\mu(y)$$

- The Probability Generating Functional of a DPP is given by

$$L[1 - \phi] = \sum \frac{(-1)^n}{n!} \int_{\mathcal{L}^n} \dots \int \det \left[K(x_i, x_j) \right]_{i,j=1}^n \phi(x_1) \dots \phi(x_n) d\mu(x_1) \dots d\mu(x_n)$$

- This can be seen as the definition of the Fredholm determinant

if \mathcal{K} is locally trace class.

- Definition: - \mathcal{K} is trace class $\Leftrightarrow \mathcal{K}$ is the composition
of 2 Hilbert-Schmidt operators
- \mathcal{K} is locally trace class $\Leftrightarrow 1_B \mathcal{K} 1_B$ is trace class
for any bounded $B \in \mathcal{B}_1$

Exercise 3: Use the Fredholm series to prove

that - $E \int \phi d\mathbb{E} = \int K(x, x) \phi(x) d\mu(x) = \text{Tr } \phi \mathcal{K}$

- $\text{Var} \int \phi d\mathbb{E} = \int \phi(x)^2 k(x, x) d\mu(x) - \int \int \phi(x) \phi(y) K(x, y) d\mu(x) d\mu(y)$

for $\phi \in \mathcal{B}_+(\mathcal{L})$.

Exercise 4: Prove that any DPP satisfies
Assumptions A-C.

- We say that the locally trace class operator γ_K on (\mathcal{H}, μ) induces the DPP IP if its Fredholm determinant

$$\det(1 - \phi \gamma_K) = L[1 - \phi] \text{ with } L \text{ the probability generating functional of IP.}$$

- We can then choose a kernel of γ_K such that

$$\text{Tr}(\phi \gamma_K) = \int K(x, x) \phi(x) d\mu(x) \text{ for } \phi \in \mathcal{B}_+(\mathcal{H}).$$

Theorem [Soshnikov '00]:

A Hermitian locally trace class operator γ_K
 induces a DPP $\Leftrightarrow 0 \leq \gamma_K \leq 1$.
 (unique)

! No such result for non-Hermitian γ_K .

Properties of DPPs induced by Hermitian loc. tr. c. \mathcal{K}

- $\text{IP}(\xi(\mathcal{L}) < \infty) = \begin{cases} 1 & \text{if } \text{Tr } \mathcal{K} < \infty \\ 0 & \text{if } \text{Tr } \mathcal{K} = \infty \end{cases}$
- $\text{IP}(\xi(\mathcal{L}) = N) = 1 \Leftrightarrow \mathcal{K} \text{ is a rank } N \text{ Orthogonal Projection}$
- $p_m(x_1, \dots, x_m) \leq p_1(x_1) p_1(x_2) \dots p_1(x_m)$
 $(\sim \text{Repulsion})$

Exercise 5: Prove these results.

(See e.g. [Soshikov '00]).

③ Subclasses of DPPs on \mathbb{R}

- Orthogonal Polynomial Ensembles

N-point DPP,

$$\frac{1}{Z_N} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2 \prod_{j=1}^N w(x_j) dx_j$$

\rightarrow GUE if $w(x) = e^{-Nx^2}$, UE if $w(x) = e^{-NV(x)}$

$$\begin{aligned} \rightarrow K_N(x, y) &= \sqrt{w(x)w(y)} \sum_{j=0}^{N-1} p_j(x)p_j(y) \\ &= \sqrt{w(x)w(y)} C_N \frac{P_N(x)P_{N-1}(y) - P_N(y)P_{N-1}(x)}{x-y} \end{aligned}$$

is the Christoffel-Darboux kernel of orthonormal polynomials p_0, p_1, \dots wrt $w(x)dx$.

- Polynomial Ensembles

N-point DPP,

$$\frac{1}{Z_N} \prod_{1 \leq j < k \leq N} (x_k - x_j) (f(x_k) - f(x_j)) \prod_{j=1}^N \omega(x_j) dx_j$$

Example: Muttalib-Borodin Ensemble for $f(x) = x^\alpha$, $\text{supp } \omega = \mathbb{R}^+$

- Eigenvalue jpdf of products of random matrices

→ Induced by a rank N projection \mathcal{K}_N ,

but not necessarily Hermitian!

- Biorthogonal Ensembles: Replace $(x_k - x_j)$ by $(h(x_k) - h(x_j))$ above

→ Also rank N projection DPP, not necessarily Hermitian.

- Orthogonal Projection DPPs,
induced by finite or infinite rank orthogonal projection

→ OPEs (finite rank)

→ Scaling Limits of OPEs (infinite rank)

- Airy kernel $K(x, y) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x-y}$

- Sine kernel $K(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$

- Bessel kernel, Poinlevé kernels

- Integrable kernel DPPs [It's - Izergin - Korepin - Slavnov
1990]

→ Defined by a kernel of the form

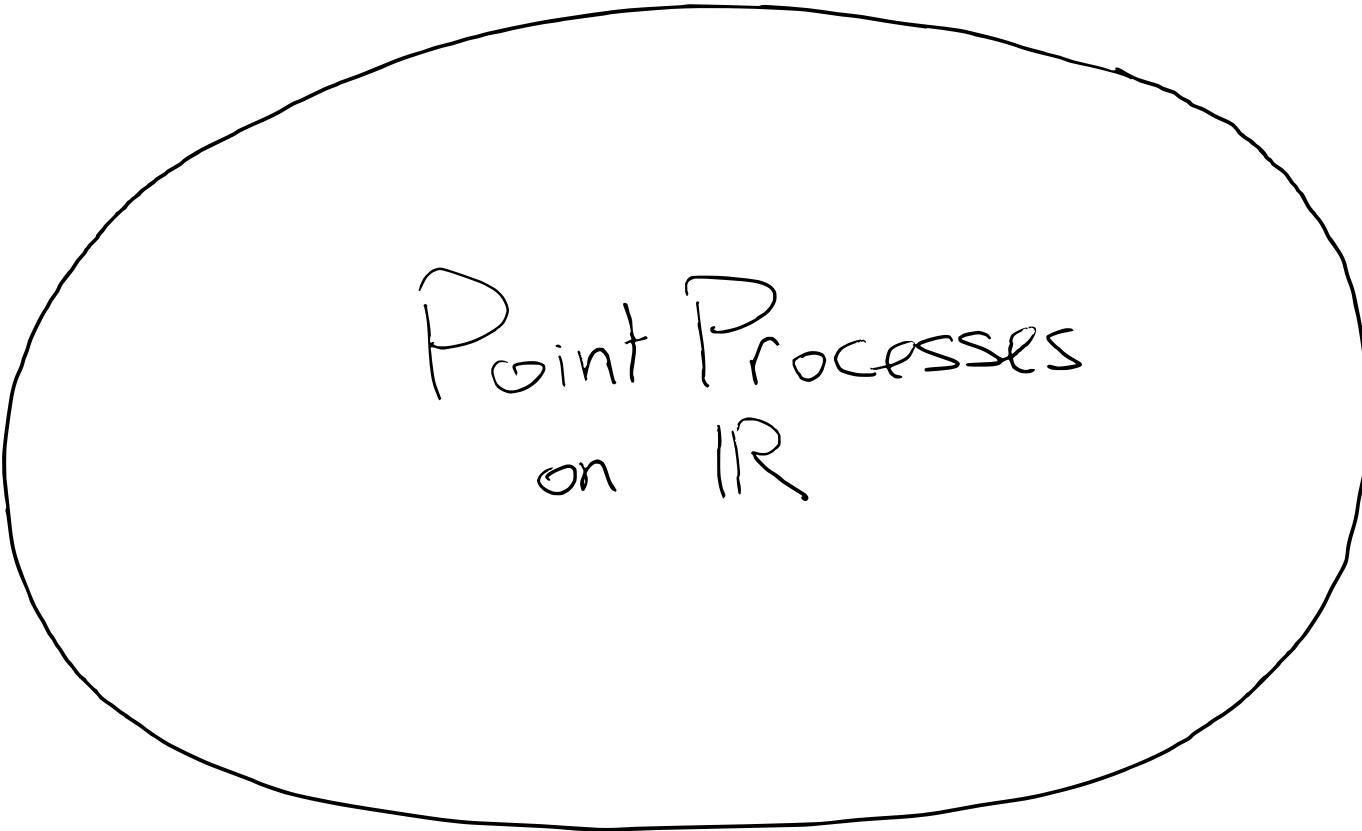
$$K(x, y) = \frac{\sum_{j=1}^k f_j(x) g_j(y)}{x - y}, \quad \sum_{j=1}^k f_j(x) g_j(x) = 0.$$

→ I Projection DPPs like OPEs and their scaling limits ($k = 2$)

→ Skew Projection DPPs like some PEs and their scaling limits ($k > 2$)

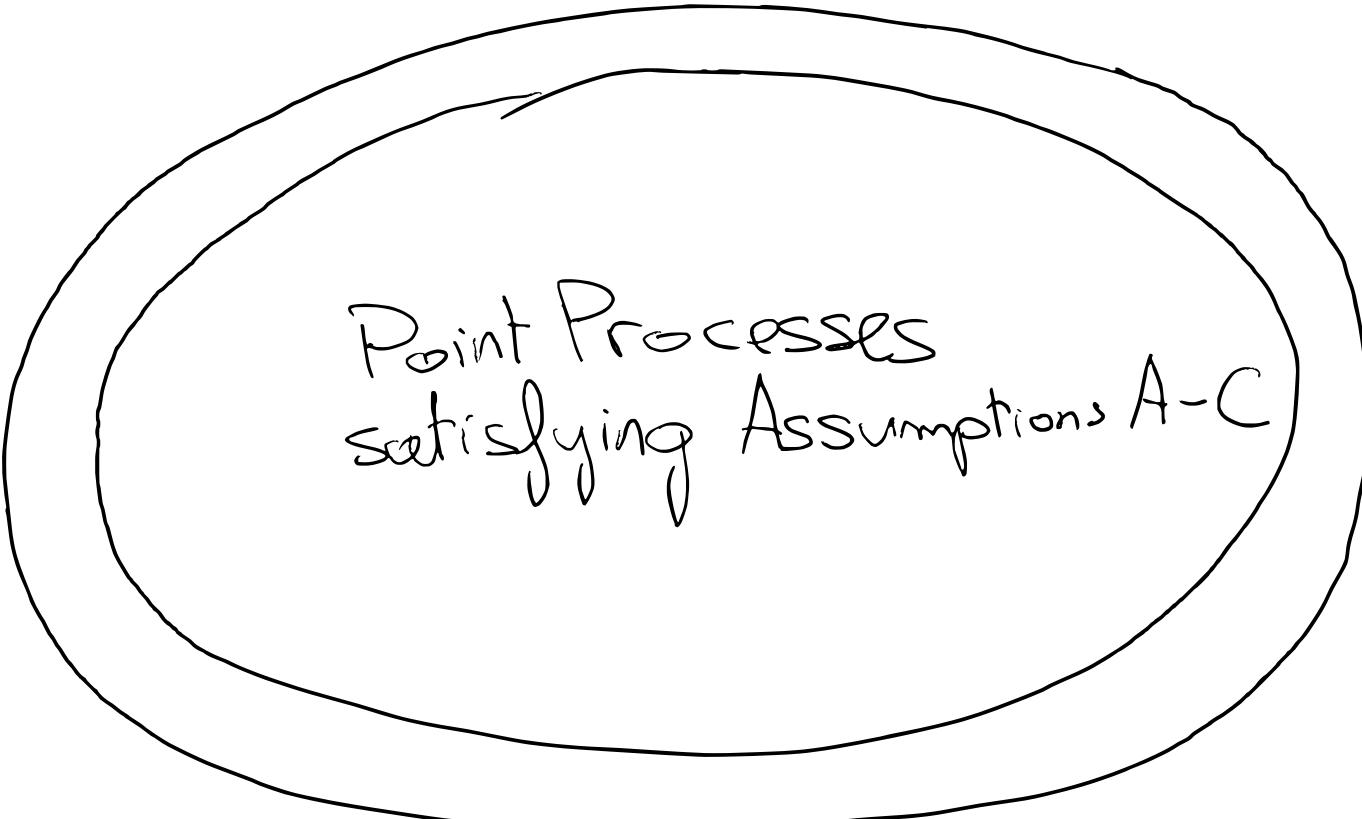
→ But also non-projection DPPs (see later).

Summary



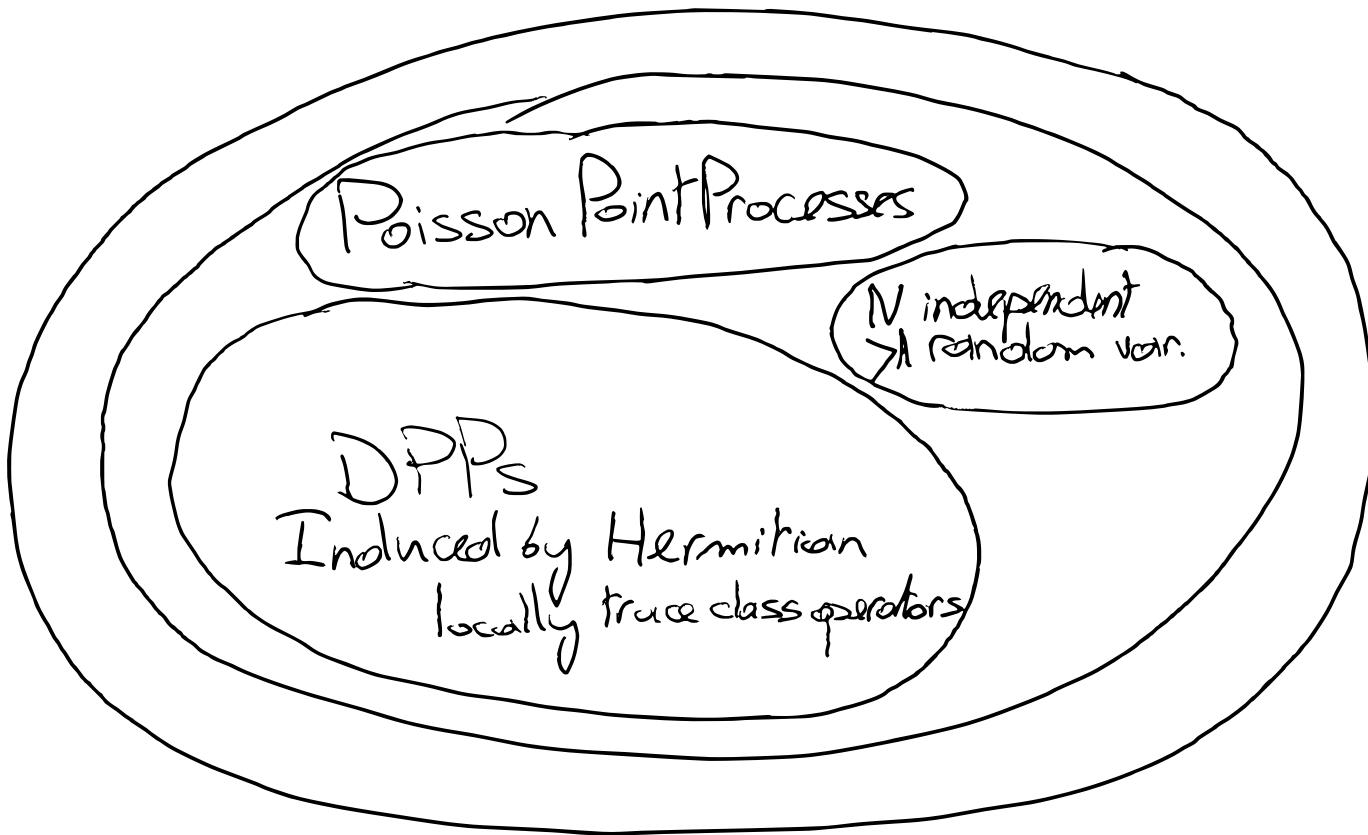
Point Processes
on $\mathbb{I}\mathbb{R}$

Summary

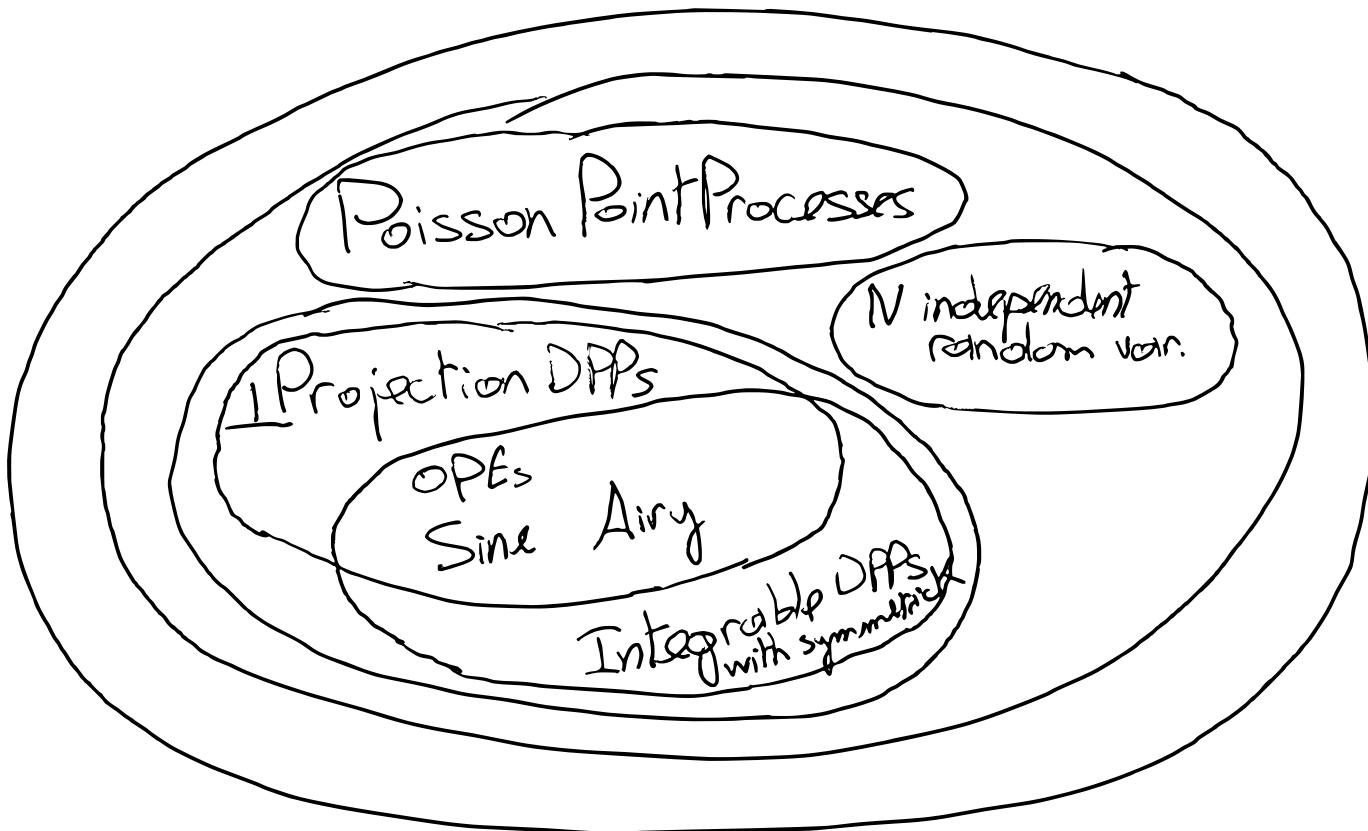


Point Processes
satisfying Assumptions A-C

Summary



Summary



References

① Point Processes

- Book by Daley - Vere Jones '03 → general theory
- Original articles by Lenard in the 1970s
→ point processes and their correlation functions

② DPPs

- Macchi 1975
- Soshnikov '00
- Johansson '06
- Lyons '03
- Borodin, Oxford Handbook of RMT '11
- Shirai - Takahashi '03
- Hough - Krishnapur - Peccati - Virág '06
- Baik - Deift - Suidan '16

Lecture 2: Transformations of DPPs

- ① Thinning of a DPP
- ② Marking a DPP
- ③ Conditioning a DPP on presence of points
- ④ Conditioning a DPP on a gap
- ⑤ Conditioning a marked DPP

→ Which subclasses of DPPs are preserved?
→ How do kernels K and operators \mathcal{J}_K behave under these transformations?