Point Processes and Multiple SLE/GFF Coupling

Makoto KATORI (Chuo Univ., Tokyo)

4th ZiF Summer School
`Randomness in Physics and Mathematics:
From Integrable Probability to Disordered Systems'

ZiF-Center for Interdisciplinary Research,
Bielefeld University, Germany,
1-13 August 2022

Plan of 4 Lectures

1. **From Bessel process to Dyson's BM model**
   (stochastic DPP)

2. **Gaussian analytic functions and their zeros**
   (permanental-determinantal PP)

3. **Schramm-Loewner evolution (SLE) and Gaussian free fields**
   (random curves and random surfaces)

4. **Multiple SLE/GFF coupling driven by Dyson's BM model**
   (random points/curves/surfaces)

Point Processes and Multiple SLE/GFF Coupling

Lecture No.1 (8 Aug. 2022)

Dyson's BM model

Makoto KATORI (Chuo Univ., Tokyo)

4th ZiF Summer School
`Randomness in Physics and Mathematics:
From Integrable Probability to Disordered Systems'

ZiF-Center for Interdisciplinary Research,
Bielefeld University, Germany,
1-13 August 2022

1 dim standard Brownian motion (BM)

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be the probability space of 1-dim. standard Brownian motion, $\{B(t) : t \geq 0\}$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

(BM1) $B(0) = 0$ **with probability 1.**

(BM2) $B(t)$ **has a continuous path almost surely.**

(BM3) Given any $t > 0$**. For an arbitrary** $n \in \mathbb{N} := \{1, 2, \dots\}$**, and for any sequence of times,** $t_0 := 0 < t_1 < \cdots < t_n := t$**,**

- $B(t_m) - B(t_{m-1}), m = 1, \dots, n$**, are independent from each other.**
- $B(t_m) - B(t_{m-1}) \sim \mathrm{N}(0, t_m - t_{m-1}), \quad m = 1, 2, \dots, n.$

It means that, for any $0 \leq s < t < \infty$ **and** $a < b$**,**

$$\mathrm{P}[B(t) - B(s) \in [a, b]] = \int_a^b p(t - s, z|0)dz,$$

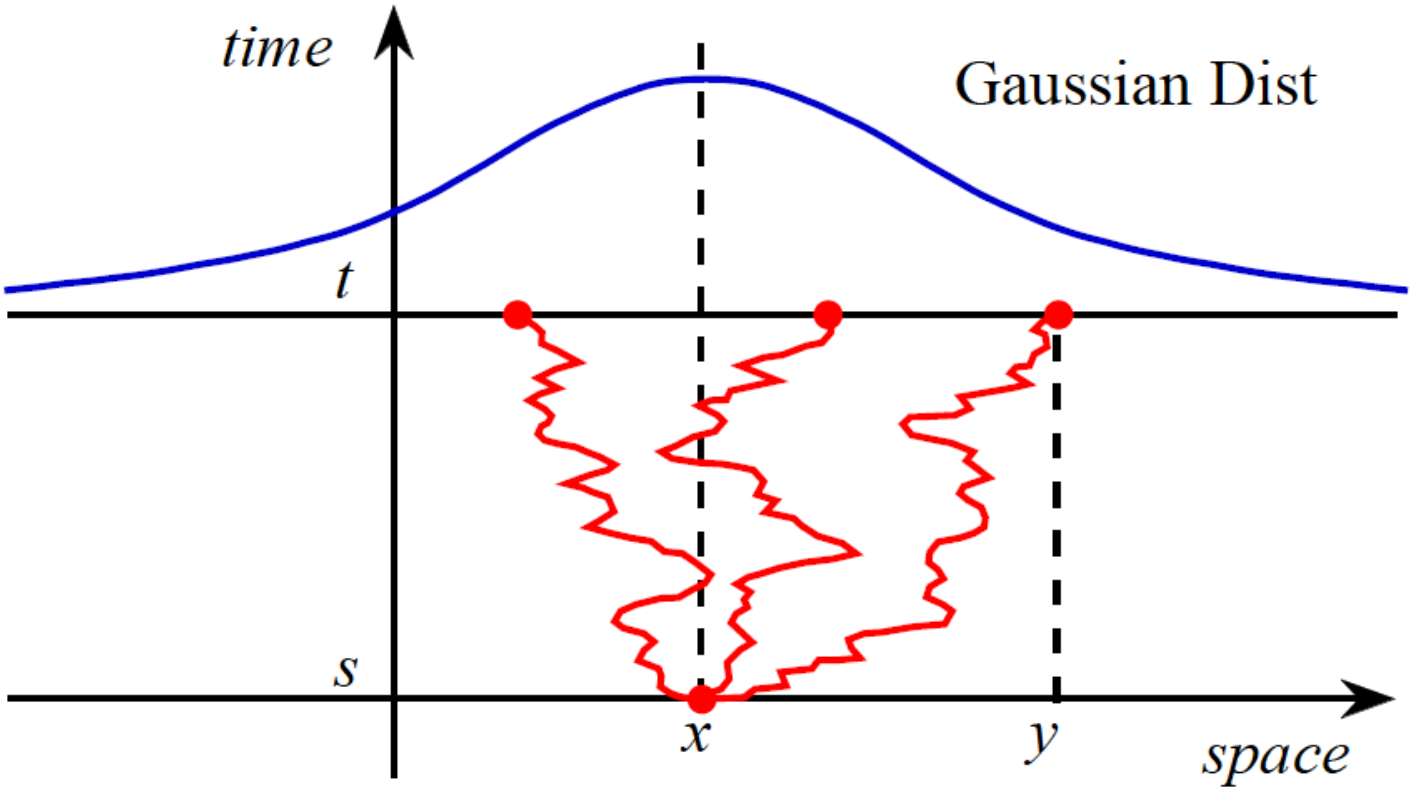
where for $x, y \in \mathbb{R}$

$$p(t, y|x) = \begin{cases} \dfrac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, & \text{for } t > 0, \\ \delta(x - y), & \text{for } t = 0, \end{cases}$$

transition prob. density.

The mean is constant in time;

$$\mathrm{E}[B(t)|\mathcal{F}_s] = \mathrm{E}[(B(t) - B(s)) + B(s)|\mathcal{F}_s] = \mathrm{E}[B(t) - B(s)|\mathcal{F}_s] + \mathrm{E}[B(s)|\mathcal{F}_s]$$
$$= 0 + B(s) = B(s), \quad 0 \le s < t < \infty, \quad \textbf{a.s.}$$



For each time interval $[0,t], t>0$, put $n \in \mathbb{N}$ and consider a subdivision of $[0,t]$

$$t_0 := 0 < t_1 < \cdots < t_n := t, \quad \textbf{s.t.} \ \max_{1\le m\le n} |t_m - t_{m-1}| \to 0 \quad \textbf{as } n \to \infty.$$

The quadratic variation of $B(t)$ is defined by

$$\langle B, B\rangle_t := \text{P-}\lim_{n\to\infty} \sum_{m=1}^{n} (B(t_m) - B(t_{m-1}))^2 \quad \textbf{convergence in probability}$$

$$\iff \lim_{n\to\infty} \mathrm{P}\left[\left| \sum_{m=1}^{n} (B(t_m) - B(t_{m-1}))^2 - \langle B, B\rangle_t \right| > \varepsilon \right] = 0, \quad \forall \varepsilon > 0.$$

We can prove

$$\langle B, B\rangle_t = t, \quad t \ge 0.$$

We can also prove that the total variation diverges,

$$\sum_{m=1}^{n} |B(t_m) - B(t_{m-1})| \to \infty \quad \textbf{as } n \to \infty \textbf{ in probability.}$$

See Section 1.2 of Lecture Note for proofs.

In summary,

$$\left.\begin{array}{l} \mathrm{E}[(B(t)|\mathcal{F}_s] = B(s), \quad 0 \le s < t < \infty, \textbf{ a.s.} \\ \sum_{m=1}^{n} |B(t_m) - B(t_{m-1})| \xrightarrow{\mathrm{P}} \infty, \\ \sum_{m=1}^{n} (B(t_m) - B(t_{m-1}))^2 \xrightarrow{\mathrm{P}} \langle B, B\rangle_t = t \uparrow \infty \textbf{ as } t \uparrow \infty \end{array}\right\} \quad \textbf{martingale}$$

Consider the process $\Big(B(t)^2 - \langle B, B\rangle_t\Big)_{t\geq 0} = \Big(B(t)^2 - t\Big)_{t\geq 0}$.

This is martingale;

$$\begin{aligned}
\mathrm{E}[B(t)^2 - t|\mathcal{F}_s] &= \mathrm{E}[(B(t) - B(s))^2 + 2(B(t) - B(s))B(s) + B(s)^2 - t|\mathcal{F}_s] \\
&= \mathrm{E}[(B(t) - B(s))^2|\mathcal{F}_s] + 2\mathrm{E}[(B(t) - B(s))B(s)|\mathcal{F}_s] + \mathrm{E}[B(s)^2|\mathcal{F}_s] - t \\
&= (t - s) + 0 + B(s)^2 - t \\
&= B(s)^2 - s, \quad 0 \leq s < t < \infty, \quad \textbf{a.s.}
\end{aligned}$$

$$\mathrm{E}[B(t)|\mathcal{F}_s] = B(s) \iff \mathrm{E}[B(t) - B(s)|\mathcal{F}_a] = 0$$
$$\implies \mathrm{E}[dB(t)] = 0 \ \because \textbf{ increment } dB(t) \sim \textbf{ symmetric dist. } \mathrm{N}(\mathbf{0},\ \mathbf{dt})$$
$$\mathrm{E}[B(t)^2 - t|\mathcal{F}_s] = B(s)^2 - s \iff \mathrm{E}[(B(t)^2 - B(s)^2) - (t-s)|\mathcal{F}_a] = 0$$
$$\implies \mathrm{E}[d(B(t)^2 - t)] = 2\mathrm{E}[B(t)dB(t)] - dt = -dt \neq 0?$$

This implies $dB(t)^2 \neq 2B(t)dB(t)$. We need additional term to cancel $-dt$.
That new term should be called the Ito term:

$$dB(t)^2 = 2B(t)dB(t) + dt = 2B(t)dB(t) + \frac{1}{2} \times 2 \times dt$$
$$\iff dB(t)^2 = df(B(t)) \qquad \textbf{(with } f(y) = y^2\textbf{)}$$
$$= \left.\frac{df(y)}{dy}\right|_{y=B(t)} dB(t) + \frac{1}{2}\left.\frac{d^2 f(y)}{dy^2}\right|_{y=B(t)} d\langle B, B\rangle_t$$
$$= 2y\Big|_{y=B(t)} dB(t) + \frac{1}{2} \times 2 \times d\langle B, B\rangle_t$$

If we follow this Ito calculus (stochastic analysis), then for $(B(t)^2 - t)_{t\geq 0}$,

$$\mathrm{E}[d(B(t)^2 - t)] = \mathrm{E}[2B(t)dB(t) + dt] - dt = 2\mathrm{E}[B(t)dB(t)] + dt - dt = 2\mathrm{E}[B(t)dB(t)] = 0,$$

and the quadratic variation is $\langle 2BdB, 2BdB\rangle_t = 4B(t)^2\langle dB, dB\rangle_t = 4B(t)^2 dt.$

Let $D \in \mathbb{N}$ and consider identical but independent D BMs, $B_1(t), \ldots, B_D(t)$. By definition, $B_i(t) + B_j(t)$ and $B_i(t) - B_j(t)$, $1 \leq i, j \leq D$ are martingales. Then their squares are not, but the followings shall be martingales;

$$(B_i(t) + B_j(t))^2 - \langle B_i + B_j, B_i + B_j \rangle_t$$
$$(B_i(t) - B_j(t))^2 - \langle B_i - B_j, B_i - B_j \rangle_t.$$

and hence, the difference of these two should be also martingale,

$$4B_i(t)B_j(t) - (\langle B_i + B_j, B_i + B_j \rangle_t - \langle B_i - B_j, B_i - B_j \rangle_t).$$

Define the mutual quadratic variation (cross variation) by

$$\langle B_i, B_j \rangle_t := \frac{1}{4}(\langle B_i + B_j, B_i + B_j \rangle_t - \langle B_i - B_j, B_i - B_j \rangle_t).$$

This is calculated as

$$\langle B_i, B_j \rangle_t = \begin{cases} (4\langle B_i, B_i \rangle_t - 0)/4 = t, & (i = j) \\ (2t - 2t)/4 = 0, & (i \neq j) \end{cases} \quad \Longrightarrow \quad d\langle B_i, B_j \rangle_t = \delta_{ij} dt.$$

Exercise 1.1

For $t > 0$, $x \in \mathbb{R}$, and $\alpha \in \mathbb{C}$, let

$$G_\alpha(t,x) = e^{\alpha x - \alpha^2 t/2}. \tag{1.1}$$

Let $(B(t))_{t\geq 0}$ be the 1 dimensional standard Brownian motion (BM) in $(\Omega, \mathrm{P}, \mathcal{F})$ with the filtration $\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t), t \geq 0$.

(1) Prove the equality

$$\frac{e^{\alpha B(t)}}{\mathrm{E}[e^{\alpha B(t)}]} = G_\alpha(t, B(t)), \quad t \geq 0. \tag{1.2}$$

(2) Prove that $(G_\alpha(t, B(t))_{t\geq 0}$ is $\mathcal{F}_t$-martingale;

$$\mathrm{E}[G_\alpha(t, B(t))|\mathcal{F}_s] = G_\alpha(s, B(s)), \quad 0 < s < t. \tag{1.3}$$

(3) Let $\widehat{G}_w(t,x)$ be the Fourier transformation of $G_\alpha(t,x)$;

$$\widehat{G}_w(t,x) := \int_{-\infty}^{\infty} \frac{e^{-\sqrt{-1}\alpha w}}{2\pi} G_\alpha(t,x) d\alpha. \tag{1.4}$$

Show that

$$\widehat{G}_w(t,x) = \frac{e^{-(\sqrt{-1}x+w)^2/2t}}{\sqrt{2\pi t}}. \tag{1.5}$$

(4) Define the integral transformation by

$$\mathcal{I}[f(W)|(t,x)] = \int_{-\infty}^{\infty} f(\sqrt{-1}w)\widehat{G}_w(t,x)dw. \tag{1.6}$$

For $n \in \mathbb{N}_0 := \{0, 1, \dots\}$, let

$$\begin{aligned} m_n(t,x) &= \mathcal{I}[W^n|(t,x)] \\ &= \int_{-\infty}^{\infty} (\sqrt{-1}w)^n \widehat{G}_w(t,x)dw, \quad t \geq 0. \end{aligned} \tag{1.7}$$

Express $m_n(t,x), n \in \mathbb{N}_0$ using the Hermite polynomials,

$$\begin{aligned} H_n(x) &:= (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n} \\ &= \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{k!(n-2k)!} (2x)^{n-2k}. \end{aligned} \tag{1.8}$$

(5) Prove the following for $\{m_n(t,x)\}_{n\in\mathbb{N}_0}$.

(i) They are monic polynomials of degrees $n \in \mathbb{N}_0$ with time-dependent coefficients:

$$m_n(t,x) = x^n + \sum_{k=0}^{n-1} c_n^{(k)}(t) x^k, \quad t \geq 0.$$

(ii) For $0 \leq k \leq n-1$, $c_n^{(k)}(0) = 0$. That is,

$$m_n(0,x) = x^n, \quad n \in \mathbb{N}_0.$$

(iii) If we set $x = B(t)$, they provide martingales:

$$\mathrm{E}[m_n(t,B(t))|\mathcal{F}_s] = m_n(s,B(s)), \quad 0 \leq s \leq t, \quad n \in \mathbb{N}_0. \tag{1.9}$$

D-dim BM and Bessel process

For the BM starting from $x \in \mathbb{R}$, we write

$$B^x(t) := x + B(t), \quad t \geq 0.$$

The probability law and the expectation are written as P^x and E^x; for $\mathcal{F}_t$-measurable function f,

$$\mathrm{E}^x[f(B(t))] = \mathrm{E}[f(B^x(t))].$$

Consider D independent BMs, $B_i^{x_i}(t)$, $t \geq 0$ starting from $x_i, i = 1, \ldots D$. Then define the D-dim. BM started at $\boldsymbol{x}$ as the vector-valued BM,

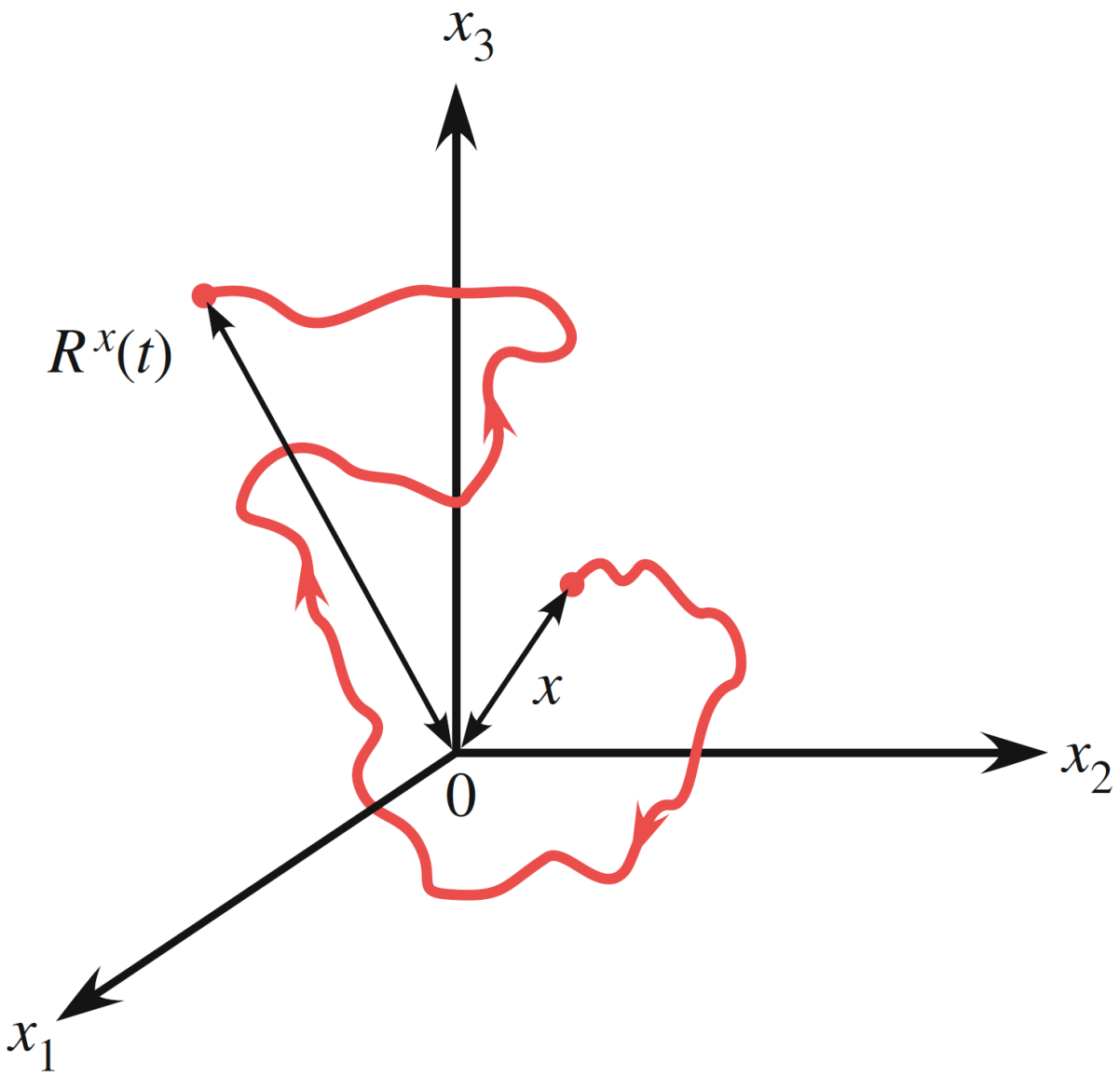
$$\boldsymbol{B}^{\boldsymbol{x}} := (B_1^{x_1}(t), B_2^{x_2}(t), \ldots, B_D^{x_D}), \quad t \geq 0.$$

Then the D-dim. Bessel process, BES_D, is defined as the radial coordinate of the D-dim. BM,

$$R^x(t) := |\boldsymbol{B}^{\boldsymbol{x}}(t)| = \sqrt{B_1^{x_1}(t)^2 + \cdots + B_D^{x_D}(t)^2}, \quad t \geq 0,$$

where the initial value is given by $R^x(0) = x := |\boldsymbol{x}| = \sqrt{x_1^2 + \cdots + x_D^2} \geq 0$.

3 dim Bessel process, BES_3



BES_D, $(R^x(t))_{t\geq 0}$, is defined as

$$R^x(t) = F(\boldsymbol{B}^{\boldsymbol{x}}(t)), \quad t \geq 0 \quad \textbf{with} \quad F(\boldsymbol{y}) = \sqrt{y_1^2 + \cdots + y_D^2}.$$

Itô's formula

- **Let $N \in \mathbb{N}$ and $\boldsymbol{y} := (y_1, \ldots, y_N) \in \mathbb{R}^N$.**
- **Let F be a function of t and $\boldsymbol{y}$; $F = F(t, \boldsymbol{y})$.**
- **Assume that a N-component stochastic process $\boldsymbol{Y}(t) = (Y_1(t), \ldots, Y_N(t)), t \geq 0$ is given.**

Then, for a stochastic process $(F(t, \boldsymbol{Y}(t)))_{t\geq 0}$, the difference is given by the following formula, which is called Itô's formula,

$$dF(t, \boldsymbol{Y}(t)) = \frac{\partial F(t, \boldsymbol{y})}{\partial t} dt + \sum_{i=1}^{N} \left. \frac{\partial F(t, \boldsymbol{y})}{\partial y_i} \right|_{\boldsymbol{y}=\boldsymbol{Y}(t)} dY_i(t)$$
$$+ \frac{1}{2} \sum_{1\leq i,j\leq N} \left. \frac{\partial^2 F(t, \boldsymbol{y})}{\partial y_i \partial y_j} \right|_{\boldsymbol{y}=\boldsymbol{Y}(t)} d\langle Y_i, Y_j \rangle_t, \quad t \geq 0.$$

BES$_D$, $(R^x(t))_{t\geq 0}$, is defined as $R^x(t) = F(\boldsymbol{B}^{\boldsymbol{x}}(t))$, $t \geq 0$, with $F(\boldsymbol{y}) = \sqrt{y_1^2 + \cdots + y_D^2}$.
We see that

$$\frac{\partial F(\boldsymbol{y})}{\partial t} = 0, \quad \frac{\partial F(\boldsymbol{y})}{\partial y_i} = \frac{y_i}{F(\boldsymbol{y})}, \quad \frac{\partial^2 F(\boldsymbol{y})}{\partial y_i \partial y_j} = \frac{\delta_{ij}}{F(\boldsymbol{y})} - \frac{y_i y_j}{F(\boldsymbol{y})^3}, \quad i, j = 1, \ldots, D,$$
$$d\langle B_i^{x_i}, B_j^{x_j} \rangle_t = \delta_{ij} dt, \quad i, j = 1, \ldots, D.$$

Then the drift terms (finite variation part) become

$$\frac{1}{2} \sum_{1 \leq i,j \leq D} \frac{\partial^2 F(t, \boldsymbol{y})}{\partial y_i \partial y_j}\bigg|_{\boldsymbol{y} = \boldsymbol{B}^{\boldsymbol{x}}(t)} d\langle B_i^{x_i}, B_j^{x_j} \rangle_t = \frac{1}{2} \sum_{1 \leq i,j \leq D} \left\{ \frac{\delta_{ij}}{F(\boldsymbol{B}^{\boldsymbol{x}}(t))} - \frac{B_i^{x_i}(t) B_j^{x_j}(t)}{F(\boldsymbol{B}^{\boldsymbol{x}}(t))^3} \right\} \delta_{ij} dt$$
$$= \frac{1}{2} \left\{ \frac{\sum_{1 \leq i \leq D} 1}{F(\boldsymbol{B}^{\boldsymbol{x}}(t))} - \frac{\sum_{1 \leq i \leq D} B_i^{x_i}(t)^2}{F(\boldsymbol{B}^{\boldsymbol{x}}(t))^3} \right\} \delta_{ij} dt$$
$$= \frac{D-1}{2} \frac{1}{F(\boldsymbol{B}^{\boldsymbol{x}}(t))} dt = \frac{D-1}{2} \frac{dt}{R^x(t)}.$$

BES$_D$, $(R^x(t))_{t\geq 0}$, is defined as $R^x(t) = F(\boldsymbol{B}^{\boldsymbol{x}}(t))$, $t \geq 0$, with $F(\boldsymbol{y}) = \sqrt{\sum_{i=1}^{D} y_i^2}$.

We see that

$$\frac{\partial F(\boldsymbol{y})}{\partial t} = 0, \quad \frac{\partial F(\boldsymbol{y})}{\partial y_i} = \frac{y_i}{F}, \quad \frac{\partial^2 F(\boldsymbol{y})}{\partial y_i \partial y_j} = \frac{\delta_{ij}}{F} - \frac{y_i y_j}{F^3}, \quad i,j = 1,\dots,D,$$
$$d\langle B_i^{x_i}, B_j^{x_j}\rangle_t = \delta_{ij} dt, \quad i,j = 1,\dots,D.$$

The martingale part is obtained as

$$\sum_{i=1}^{D} \frac{\partial F(t,\boldsymbol{y})}{\partial y_i}\bigg|_{\boldsymbol{y}=\boldsymbol{B}^{\boldsymbol{x}}(t)} dB_i^{x_i}(t) = \frac{1}{R^x(t)} \sum_{i=1}^{D} B_i^{x_i}(t) dB_i^{x_i}(t).$$

Its quadratic variation is calculated as

$$\left\langle \frac{1}{R^x} \sum_{i=1}^{D} B_i^{x_i} dB_i^{x_i}, \frac{1}{R^x} \sum_{j=1}^{D} B_j^{x_j} dB_j^{x_j} \right\rangle_t = \frac{1}{R^x(t)^2} \sum_{i=1}^{D}\sum_{j=1}^{D} B_i^{x_i}(t) B_j^{x_j}(t) \langle dB_i^{x_i}, dB_j^{x_j}\rangle_t$$
$$= \frac{1}{R^x(t)^2} \sum_{i=1}^{D}\sum_{j=1}^{D} B_i^{x_i}(t) B_j^{x_j}(t) \delta_{ij} dt = \frac{1}{R^x(t)^2} \sum_{i=1}^{D} B_i^{x_i}(t)^2 \delta_{ij} dt = dt.$$

This implies that, with another BM $(B(t))_{t\geq 0}$, we have

$$\left(\frac{1}{R^x(t)} \sum_{i=1}^{D} B_i^{x_i}(t) dB_i^{x_i}(t) \right)_{t\geq 0} \overset{\text{(law)}}{=} (B(t))_{t\geq 0}.$$

Now we assume $D > 1$ (not only integers). Consider BES_D starting from $x \geq 0$ as the solution of the following stochastic differential equaiton (SDE),

$$dR^x(t) = dB^x(t) + \frac{D-1}{2}\frac{dt}{R^x(t)}, \quad 0 \leq t < T^x,$$

where $T^x = \inf\{t > 0 : R^x(t) = 0\}$.

The first term in the RHS is a martingale part
and the second term is a finite variation part (drift term).
The process consisting of these two parts is called a semi-martingale.

Dyson's BM model

- **Fix** $N \in \mathbb{N}$.
- **Prepare** N **BMs,**

$$B_{ii}^{x_i}(t) = x_i + B_{ii}(t), \quad t \geq 0, \quad x_i \in \mathbb{R}, \quad i = 1, \ldots, N,$$

and $N(N-1)/2$ **pairs of BMs**

$$\{B_{ij}(t), \widetilde{B}_{ij}(t)\}_{1 \leq i < j \leq N}, \quad t \geq 0, \quad \textbf{starting from the origin.}$$

- **Then, we have a total of** $N + 2 \times N(N-1)/2 = N^2$ **BMs, each of them independent from the rest.**
- **Then consider an** $N \times N$ **Hermitian-matrix-valued Brownian motion,**

$$\mathsf{B}^{\boldsymbol{x}}(t) = \begin{pmatrix} B_{11}^{x_1}(t) & \frac{B_{12}(t) + \sqrt{-1}\widetilde{B}_{12}(t)}{\sqrt{2}} & \cdots & \frac{B_{1N}(t) + \sqrt{-1}\widetilde{B}_{1N}(t)}{\sqrt{2}} \\ \frac{B_{12}(t) - \sqrt{-1}\widetilde{B}_{12}(t)}{\sqrt{2}} & B_{22}^{x_2}(t) & \cdots & \frac{B_{2N}(t) + \sqrt{-1}\widetilde{B}_{2N}(t)}{\sqrt{2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{B_{1N}(t) - \sqrt{-1}\widetilde{B}_{1N}(t)}{\sqrt{2}} & \frac{B_{2N}(t) - \sqrt{-1}\widetilde{B}_{2N}(t)}{\sqrt{2}} & \cdots & B_{NN}^{x_N}(t). \end{pmatrix}$$

- **By definition, the initial state of this Brownian motion is the diagonal matrix** $\mathsf{B}^{\boldsymbol{x}}(0) = \mathrm{diag}(x_1, x_2, \ldots, x_N)$. **We assume** $x_1 \leq x_2 \leq \cdots \leq x_N$.

- **Corresponding to calculating the absolute value of $B^{\boldsymbol{x}}(t)$, by which $\mathbf{BES}_D$ was introduced, here we calculate the eigenvalues of $\mathsf{B}^{\boldsymbol{x}}(t)$.**

- **For any $t \geq 0$, there is a family of $N \times N$ unitary-matrix-valued process, $\{\mathsf{U}(t)\}$ which diagonalizes $\mathsf{B}^{\boldsymbol{x}}(t)$,**

$$\mathsf{U}^*(t)\mathsf{B}^{\boldsymbol{x}}(t)\mathsf{U}(t) = \mathrm{diag}(\lambda_1(t), \ldots, \lambda_N(t)) =: \Lambda(t), \quad t \geq 0.$$

- **Consider a subspace of $\mathbb{R}^N$ defined by**

$$\mathbb{W}_N := \{\boldsymbol{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\},$$

which is called the Weyl chamber. If we impose the condition $(\lambda_i(t))_{i=1}^N \in \mathbb{W}_N$, $\mathsf{U}(t)$ is uniquely determined at each time $t \geq 0$.

Itô's formula

If $\mathsf{X}(t) = (X_{ij}(t))$ and $\mathsf{Y}(t) = (Y_{ij}(t))$ are $N \times N$ matrices with semi-martingale elements, then

$$d(\mathsf{X}^*(t)\mathsf{Y}(t)) = d\mathsf{X}^*(t)\mathsf{Y}(t) + \mathsf{X}^*(t)d\mathsf{Y}(t) + \langle d\mathsf{X}^*, d\mathsf{Y}\rangle_t, \quad t \geq 0,$$

where $\langle d\mathsf{X}^*, d\mathsf{Y}\rangle_t$ denotes an $N \times N$ matrix-valued process, whose (i,j)-th element is given by the finite-variation process $\sum_{k=1}^{N} \langle d\overline{X_{ki}}, dY_{kj}\rangle_t$, $1 \leq i, j \leq N$.

Applying Itô's formula to $\Lambda(t) = \mathsf{U}^*(t)B^{\boldsymbol{x}}(t)\mathsf{U}(t), t \geq 0$, we have the equality

$$d\Lambda(t) = d\mathsf{U}^*(t)\mathsf{B}^{\boldsymbol{x}}(t)\mathsf{U}(t) + \mathsf{U}^*(t)d\mathsf{B}^{\boldsymbol{x}}(t)\mathsf{U}(t) + \mathsf{U}^*(t)\mathsf{B}^{\boldsymbol{x}}(t)d\mathsf{U}(t)$$
$$+ \ \langle d\mathsf{U}^*, d\mathsf{B}^{\boldsymbol{x}}\mathsf{U}\rangle_t + \langle d\mathsf{U}^*, \mathsf{B}^{\boldsymbol{x}}d\mathsf{U}\rangle_t + \langle \mathsf{U}^*d\mathsf{B}^{\boldsymbol{x}}, d\mathsf{U}\rangle_t, \quad t \geq 0.$$

Since $\mathsf{U}^*(t)\mathsf{U}(t) = \mathsf{I}$ for each time t, Itô's formula gives,

$$0 = d(\mathsf{U}^*(t)\mathsf{U}(t)) = d\mathsf{U}^*(t)\mathsf{U}(t) + \mathsf{U}^*(t)d\mathsf{U}(t) + \langle d\mathsf{U}^*, d\mathsf{U}\rangle_t.$$

The following lemma is established.

Lemma 1.1 The eigenvalue process $(\lambda_i(t))_{i=1}^N$, $t \geq 0$ of the Hermitian-matrix-valued Brownian motion $(\mathsf{B}(t))_{t\geq 0}$ starting from $\mathrm{diag}(x_1, \ldots, x_N)$, satisfies the SDEs,

$$d\lambda_i(t) = dB_i^{x_i}(t) + \sum_{\substack{1\leq j\leq N,\\ j\neq i}} \frac{dt}{\lambda_i(t) - \lambda_j(t)}, \quad t \geq 0, \quad i = 1, \ldots, N,$$

where $(B_i^{x_i}(t))_{i=1}^N$, $t \geq 0$ are independent BMs different from the N^2 BMs used to define $\mathsf{B}^{\boldsymbol{x}}(t)$ above.

We consider the following one-parameter ($\beta > 0$) extension of the above eigenvalue-process.

Define an interacting Brownian motions, $\boldsymbol{X}(t) = (X_1(t), X_2(t), \ldots, X_N(t))$, $t \geq 0$ as a solution of the following system of SDEs, with the condition $x_1 < x_2 < \cdots < x_N$ for initial positions $x_i = X_i(0), 1 \leq i \leq N$,

$$dX_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{\substack{1 \leq j \leq N, \\ j \neq i}} \frac{dt}{X_i(t) - X_j(t)}, \quad t \in [0, T^{\boldsymbol{x}}), \quad 1 \leq i \leq N,$$

where $\{B_i(t)\}_{i=1}^N$, $t \geq 0$ are independent BMs and

$$\begin{aligned} T_{ij}^{\boldsymbol{x}} &= \inf\{t > 0 : X_i(t) = X_j(t)\}, \quad 1 \leq i < j \leq N, \\ T^{\boldsymbol{x}} &= \min_{1 \leq i < j \leq N} T_{ij}^{\boldsymbol{x}}. \end{aligned}$$

It is called Dyson's Brownian motion model with parameter β with N particles, abbreviated as DYS_β^N here.

Let $X(t), t \geq 0$ be a one-dimensional semi-martingale, which satisfies the SDE

$$dX(t) = \sigma(X(t))dB(t) + b(X(t))dt.$$

Then its transition probability density $p_X(t-s, y|x), 0 \leq s \leq t$ satisfies the partial differential equation (PDE) called the backward Kolmogorov equation,

$$\frac{\partial}{\partial s} p_X(t-s, y|x) = \left[\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}\right] p_X(t-s, y|x) \quad \textbf{with } a(x) = \sigma(x)^2$$

under the initial condition $\lim_{s\uparrow t} p_X(t-s, y|x) = \delta(x-y)$.
This fact is proved using again Itô's formula.

SDE for BES$_D$ $R^x(t) = dB(t) + \dfrac{D-1}{2}\dfrac{dt}{R^x(t)}dt, \quad t \geq 0.$

PDE for BES$_D$ $\dfrac{\partial}{\partial t}p_{\mathrm{BES}_D}(t,y|x) = \left[\dfrac{1}{2}\dfrac{\partial^2}{\partial x^2} + \dfrac{D-1}{2x}\dfrac{\partial}{\partial x}\right] p_{\mathrm{BES}_D}(t,y|x), \quad x,y>0, \quad t \geq 0.$

The latter is solved under $p_{\mathrm{BES}_D}(0,y|x) = \delta(y-x)$ as

$$p_{\mathrm{BES}_D}(t,y|x) = \begin{cases} \dfrac{1}{t}\dfrac{y^{\nu+1}}{x^\nu}e^{-(x^2+y^2)/2t}I_\nu\left(\dfrac{xy}{t}\right), & t>0,\ x>0,\ y\geq 0, \\ \dfrac{y^{2\nu+1}}{2^\nu t^{\nu+1}\Gamma(\nu+1)}e^{-y^2/2t}, & t>0,\ x=0,\ y\geq 0 \\ \\ \delta(y-x), & t=0,\ x,y\geq 0, \end{cases}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind defined by

$$I_\nu(z) = \sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)\Gamma(n+1+\nu)}\left(\frac{z}{2}\right)^{2n+\nu} \quad \textbf{with } \nu = \frac{D-2}{2} \iff D = 2(\nu+1)$$

with the gamma function $\Gamma(z) = \displaystyle\int_0^\infty e^{-u}u^{z-1}du,\ \mathrm{Re}\, z > 0.$

When

$$D = 3 \iff \nu = \frac{1}{2},$$

we can use the equality

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z = \frac{e^z - e^{-z}}{\sqrt{2\pi z}}.$$

Then

$$p_{\mathrm{BES}_3} = \frac{y}{x}\Big\{p(t, y|x) - p(t, y| - x)\Big\}, \quad x > 0, \quad y \geq 0, \quad t > 0,$$

where $p(t, y|x)$ is the transition probability density of BM started at x.

SDEs for DYS_β^N

$$dX_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{\substack{1 \le j \le N, \\ j \ne i}} \frac{dt}{X_i(t) - X_j(t)}, \qquad t \ge 0, \qquad i = 1, \dots, N.$$

PDEs for DYS_β^N

$$\frac{\partial}{\partial t} p_{\mathrm{DYS}_\beta^N}(t, \boldsymbol{y}|\boldsymbol{x}) = \left[\frac{1}{2} \Delta^{(N)} + \frac{\beta}{2} \sum_{\substack{1 \le j \le N, \\ j \ne i}} \frac{1}{x_i - x_j} \frac{\partial}{\partial x_i} \right] p_{\mathrm{DYS}_\beta^N}(y, \boldsymbol{y}|\boldsymbol{x}), \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N,$$

where $\Delta^{(N)} := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$.

SDEs for DYS_β^N

$$dX_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{\substack{1 \le j \le N, \\ j \neq i}} \frac{dt}{X_i(t) - X_j(t)}, \quad t \ge 0, \quad i = 1, \ldots, N.$$

PDE for DYS_β^N

$$\frac{\partial}{\partial t} p_{\mathrm{DYS}_\beta^N}(t, \boldsymbol{y}|\boldsymbol{x}) = \left[\frac{1}{2} \Delta^{(N)} + \frac{\beta}{2} \sum_{\substack{1 \le j \le N, \\ j \neq i}} \frac{1}{x_i - x_j} \frac{\partial}{\partial x_i} \right] p_{\mathrm{DYS}_\beta^N}(t, \boldsymbol{y}|\boldsymbol{x}), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N, \quad t \ge 0,$$

where $\Delta^{(N)} := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$.

Consider the case $\beta = 2$, in which

$$\mathrm{DYS}_2^N \overset{(\mathrm{law})}{=} \text{eigenvalue process of Hermitian-matrix-valued BM.}$$

PDE for DYS_2^N

$$\frac{\partial}{\partial t} p_{\mathrm{DYS}_2^N}(t, \boldsymbol{y}|\boldsymbol{x}) = \left[\frac{1}{2}\Delta^{(N)} + \sum_{\substack{1 \le j \le N, \\ j \ne i}} \frac{1}{x_i - x_j} \frac{\partial}{\partial x_i} \right] p_{\mathrm{DYS}_2^N}(t, \boldsymbol{y}|\boldsymbol{x}), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N, \quad t \ge 0,$$

Under the initial condition $p_{\mathrm{DYS}_2^N}(0, \boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^N \delta(x_i - y_i)$, It is solved as

$$p_{\mathrm{DYS}_2^N}(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} \det_{1 \le i,j \le N} [p(t, y_i|x_j)], \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N, \quad t \ge 0,$$

where

$$h_N(\boldsymbol{x}) := \prod_{1 \le i < j \le N} (x_j - x_i) = \det_{1 \le i,j \le N} [x_i^{j-1}] \quad \textbf{(Vandermonde determinant)},$$

and $p(t, y|x)$ is the transition probability density of BM started at x.

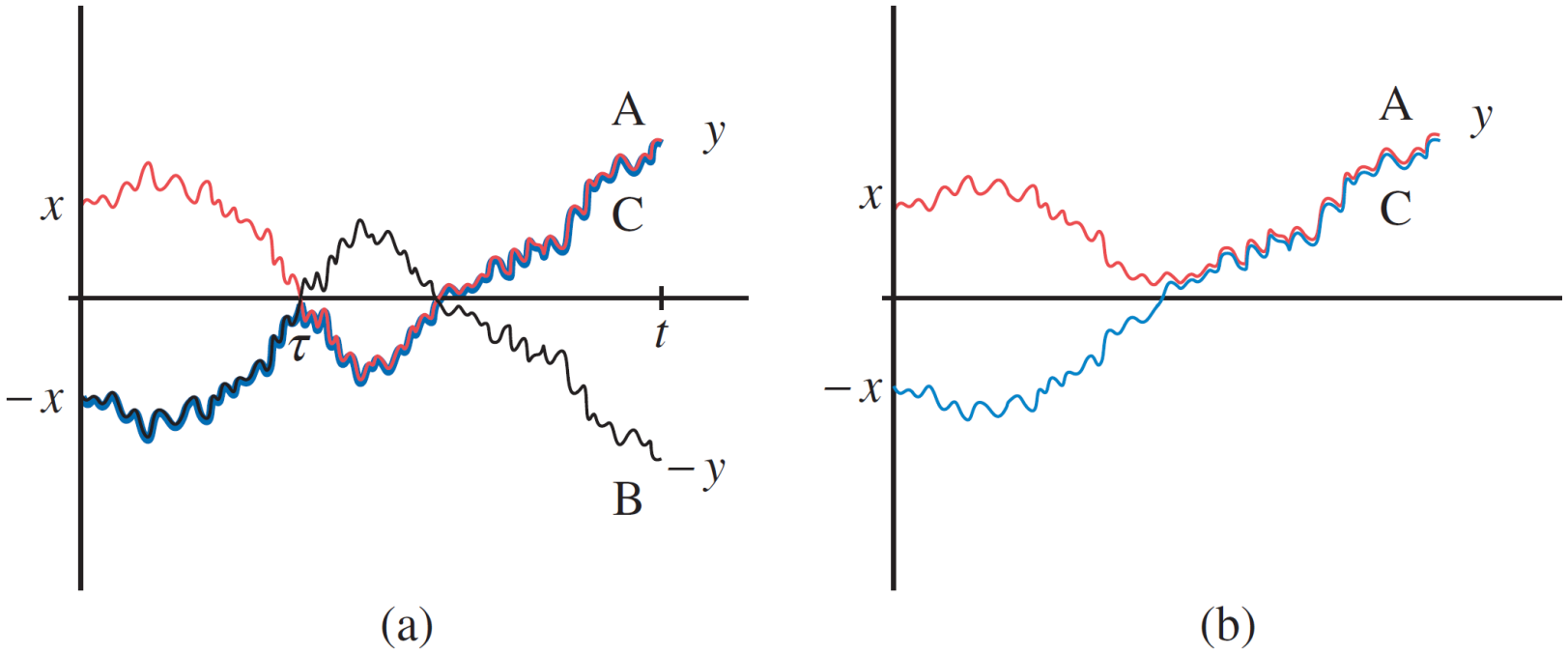
$$p_{\mathrm{BES}_3} = \frac{y}{x} q_{\mathbb{R}_+}(t, y|x), \quad x > 0, \quad y \geq 0, \quad t > 0,$$

where, $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ **and**

$$q_{\mathbb{R}_+}(t, y|x) := p(t, y|x) - p(t, y| - x)$$

= transition probability density

of BM staying in positive region $\mathbb{R}_+$.



$\mathbb{P}^x_{\mathrm{BES}_3}$:= **prob. measure of BES$_3$ starting from** $x \geq 0$

P^x := **prob. measure of BM starting from** $x \geq 0$

F := $\mathcal{F}_t$**-measurable bounded function**, $0 \leq t < \infty$,

$$\mathbb{E}^x_{\mathrm{BES}_3}[F(R(t))] = \mathrm{E}^x\left[F(B(t))\mathbf{1}_{(\tau>t)}\frac{B(t)}{x}\right],$$

where $\tau := \inf\{t > 0 : B(t) = 0\}$.

If F **is even;** $F(-x) = F(x)$**, then**

$$\mathbb{E}^x_{\mathrm{BES}_3}[F(R(t))] = \mathrm{E}^x\left[F(B(t))\frac{B(t)}{x}\right], \quad 0 \leq t < \infty.$$

From now on, we regard DYS_β as a time-dependent point process. The configuration space of point processes is given by

$$\mathrm{Conf}(\mathbb{R}) := \left\{ \xi = \sum_i \delta_{x_i} : x_i \in \mathbb{R}, \xi(K) < \infty \text{ for all bounded set } K \subset \mathbb{R} \right\}.$$

The set of configurations of simple point processes is denoted by

$$\mathrm{Conf}_0(\mathbb{R}) := \{\xi \in \mathrm{Conf}(\mathbb{R}) : \xi(\{x\}) \leq 1, {}^\forall x \in \mathbb{R}\}.$$

There is a trivial correspondence between $\mathbb{W}_N$ and $\mathrm{Conf}_0(\mathbb{R})$. DYS_2 is described as an $\mathrm{Conf}_0(\mathbb{R})$-valued process,

$$\Xi(t, \cdot) = \sum_{i=1}^N \delta_{X_i(t)}(\cdot), \quad t \geq 0,$$

starting from the initial configuration $\xi = \sum_{i=1}^N \delta_{x_i}$. We consider a natural filtration associated with $(\Xi(t))_{t\geq 0}$; $(\mathcal{F}_\Xi)_t := \sigma(\Xi(s); s \in [0, t])$, $t \geq 0$.

$$p_{\mathrm{DYS}_2^N}(t,\boldsymbol{y}|\boldsymbol{x}) = \frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} q_{\mathbb{W}_N}(t,\boldsymbol{y}|\boldsymbol{x}), \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{W}_N, \quad t \geq 0,$$

$$q_{\mathbb{W}_B}(t,\boldsymbol{y}|\boldsymbol{x}) := \det_{1\leq i,j\leq N}[p(t,y_i|x_j)]$$

= transition probability density of N BMs staying in the Weyl chamber $\mathbb{W}_N$

$\mathbb{P}^{\xi}_{\mathrm{DYS}_2^N}$:= **prob. measure of DYS_2^N starting from $x \geq 0$**

$\mathrm{P}^{\boldsymbol{x}}$:= **prob. measure of BM starting from $\boldsymbol{x} \in \mathbb{W}_N$**

F := **$(\mathcal{F}_\Xi)_t$-measurable bounded function,** $0 \leq t < T < \infty$,

$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}[F(\Xi(\cdot))] = \mathrm{E}^{\boldsymbol{x}}\left[F\left(\sum_{i=1}^N \delta_{B_i(\cdot)}\right) \mathbf{1}_{(\tau>T)} \frac{h_N(\boldsymbol{B}(T))}{h_N(\boldsymbol{x})}\right],$$

where $\tau := \inf\{t > 0 : \boldsymbol{B}^{\boldsymbol{x}}(T) \notin \mathbb{W}_N\}$.

Since $\Xi(\cdot)$ is invariant under exchange of particles,

$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}[F(\Xi(\cdot))] = \mathrm{E}^{\boldsymbol{x}}\left[F\left(\sum_{i=1}^N \delta_{B_i(\cdot)}\right) \frac{h_N(\boldsymbol{B}(T))}{h_N(\boldsymbol{x})}\right].$$

Remark 1.1 **Consider BES_3. For even function F,**

$$\mathbb{E}^x_{\mathrm{BES}_3}[F(R(t))] = \mathrm{E}^x\left[F(B(t))\frac{B(t)}{x}\right]$$
$$= \mathrm{E}^x\left[F(B(t))\frac{h_1(B(t))}{h_1(x)}\right], \quad h_1(x) := x.$$
$$\mathbb{E}^\xi_{\mathrm{DYS}^N_2}[F(\Xi(\cdot))] = \mathrm{E}^{\boldsymbol{x}}\left[F\left(\sum_{i=1}^N \delta_{B_i(\cdot)}\right)\frac{h_N(\boldsymbol{B}(T))}{h_N(\boldsymbol{x})}\right].$$

$$h_1(x) = x \in \mathbb{R}_+ \iff \Delta^{(1)}h_1(x) := \frac{d^2}{dx^2}h_1(x) = 0, \quad h_1(0) = 0$$

$$h_N(\boldsymbol{x}) = \prod_{1\le j<k\le N}(x_k - x_j) \in \mathbb{W}_N \iff \Delta^{(N)}h_N(\boldsymbol{x}) := \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}h_N(\boldsymbol{x}) = 0, \quad h_N(\boldsymbol{x}) = 0, \boldsymbol{x} \in \partial\mathbb{W}_N.$$

$$h_1(x), h_N(\boldsymbol{x}) : \textbf{harmonic functions}$$

harmonic transformations

Remark 1.1 **Consider BES_3. For even function F,**

$$\mathbb{E}^x_{\mathrm{BES}_3}[F(R(t))] = \mathrm{E}^x\left[F(B(t))\frac{B(t)}{x}\right]$$

$$= \mathrm{E}^x\left[F(B(t))\frac{h_1(B(t))}{h_1(x)}\right], \quad h_1(x) := x.$$

$$\mathbb{E}^\xi_{\mathrm{DYS}^N_2}[F(\Xi(\cdot))] = \mathrm{E}^{\boldsymbol{x}}\left[F\left(\sum_{i=1}^N \delta_{B_i(\cdot)}\right)\frac{h_N(\boldsymbol{B}(T))}{h_N(\boldsymbol{x})}\right].$$

$$h_1(x) = x \in \mathbb{R}_+ \iff \Delta^{(1)}h_1(x) := \frac{d^2}{dx^2}h_1(x) = 0, \quad h_1(0) = 0$$

$$h_N(\boldsymbol{x}) = \prod_{1\le j<k\le N}(x_k - x_j) \in \mathbb{W}_N \iff \Delta^{(N)}h_N(\boldsymbol{x}) := \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}h_N(\boldsymbol{x}) = 0, \quad h_N(\boldsymbol{x}) = 0, \boldsymbol{x} \in \partial\mathbb{W}_N.$$

$h_1(x), h_N(\boldsymbol{x})$: **harmonic functions**

From h-transform to Determinantal Martingale Representation (DMR)

- **Assume that** $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{W}_N \iff \xi := \sum_{i=1}^N \delta_{x_i} \in \mathrm{Conf}_0(\mathbb{R})$**. Define**

$$\Phi_\xi^u(z) := \prod_{\substack{1 \le k \le N, \\ x_k \neq u}} \frac{z - x_k}{u - x_k}, \qquad z, u \in \mathbb{C}.$$

- **For** $(t, x) \in [0, \infty) \times \mathbb{R}$**, consider the function**

$$\widehat{G}_w(t, x) := \frac{e^{-(\sqrt{-1}x + w)^2/2t}}{\sqrt{2\pi t}}, \qquad w \in \mathbb{R},$$

and introduce an integral transformation

$$\mathcal{I}[f(W)|(t, x)] := \int_{-\infty}^{\infty} f(\sqrt{-1}w) \widehat{G}_w(t, x) dw,$$

where W **in the LHS is a dummy variable.**

- **Define**

$$\mathcal{M}_\xi^x(t, y) := \mathcal{I}[\Phi_\xi^x(W)|(t, y)], \qquad x, y \in \mathbb{R}, \qquad t \ge 0.$$

In summary,

$$\mathcal{M}_\xi^x(t,y) = \mathcal{I}[\Phi_\xi^x(W)|(t,y)] = \int_{-\infty}^{\infty} \Phi_\xi^x(w)\widehat{G}_w(t,y)dw$$
$$= \int_{-\infty}^{\infty} \Phi_\xi^x(\sqrt{-1}w)\frac{e^{-(\sqrt{-1}y+w)^2/2t}}{\sqrt{2\pi t}}dw = \int_{-\infty}^{\infty} \prod_{\substack{1\le k\le N,\\ x_k\neq u}} \frac{\sqrt{-1}w - x_k}{u - x_k}\frac{e^{-(\sqrt{-1}y+w)^2/2t}}{\sqrt{2\pi t}}dw.$$

We can verify the following identity; for $\xi = \sum_{i=1}^N \delta_{x_i}, \boldsymbol{y} \in \mathbb{W}_N, t \ge 0$**,**

$$\frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} = \det_{1\le i,j\le N}[\mathcal{M}_\xi^{x_i}(t,y_j)] =: \mathcal{D}_\xi(t,\boldsymbol{y}).$$

The following are also proved.

Proposition 1.2 Assume that $\xi = \sum_{i=1}^N \delta_{x_i} \in \mathrm{Conf}_0(\mathbb{R})$**.**

(i) $(\mathcal{M}_\xi^{x_i}(t,B(t)))_{t\ge 0}, i = 1,\dots N$ **are continuous martingales.**

(ii) **For any time** $t \ge 0$**,** $\mathcal{M}_\xi^{x_i}(t,y), i = 1,\dots,N$ **are linearly independent functions of** y**.**

(iii) **For** $1 \le i,j \le N$**,** $\lim_{t\downarrow 0} \mathrm{E}^{x_i}[\mathcal{M}_\xi^{x_j}(t,B(t))] = \delta_{ij}$**.**

Then for any $(\mathcal{F}_\Xi)_t$-measurable bounded function F, $0 \le t \le T < \infty$, the equality

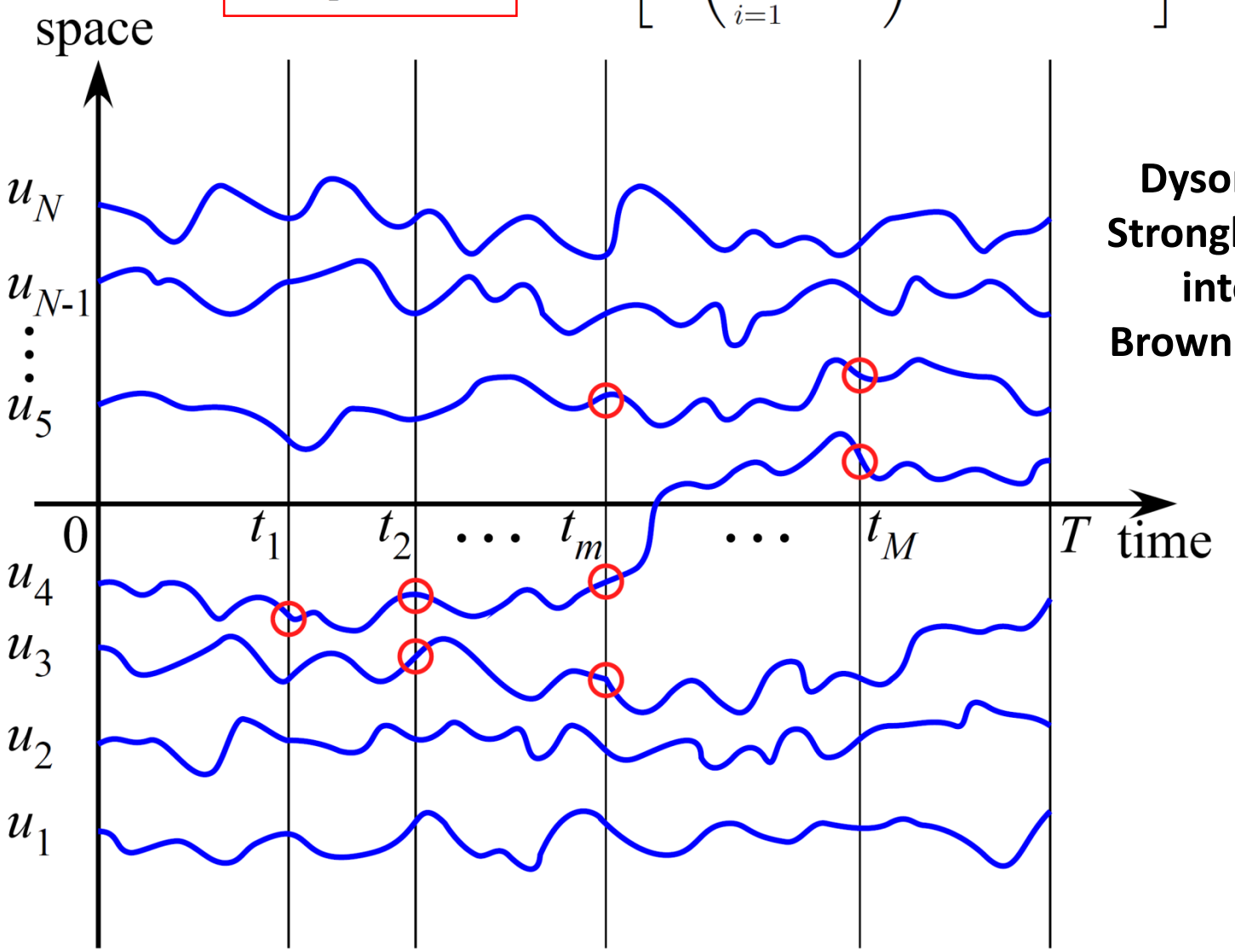
$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}[F(\Xi(\cdot))] = \mathrm{E}^{\boldsymbol{x}}\left[F\left(\sum_{i=1}^{N}\delta_{B_i(\cdot)}\right)\mathcal{D}_\xi(T, \boldsymbol{B}(T))\right]$$

holds, where

$$\mathcal{D}_\xi(t, \boldsymbol{y}) = \det_{1\le i,j\le N}[\mathcal{M}^{y_j}_\xi(t, y_i)], \quad \boldsymbol{y} = (y_1, \dots, y_N) \in \mathbb{W}_N, \quad t \ge 0.$$

We call $\mathcal{D}_\xi(t, \boldsymbol{B}(t)),\ t \ge 0$ a determinantal martingale and the above equality the determinantal martingale representation (DMR) of $\mathbf{DYS}_2^N$.

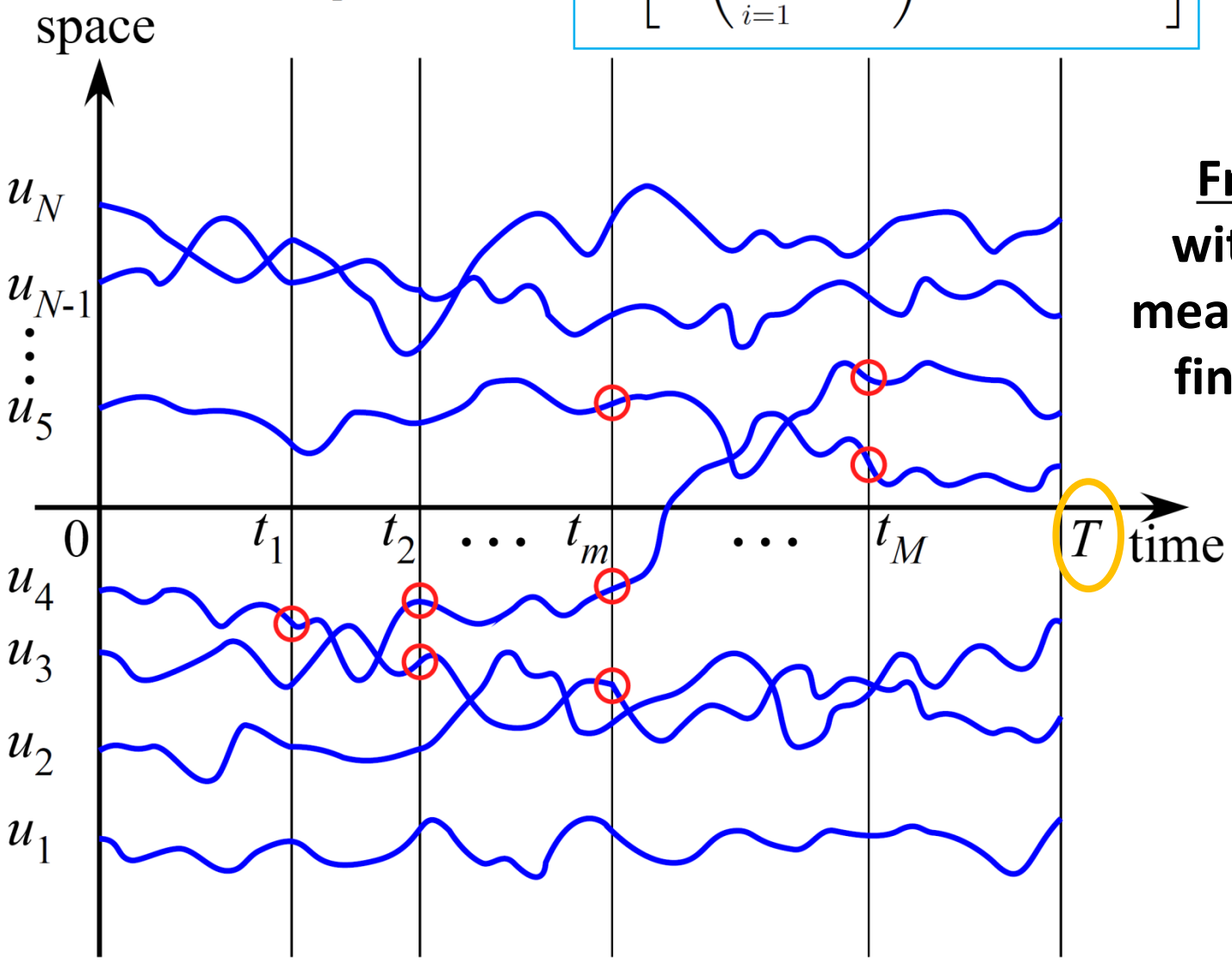
$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}[F(\Xi(\cdot))] = \mathrm{E}^{\boldsymbol{x}}\left[F\left(\sum_{i=1}^{N}\delta_{B_i(\cdot)}\right)\mathcal{D}_{\xi}(T,\boldsymbol{B}(T))\right]$$



Dyson's model: Strongly repulsive-interacting Brownian motions

We observe particles at points ◯ on the spatio-temporal plane.

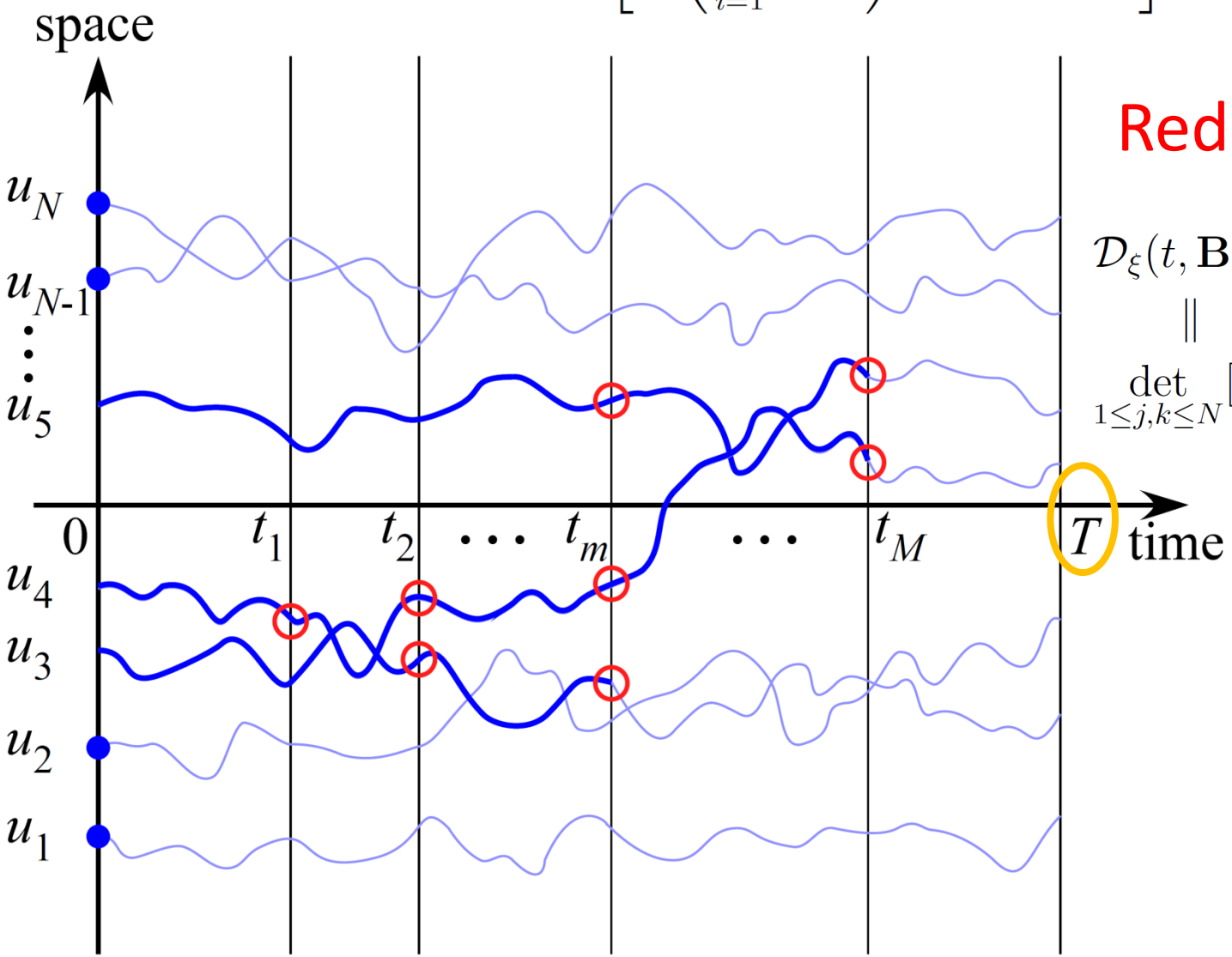
$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}[F(\Xi(\cdot))] = \mathrm{E}^{\boldsymbol{x}}\left[F\left(\sum_{i=1}^{N}\delta_{B_i(\cdot)}\right)\mathcal{D}_{\xi}(T,\boldsymbol{B}(T))\right]$$



Free BMs with signed measure at the final time *T*

We observe particles at points ◯ on the spatio-temporal plane.

$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}[F(\Xi(\cdot))] = \mathrm{E}^{\boldsymbol{x}}\left[F\left(\sum_{i=1}^{N}\delta_{B_i(\cdot)}\right)\mathcal{D}_{\xi}(T,\boldsymbol{B}(T))\right]$$



Reducibility

$$\mathcal{D}_{\xi}(t,\mathbf{B}(t)) \;\|\; \det_{1\le j,k\le N}[\mathcal{M}_{\xi}^{u_k}(t,B_j(t))]$$

We observe particles at points ◯ on the spatio-temporal plane.

From Determinantal Martingale Representation (DMR) to Stochastic Determinanal Process (SDP)

time-dependent density function

- The density function at a single time for $(\Xi, \mathbb{P}^{\xi}_{\mathrm{DYS}_2^N}), \xi \in \mathrm{Conf}_0(\mathbb{R})$ is denoted by $\rho_\xi(t,x)$.

- It is defined as a continuous function of $x \in \mathbb{R}$ for $0 \le t \le T < \infty$ such that for any test function, $\chi \in \mathrm{C_c}(\mathbb{R})$,

$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}\left[\int_{\mathbb{R}} \chi(x)\Xi(t,dx)\right] = \int_{\mathbb{R}} dx\, \chi(x)\rho_\xi(t,x).$$

- The test function χ is symmetrized as $g(\boldsymbol{x}) = \sum_{i=1}^{N} \chi(x_i)$, which is applied as F to our DMR, and we obtain the equality

$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}\left[\sum_{i=1}^{N} \chi(X_i(t))\right] = \mathrm{E}^{\boldsymbol{x}}\left[\sum_{i=1}^{N} \chi(B_i(t))\mathcal{D}_\xi(T, \boldsymbol{B}(T))\right], \quad 0 \le t \le T < \infty.$$

- **The LHS gives**

$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}\left[\sum_{i=1}^{N}\chi(X_i(t))\right]=\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}\left[\int_{\mathbb{R}}\chi(x)\Xi(t,dx)\right]$$

  by the definition $\Xi(t,\cdot):=\sum_{i=1}^{N}\delta_{X_i(t)}(\cdot)$.

- **On the other hand, RHS is reduced as**

$$\begin{aligned}\sum_{i=1}^{N}\mathrm{E}^{\boldsymbol{x}}[\chi(B_i(t))\mathcal{D}_\xi(T,\boldsymbol{B}(T))]&=\int_{\mathbb{R}}\xi(dv)\mathrm{E}^{v}[\chi(B(t))\mathcal{M}^{v}_{\xi}(t,B(t))]\\&=\int_{\mathbb{R}}\xi(dv)\int_{\mathbb{R}}dx\,\chi(x)p(t,x|v)\mathcal{M}^{v}_{\xi}(t,x).\end{aligned}$$

- **By Fubini's theorem, we can rewrite it as**

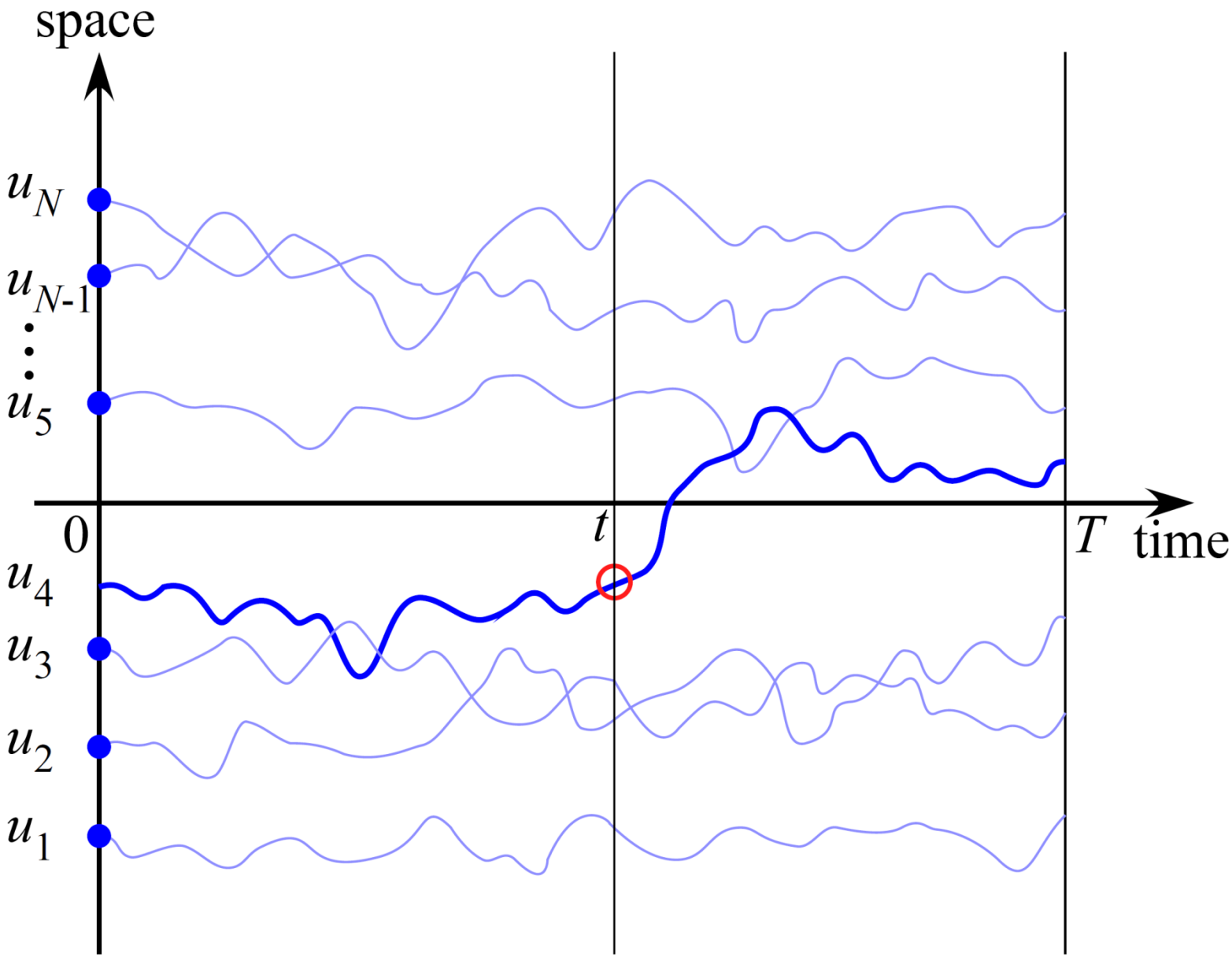
$$\int_{\mathbb{R}}dx\,\chi(x)\mathcal{G}_\xi(t,x;t,x),$$

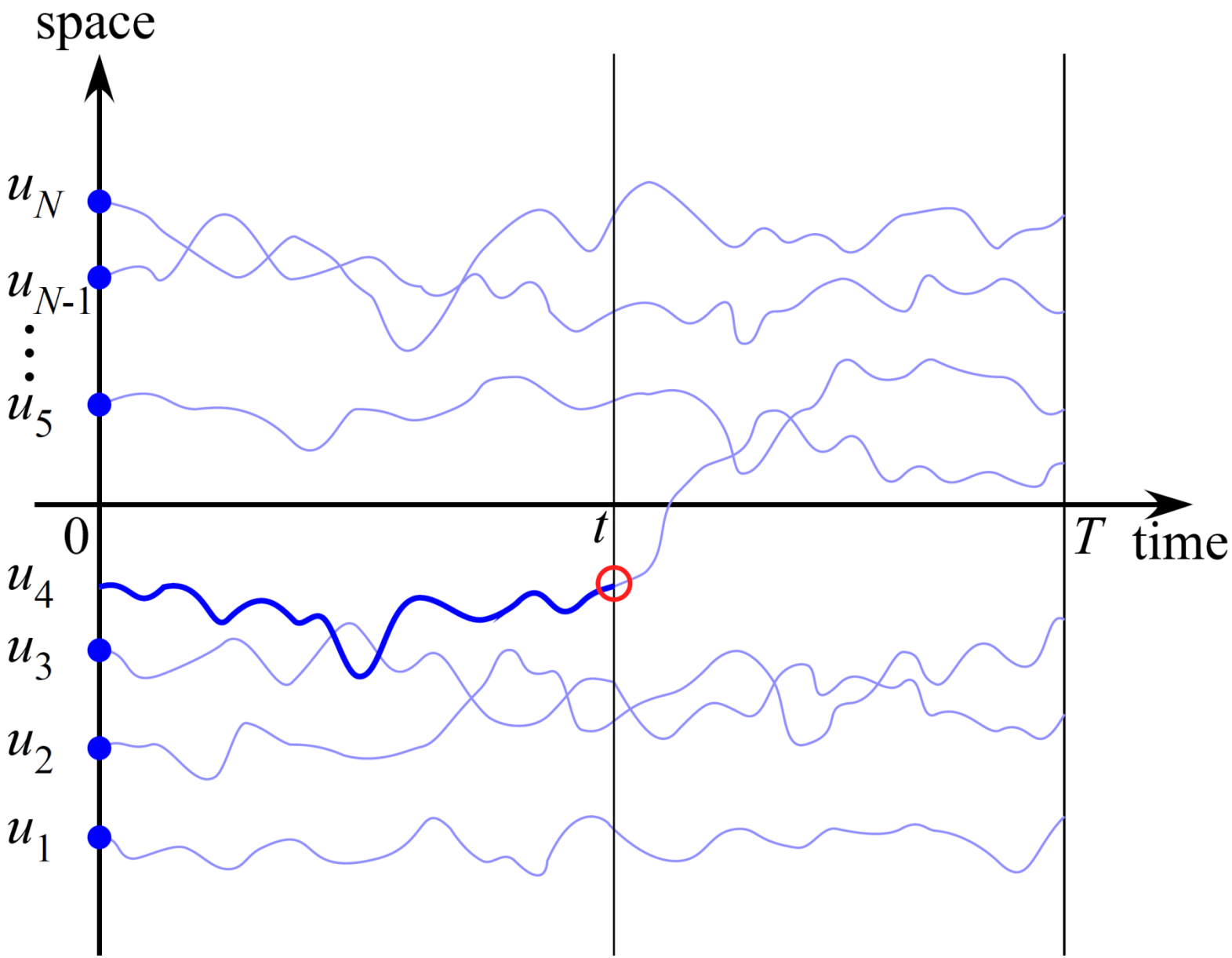
  where

$$\boxed{\mathcal{G}_\xi(s,x;t,y)=\int_{\mathbb{R}}\xi(dv)p(s,x|v)\mathcal{M}^{v}_{\xi}(t,y).}$$

  Then the time-dependent density function is determined as

$$\rho_\xi(t,x)=\mathcal{G}_\xi(t,x;t,x),\quad x\in\mathbb{R},\quad t\geq 0.$$


space
u_N
u_{N-1}
u_5
0
t
T
time
u_4
u_3
u_2
u_1


space
u_N
u_{N-1}
u_5
0
t
T
time
u_4
u_3
u_2
u_1

two-time correlation function

- **For $0 \leq t_1 < t_2 \leq T < \infty$, set the symmetrized test functions,**

$$g_1(\boldsymbol{x}) = \sum_{i=1}^{N} \chi_1(x_i), \quad g_2(\boldsymbol{x}) = \sum_{i=1}^{N} \chi_2(x_i),$$

where $\chi_m \in \mathrm{C_c}(\mathbb{R}), m = 1, 2$, and put

$$F(\Xi(\cdot)) = \prod_{m=1}^{2} g_m(\boldsymbol{X}(t_m)).$$

- **If we apply this to DMR, we obtain the equality**

$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N} \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \chi_1(X_i(t_1)) \chi_2(X_j(t_2)) \right]$$

$$= \mathrm{E}^{\boldsymbol{x}} \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \chi_1(B_i(t_1)) \chi_2(B_j(t_2)) \mathcal{D}_{\xi}(T, \boldsymbol{B}(T)) \right], \ 0 \leq t \leq T < \infty.$$

- **The LHS defines the two-time correlation function $\rho_\xi(s,x;t,y)$ as**

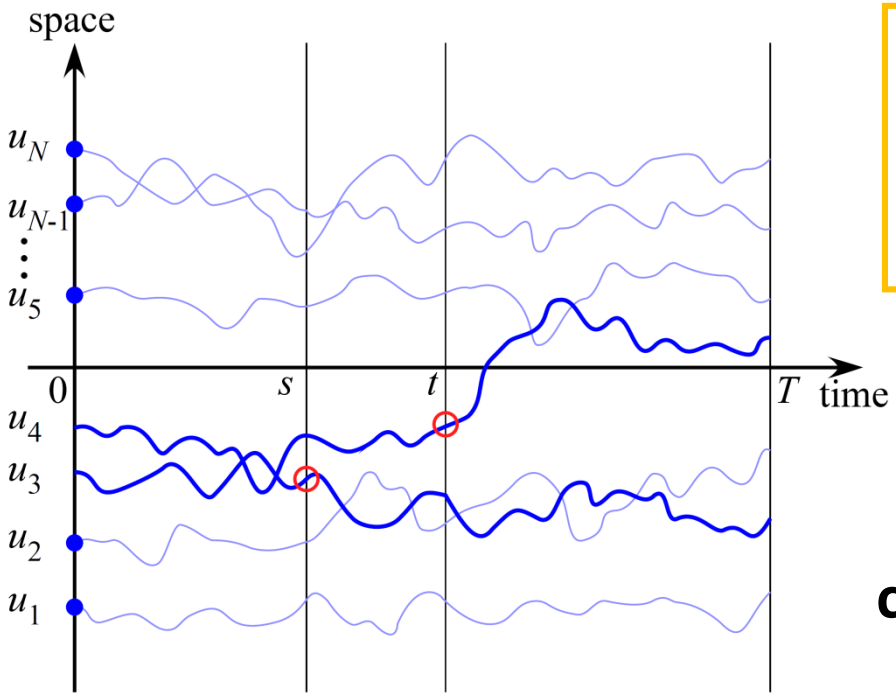
$$\mathbb{E}^{\xi}_{\mathrm{DYS}_2^N}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\chi_1(X_i(t_1))\chi_2(X_j(t_2))\right]=\int_{\mathbb{R}^2}dx_1dx_2\,\chi_1(x_1)\chi_2(x_2)\rho_\xi(t_1,x_1;t_2,x_2).$$

- **On the other hand, RHS gives**

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{E}^{\boldsymbol{x}}[\chi_1(B_i(t_1))\chi_2(B_j(t_2))\mathcal{D}_\xi(T,\boldsymbol{B}(T))]\\&\quad=\sum_{\substack{1\le i,j\le N,\\ i\neq j}}\mathrm{E}^{\boldsymbol{x}}[\chi_1(B_i(t_1))\chi_2(B_j(t_2))\mathcal{D}_\xi(T,\boldsymbol{B}(T))]\\&\quad+\sum_{1\le i\le N}\mathrm{E}^{\boldsymbol{x}}[\chi_1(B_i(t_1))\chi_2(B_i(t_2))\mathcal{D}_\xi(T,\boldsymbol{B}(T))].\end{aligned}$$

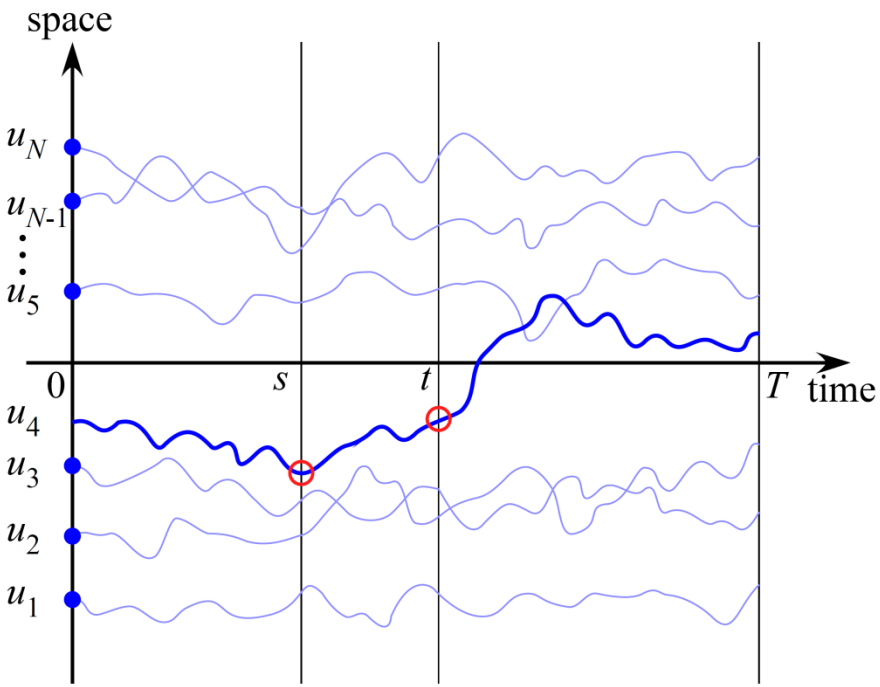
- **By the reducibility of DMR, the last expression becomes**

$$\begin{aligned}&\int_{\mathbb{R}^2}\xi^{\otimes 2}(d\boldsymbol{v})\\&\quad\times\mathrm{E}^{(v_1,v_2)}\left[\chi_1(B_1(t_1))\chi_2(B_2(t_2))\det\begin{pmatrix}\mathcal{M}^{v_1}_\xi(T,B_1(T)) & \mathcal{M}^{v_1}_\xi(T,B_2(T))\\ \mathcal{M}^{v_2}_\xi(T,B_1(T)) & \mathcal{M}^{v_2}_\xi(T,B_2(T))\end{pmatrix}\right]\\&+\int_{\mathbb{R}}\xi(dv)\mathrm{E}^{v}[\chi_1(B(t_1))\chi_2(B(t_2))\mathcal{M}^{v}_\xi(T,B(T))].\end{aligned}$$

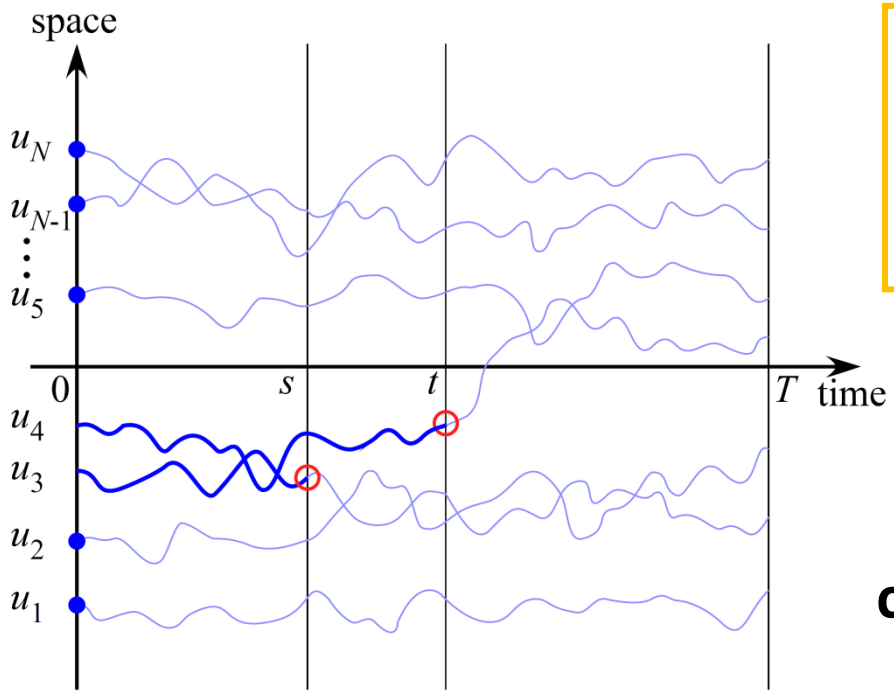


We observe at the TWO spatio-temporal points marked by ○.

The case that two paths contribute to the observation.

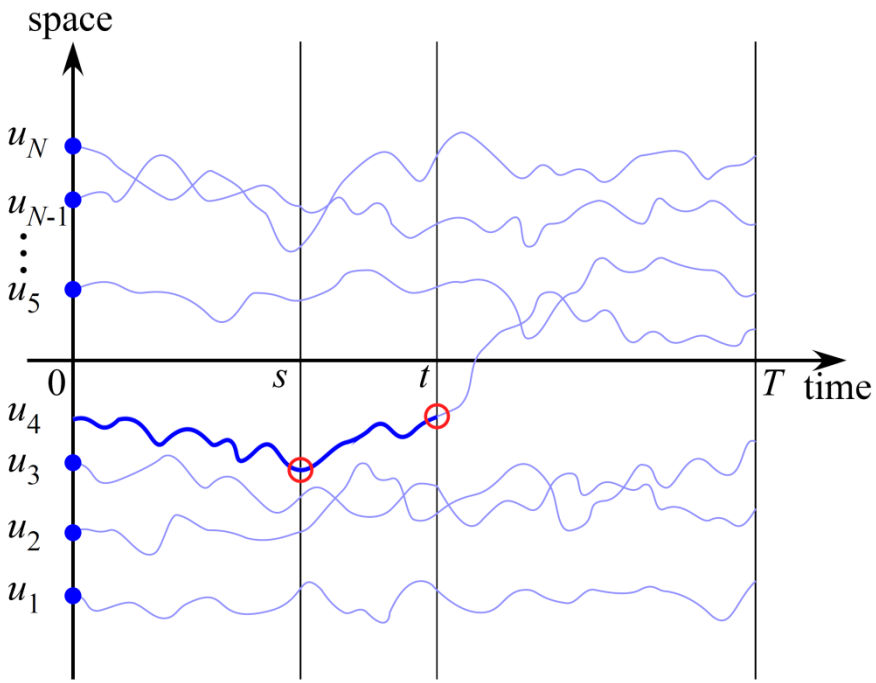


The case that only one path contributes to the observation.



We observe at the TWO spatio-temporal points marked by ◯.

The case that two paths contribute to the observation.



The case that only one path contributes to the observation.

- **If we use the martingale property (i) of $\mathcal{M}_\xi^v$ in Proposition 1.2, it is written as**

$$\int_{\mathbb{R}^2} \xi^{\otimes 2}(d\boldsymbol{v}) \times \mathrm{E}^{(v_1,v_2)}\left[\chi_1(B_1(t_1))\chi_2(B_2(t_2))\det\begin{pmatrix} \mathcal{M}_\xi^{v_1}(t_1,B_1(t_1)) & \mathcal{M}_\xi^{v_1}(t_2,B_2(t_2)) \\ \mathcal{M}_\xi^{v_2}(t_1,B_1(t_1)) & \mathcal{M}_\xi^{v_2}(t_2,B_2(t_2)) \end{pmatrix}\right]$$
$$+\int_{\mathbb{R}} \xi(dv)\mathrm{E}^v[\chi_1(B(t_1))\chi_2(B(t_2))\mathcal{M}_\xi^v(t_2,B(t_2))].$$

- **By Fubini's theorem, this is equal to**

$$\int_{\mathbb{R}^2} dx_1dx_2\,\chi_1(x_1)\chi_2(x_2)\det\begin{pmatrix} \mathcal{G}_\xi(t_1,x_1;t_1,x_1) & \mathcal{G}_\xi(t_1,x_1;t_2,x_2) \\ \mathcal{G}_\xi(t_2,x_2;t_1,x_1) & \mathcal{G}_\xi(t_2,x_2;t_2,x_2) \end{pmatrix}$$
$$+\int_{\mathbb{R}^2} dx_1dx_2\,\chi_1(x_1)\chi_2(x_2)\mathcal{G}_\xi(t_1,x_1;t_2,x_2)p(t_2-t_1,x_2|x_1)$$
$$= \int_{\mathbb{R}^2} dx_1dx_2\,\chi_1(x_1)\chi_2(x_2) \times \det\begin{pmatrix} \mathcal{G}_\xi(t_1,x_1;t_1,x_1) & \mathcal{G}_\xi(t_1,x_1;t_2,x_2) \\ \mathcal{G}_\xi(t_2,x_2;t_1,x_1)-p(t_2-t_1,x_2|x_1) & \mathcal{G}_\xi(t_2,x_2;t_2,x_2) \end{pmatrix}.$$

- **Since this is equal to**

$$\int_{\mathbb{R}^2} dx_1 dx_2 \, \chi_1(x_1)\chi_2(x_2)\rho_\xi(t_1, x_1; t_2, x_2),$$

the two-time correlation function is determined as

$$\rho_\xi(s, x; t, y) = \det \begin{pmatrix} \mathbb{K}_\xi(s, x; s, x) & \mathbb{K}_\xi(s, x; t, y) \\ \mathbb{K}_\xi(t, y; s, x) & \mathbb{K}_\xi(t, y; t, y) \end{pmatrix}$$

for $0 \leq s < t < \infty$, $x, y \in \mathbb{R}$, **where**

$$\mathbb{K}_\xi(s, x; t, y) = \mathcal{G}_\xi(s, x; t, y) - \mathbf{1}_{(s>t)} p(s - t, x|y).$$

The following is the main theorem in this topic.

Theorem 1.3 For any finite and fixed initial configuration without multiple points, that is, for $\xi \in \mathrm{Conf}_0(\mathbb{R}), \xi(\mathbb{R}) = N \in \mathbb{N}$, $\mathbf{DYS}_2^N$ is a determinantal stochastic process (DSP). Its spatio-temporal correlation kerne is given by

$$\mathbb{K}_\xi(s,x;t,y) = \mathcal{G}_\xi(s,x;t,y) - \mathbf{1}_{(s>t)}p(s-t,x|y), \qquad (s,x),(t,y) \in [0,\infty)\times\mathbb{R}$$

with

$$\mathcal{G}_\xi(s,x;t,y) = \int_{\mathbb{R}} \xi(dv)p(s,x|v)\mathcal{M}_\xi^v(t,y).$$

- General definition of the DSP =spatio-temporal DPP is given in Section 1.7 of Lecture Note.
- An extension to $\xi \in \mathrm{Conf}(\mathbb{R})$ is also mentioned in Lecture Note. The DPS with the extended Hermite kernel is given as an example of such a case $\xi = N\delta_0$.
- The infinite-particle limits, $N \to \infty$, can be also formulated by DMR.

Exercise 1.2

Let

$$h_N(\boldsymbol{x}) := \prod_{1\le i<j\le N}(x_j - x_i) = \det_{1\le i,j\le N}[x_j^{i-1}]. \tag{1.10}$$

(1) Using the basic properties of determinant, verify the equalities,

$$\begin{aligned}\frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} &= \frac{1}{h_N(\boldsymbol{x})}\det_{1\le i,j\le N}[y_j^{i-1}] \\ &= \frac{1}{h_N(\boldsymbol{x})}\det_{1\le i,j\le N}[m_{i-1}(t,y_j)],\end{aligned} \tag{1.11}$$

where $\{m_n(t,x)\}_{n\in\mathbb{N}_0}$ are given by (1.7) in Exercise 1.1.

(2) We extend the integral transformation $\mathcal{I}$ defined by (1.6) in Exercise 1.1 to a linear integral transformation of multivariate functions as follows: When $F^{(i)}(\boldsymbol{x}) = \prod_{j=1}^{N} f_j^{(i)}(x_j)$, $i = 1, 2$ are given for $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$, then we define

$$\mathcal{I}\left[F^{(i)}(\mathbf{W}) \,\middle|\, \{(t_\ell, x_\ell)\}_{\ell=1}^{N}\right] := \prod_{j=1}^{N} \mathcal{I}\left[f_j^{(i)}(W_j) \,\middle|\, (t_j, x_j)\right], \quad i = 1, 2, \tag{1.12}$$

and

$$\begin{aligned}
&\mathcal{I}\Big[c_1 F^{(1)}(\mathbf{W}) + c_2 F^{(2)}(\mathbf{W}) \,\Big|\, \{(t_\ell, x_\ell)\}_{\ell=1}^{N}\Big] \\
&\quad := c_1 \mathcal{I}\Big[F^{(1)}(\mathbf{W}) \,\Big|\, \{(t_\ell, x_\ell)\}_{\ell=1}^{N}\Big] + c_2 \mathcal{I}\Big[F^{(2)}(\mathbf{W}) \,\Big|\, \{(t_\ell, x_\ell)\}_{\ell=1}^{N}\Big],
\end{aligned} \tag{1.13}$$

$c_1, c_2 \in \mathbb{C}$, for $0 < t_i < \infty, i = 1, \ldots, N$, where $\mathbf{W} = (W_1, \ldots, W_N) \in \mathbb{R}^N$. In particular, if $t_\ell = t, 1 \leq {}^{\forall}\ell \leq N$, we write $\mathcal{I}[\,\cdot\,|\{(t_\ell, x_\ell)\}_{\ell=1}^{N}]$ simply as $\mathcal{I}[\,\cdot\,|(t, \boldsymbol{x})]$ with $\boldsymbol{x} = (x_1, \ldots, x_N)$.

Using multilinearity of determinants, verify the following equalities,

$$\begin{aligned}
\frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} &= \frac{1}{h_N(\boldsymbol{x})} \det_{1 \leq i,j \leq N} \Big[\mathcal{I}[(W_j)^{i-1} | (t, y_j)]\Big] \\
&= \mathcal{I}\left[\frac{1}{h_N(\boldsymbol{x})} \det_{1 \leq i,j \leq N} [(W_j)^{i-1}] \,\middle|\, (t, \boldsymbol{y})\right] \\
&= \mathcal{I}\left[\frac{h_N(\mathbf{W})}{h_N(\boldsymbol{x})} \,\middle|\, (t, \boldsymbol{y})\right].
\end{aligned} \tag{1.14}$$

(3) Prove the equality,

$$\frac{h_N(\boldsymbol{z})}{h_N(\boldsymbol{x})} = \det_{1\le i,j\le N}\left[\Phi_\xi^{x_i}(z_j)\right], \tag{1.15}$$

where

$$\Phi_\xi^u(z) = \prod_{\substack{1\le k\le N,\\ x_k\ne u}} \frac{z-x_k}{u-x_k}, \tag{1.16}$$

$\xi = \sum_{i=1}^N \delta_{x_i} \in \mathrm{Conf}_0(\mathbb{R}), z, u \in \mathbb{C}$**, following the steps given below.**

(i) Let

$$H(\boldsymbol{x},\boldsymbol{z}) = \det_{1\le i,j\le N}\left[\prod_{1\le k\le N, k\ne i}(z_j - x_k)\right]. \tag{1.17}$$

Show that (a) it is a polynomial function of $x_i, z_i, 1 \le i \le N$ **with degree** $N(N-1)$**, (b)** $H(\boldsymbol{x},\boldsymbol{x}) = h(\boldsymbol{x})^2$**, and (c)** $H(\boldsymbol{x},\boldsymbol{z}) = 0$**, if** $x_i = x_j$ **or** $z_i = z_j$ **for some** i, j **with** $1 \le i < j \le N$**. It is implied by (a)–(c) that** $H(\boldsymbol{x},\boldsymbol{z}) = h_N(\boldsymbol{x})h_N(\boldsymbol{z})$**.**

(ii) By the definition (1.16), hence the RHS of (1.15) is equal to $H(\boldsymbol{x},\boldsymbol{z})/h(\boldsymbol{x})^2$ **for** $\xi \in \mathrm{Conf}_0(\mathbb{R})$**.**

Exercise 1.3

For a pair of space-time coordinates $(s,x),(t,y)\in[0,\infty)\times\mathbb{R}$, consider the function,

$$\mathcal{M}_N((s,x)|(t,y))=\sum_{n=0}^{N-1}\frac{1}{n!2^n}m_n(s,x)m_n(t,y), \tag{1.18}$$

where $\{m_n(t,x)\}_{n\in\mathbb{N}_0}$ are given by (1.7) in Exercise 1.1.

(1) The Hermite orthonormal functions are defined by

$$\varphi_n(x)=\frac{1}{\sqrt{\sqrt{\pi}2^n n!}}H_n(x)e^{-x^2/2},\quad x\in\mathbb{R},\quad n\in\mathbb{N}_0, \tag{1.19}$$

where $\{H_n(x)\}_{n\in\mathbb{N}_0}$ are the Hermite polynomials defined by (1.8). Prove the following expression of $\mathcal{M}_N((s,x)|(t,y)$,

$$\mathcal{M}_N((s,x)|(t,y))=\sqrt{\pi}e^{x^2/4s+y^2/4t}\sum_{n=0}^{N-1}\left(\frac{t}{s}\right)^{n/2}\varphi_n\left(\frac{x}{\sqrt{2s}}\right)\varphi_n\left(\frac{y}{\sqrt{2t}}\right). \tag{1.20}$$

(2) Let $(B(t))_{t\geq0}$ be BM adapted to the filtration $\mathcal{F}_t, t\geq0$. Then prove that, for fixed $(s,x)\in[0,\infty)\times\mathbb{R}$, $(\mathcal{M}_N((s,x)|(t,B(t))))_{t\geq0}$ is an $\mathcal{F}_t$-martingale, and

$$\mathrm{E}\left[\mathcal{M}_N((s,x)|(t,\boldsymbol{B}(t)))\right]=\mathrm{E}\left[\mathcal{M}_N((s,x)|(0,\boldsymbol{B}(0)))\right]=1. \tag{1.21}$$

(3) The following identity is known as Mehler's formula,

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} a^n = \frac{e^{[2xya-(x^2+y^2)a^2]/(1-a^2)}}{\sqrt{1-a^2}}, \quad |a| < 1. \tag{1.22}$$

Prove the following limit,

$$\lim_{N\to\infty} \mathcal{M}_N((s,x)|(t,y)) = \frac{p(s-t,x|y)}{p(s,x|0)}, \quad 0 < t < s, \quad x, y \in \mathbb{R}, \tag{1.23}$$

where $p(t,y|x)$ denotes the transition probability density of BM,

$$p(t,y|x) = \frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}}, \quad t > 0, \quad x, y \in \mathbb{R}. \tag{1.24}$$

(4) **Define the spatio-temporal kernel as follows,**

$$K(s,x;t,y) := p(s,x|0)\mathcal{M}_N((s,x)|(t,y)) - \mathbf{1}(s>t)p(s-t,x|y), \tag{1.25}$$

$(s,x),(t,y) \in [0,\infty)\times\mathbb{R}$, where $\mathbf{1}(\omega)$ is an indicator; $\mathbf{1}(\omega)=1$, if ω is satisfied, and $\mathbf{1}(\omega)=0$, otherwise. Then prove the equality

$$K(s,x;t,y) = \frac{e^{-x^2/4s}}{e^{-y^2/4t}}\mathbf{K}^{(N)}_{\mathrm{Hermite}}(s,x;t,y), \tag{1.26}$$

where

$$\mathbf{K}^{(N)}_{\mathrm{Hermite}}(s,x;t,y) = \begin{cases} \dfrac{1}{\sqrt{2s}}\displaystyle\sum_{n=0}^{N-1}\left(\frac{t}{s}\right)^{n/2}\varphi_n\left(\frac{x}{\sqrt{2s}}\right)\varphi_n\left(\frac{y}{\sqrt{2t}}\right) & \text{for } s\le t, \\ -\dfrac{1}{\sqrt{2s}}\displaystyle\sum_{n=N}^{\infty}\left(\frac{t}{s}\right)^{n/2}\varphi_n\left(\frac{x}{\sqrt{2s}}\right)\varphi_n\left(\frac{y}{\sqrt{2t}}\right) & \text{for } s>t. \end{cases} \tag{1.27}$$

The spatio-temporal kernel $\mathbf{K}^{(N)}_{\mathrm{Hermite}}(s,x;t,y)$ is known as the extended Hermite kernel giving the correlation kernel for $\mathbf{DYS}^N_2$ starting from the configuration such that all N particles are at the origin, $N\delta_0 \in \mathrm{Conf}(\mathbb{R}) \setminus \mathrm{Conf}_0(\mathbb{R})$.