I. Motivations: What are spin glasses?

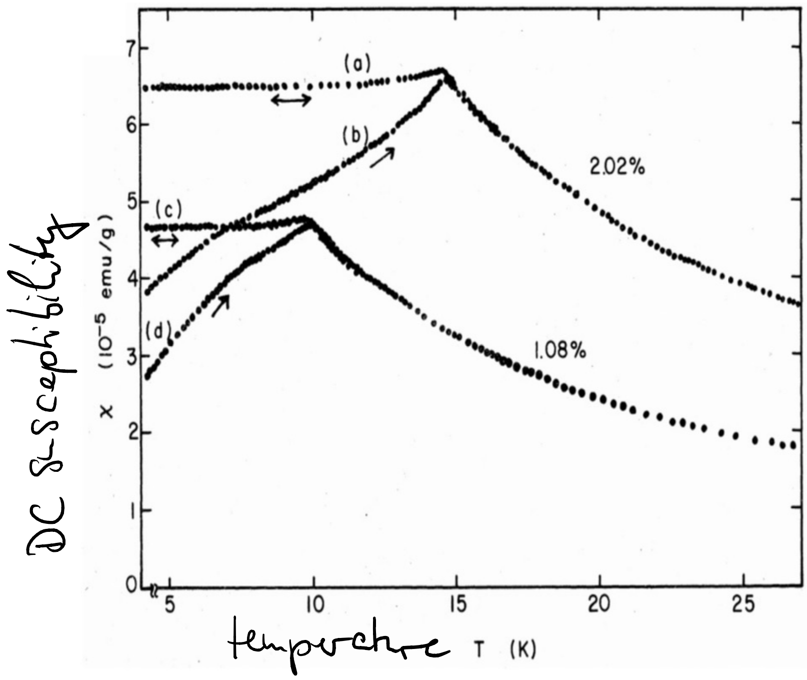
①

Physics: Substitutional alloys with atomic magnetic moments interacting in a distant-dependent way ferro- and antiferromagnetically.

Spin glass freezing transition

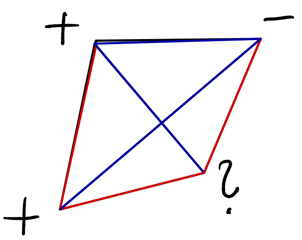
Zero / non-zero field cooling of CuMn

(Nagata / Keesom / Harrison)



Key feature of any mathematical model: Frustration

e.g. Ising spins $\underline{\sigma} = (\sigma_1 \ldots \sigma_N) \in \{-1,1\}^N$



$$\mathcal{U}(\underline{\sigma}) = \sum_{jk} J_{jk}\, \sigma_j \sigma_k$$

Q1 $\min \mathcal{U}(\underline{\sigma}) = ?$ $\quad \operatorname{argmin} \mathcal{U} = ?$ Optimization

Q2 Free energy at inv. temperature β

$$\Phi(\beta) = \lim_{N\to\infty} \frac{1}{N} \sum_{\underline{\sigma}} e^{-\beta\,\mathcal{U}(\underline{\sigma})} = ?$$ Stat Mech

Ising spin glass landscapes (mean-field models)

$$\boxed{U(\underline{\sigma}) = \frac{1}{\sqrt{N^{p-1}}} \sum_{j_1 \ldots j_p = 1}^{N} g_{j_1 \ldots j_p} \sigma_{j_1} \cdots \sigma_{j_p}}$$

e.g. iid Gaussian $\mathbb{E} g = 0$ $\mathbb{E} g^2 = 1$.

Gaussian random process on $\{-1,1\}^N =: Q_N$

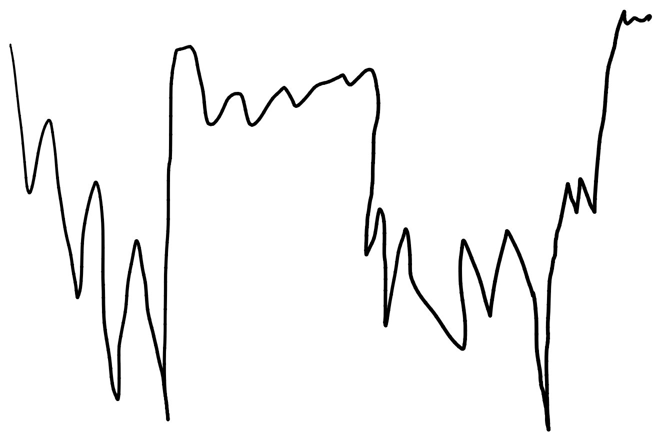
$\mathbb{E}\, U(\underline{\sigma}) = 0$ $\quad \mathbb{E}\, U(\underline{\sigma}) U(\underline{\sigma}') = N \, r_N(\underline{\sigma}, \underline{\sigma}')^p = (*)$

Overlap $r_N(\sigma, \sigma') := \frac{1}{N} \sum_{j=1}^{N} \sigma_j \sigma_j'$.

- $p = 2$ Sherrington-Kirkpatrick (SK)
- $p = \infty$ random energy model (REM) Derrida '80

$(*) = N \delta_{\sigma, \sigma'}$

Pattern: Deep valleys with high separating barriers.



Probability excercise: Extremal stat. of REM $\mathcal{U}$

$u_N(x)$ be unique solution of $2^N \int_{\sqrt{N}u_N(x)}^{\infty} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} = e^{-x}$

with $x > -\frac{\ln N}{\ln 2}$. Then:

1. $u_N(x) = \beta_c + \frac{1}{N\beta_c}\left(x - \frac{\ln(4\pi \ln 2^N)}{2}\right) + o\left(\frac{1}{N^{3/2}}\right)$

   with $\beta_c := \sqrt{2\ln 2}$

2. $\mathbb{P}(\min \mathcal{U} \geq -N u_N(x)) = (1 - 2^{-N}e^{-x})^{2^N} \to e^{-e^{-x}}$

Rough summary: $\|\mathcal{U}\|_\infty \approx -N\beta_c + o(N)$

Q1: ✓

Q2: $\frac{1}{2^N}\sum_{\sigma} e^{-\beta \mathcal{U}(\sigma)} \overset{\text{LLN ?? not correct...}}{\approx} \mathbb{E}\, e^{-\beta\mathcal{U}(\sigma)} = e^{\frac{\beta^2}{2}\mathbb{E}[\mathcal{U}(\sigma)^2]} = e^{N\frac{\beta^2}{2}}$

Only correct for $\beta \leq \beta_c$ – otherwise $\min \mathcal{U} \approx -N\beta_c$ dominates the behavior

Thm Freezing transition: Almost surely

$$\Phi(\beta) = \lim_{N\to\infty} \frac{1}{N} \ln \sum_{\sigma} e^{-\beta\mathcal{U}(\sigma)} = \begin{cases} \frac{\beta_c^2}{2} + \frac{\beta^2}{2} & \beta \leq \beta_c \\ \beta\beta_c & \beta > \beta_c \end{cases}$$

- Entropy vanishes for $\beta > \beta_c$! (only for REM!)
- For general p-spin glass → Parisi formula

Solution to computation of min:

$$\mathbb{P}(\min U \geq -N u_N) = \mathbb{P}\Big(\bigcap_{\underline{\sigma}} \{U(\underline{\sigma}) \geq -N u_N\}\Big)$$

$$= \prod_{\sigma} \mathbb{P}(U(\sigma) \geq -N u_N) = \prod_{\sigma} \big(1 - \mathbb{P}(U(\sigma) < -N u_N)\big)$$

$$= \Big(1 - \int_{\sqrt{N} u_N(x)}^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-y^2/2}\Big)^{2^N}$$

$$= \Big(1 - \frac{1}{2^N} e^{-x}\Big)^{2^N} \longrightarrow e^{-e^{-x}}.$$

$$2^N \int_{\sqrt{N} u_N(x)}^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-y^2/2} = 2^N e^{-N u_N(x)^2/2} \left(\frac{1}{\sqrt{N} u_N(x)} + O\Big(\frac{1}{(\,)^3}\Big)\right)$$

$$u_N(x)^2 = 2 \ln 2 + \dots$$

Sketch pf: $\Phi_N(\beta) := \frac{1}{N} \ln Z_N(\beta)$, $Z_N(\beta) = \sum_\sigma e^{-\beta U(\sigma)}$

I. Average: • $\mathbb{E}\,\Phi_N(0) = \ln 2 = \frac{\beta_c^2}{2}$

Upper bds: • $\frac{d}{d\beta} \mathbb{E}\,\Phi_N(\beta) = -\mathbb{E}\left(\left\langle \frac{U}{N} \right\rangle_\beta\right)$ $\qquad \langle \cdot \rangle_\beta = \frac{1}{Z} \sum_\sigma (\cdot) e^{-\beta U(\sigma)}$

$\le \beta_c$ from Exercise $(*)$

$$\left[\text{Gaussian Int. by parts} = \beta \left(1 - \sum_\sigma \mathbb{E} \frac{1}{Z^2} e^{-2\beta U(\sigma)}\right) = \beta \left(1 - \mathbb{E}\left[\frac{Z(2\beta)}{Z(\beta)^2}\right]\right) \right]$$

• $\mathbb{E}\,\Phi_N(\beta) \overset{\text{Jensen}}{\le} \frac{1}{N} \ln \mathbb{E}\, Z_N(\beta) = \frac{\beta_c^2}{2} + \frac{\beta^2}{2}$

Lower bd. for $\beta < \beta_c$:

$\mathbb{E}\,\Phi_N(\beta) \ge \frac{1}{N} \mathbb{E} \ln Z_N(\beta, c) \gtrsim \frac{\beta_c^2}{2} + \frac{\beta^2}{2}$.

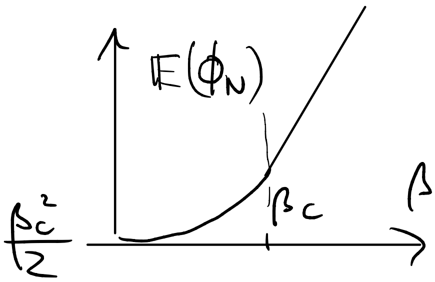
$Z_N(\beta, c) = \sum_\sigma \mathbb{1}[U(\sigma) \ge -Nc]\, e^{-\beta U(\sigma)}$

Key idea: For $\frac{c}{2} < \beta < c < \beta_c$

$$\mathbb{P}\left(|Z(\beta,c) - \mathbb{E}(Z(\beta,c))| > \delta\, \mathbb{E}\, Z(\beta,c)\right) \le \frac{e^{-NK}}{\delta^2}$$

at some $K > 0$. And $\mathbb{E}(Z(\beta,c)) = 2^N e^{\frac{\beta^2}{2} N} (1 + o(1))$

For $\beta > \beta_c$: Convexity argument + $(*)$



Summary:
$$\mathbb{E}\,\Phi_N(\beta) = \begin{cases} \frac{\beta_c^2}{2} + \frac{\beta^2}{2} & \beta < \beta_c \\ \beta \beta_c & \text{else.} \end{cases}$$

II. Fluctuations:

Thm Gaussian concentration

Let $F:\mathbb{R}^N \to \mathbb{R}$ be 1-Lipshitz and $X_1 \dots X_N$ be iid standard Gaussian r.v. Then

$$\mathbb{P}\left(|F(X)-\mathbb{E}(F(X))| > \lambda\right) \le 2\, e^{-\frac{2}{\pi^2}\lambda^2}.$$

Pf: wlog $\mathbb{E}F(X)=0$

indep. identically distributed Y

$$\mathbb{P}(F(X)>\lambda) \le \inf_{t>0} e^{-t\lambda}\, \underbrace{\mathbb{E}\, e^{t(F(X)-\mathbb{E}F(Y))}}_{=(*)}$$

$$(*) \le \mathbb{E}\exp\big(t\,\underbrace{(F(X)-F(Y))}\big)$$

Jensen

$$= \int_0^{\pi/2} \frac{d}{d\theta} F(X\cos\theta + Y\sin\theta)\, d\theta$$

$$= \int_0^{\pi/2} \nabla F(\underbrace{\quad}_{G_2(\theta)}) \cdot \underbrace{(-X\sin\theta + Y\cos\theta)}_{G_1(\theta)}\, d\theta$$

$$\le \int_0^{\pi/2} \frac{d\theta}{\pi/2}\, \mathbb{E}\, e^{t\frac{\pi}{2}\nabla F(G_2(\theta))\cdot G_1(\theta)} = \int_0^{\pi/2} \frac{d\theta}{\pi/2}\, \mathbb{E}\, e^{t^2\frac{\pi^2}{8}\|\nabla F(G_2(\theta))\|^2}$$

Key: $G_1(\theta), G_2(\theta)$ independent standard Gaussian

$$\le \exp\left(t^2\frac{\pi^2}{8}\right)$$

Optimize $\quad t^2\frac{\pi^2}{8} - t\lambda = \frac{\pi^2}{8}\left(t^2 - \frac{8}{\pi^2}\lambda t\right)$

$$= \frac{\pi^2}{8}\left(t - \left(\frac{2}{\pi}\right)^2\lambda\right)^2 - \frac{\pi^2}{8}\frac{16}{\pi^4}\cdot\lambda^2$$

□

Extension: Talagrand concentration

F as above and convex

$X_1, \dots, X_N$ independent r.v. bounded by K.

$$\mathbb{P}\left(|F(X) - \mathbb{E}F(X)| > \lambda K\right) \le C e^{-c\lambda^2}$$

for some $c, C \in (0,\infty)$ and all $\lambda > 0$.

Talagrand, Publ. Math. IHES 81, 73-205 (1995)

Application of Gaussian concentration:

$$F(\underline{X}) = \frac{1}{N} \ln \sum_{j=1}^{2^N} e^{-\beta\sqrt{N} X_j} \quad \left(= \Phi_N(\beta)\right)$$

$$\Rightarrow \quad \frac{\partial F}{\partial X_j}(\underline{X}) = \frac{1}{N} \frac{(-\beta\sqrt{N})}{\sum_j e^{-\beta\sqrt{N} X_j}} \cdot e^{-\beta\sqrt{N} X_j}$$

$$\sum_{j=1}^{2^N} \left|\frac{\partial F}{\partial X_j}(\underline{X})\right|^2 = \frac{\beta^2}{N} \frac{\sum_j e^{-\beta\sqrt{N} X_j 2}}{\left(\sum_j e^{-\beta\sqrt{N} X_j}\right)^2} \le \frac{\beta^2}{N}$$

Summary: $\mathbb{P}\left(|\Phi_N(\beta) - \mathbb{E}\Phi_N(\beta)| > \frac{\beta}{\sqrt{N}} \lambda\right) \le 2 e^{-\frac{2}{\pi^2}\lambda^2}$.

Remark on p. 4   Green box

$$\frac{Z(2\beta)}{Z(\beta)^2} = \sum_{\sigma,\sigma'} 1[r_N(\sigma,\sigma')=1] \frac{e^{-\beta H(\sigma)}}{Z(\beta)} \frac{e^{-\beta H(\sigma')}}{Z(\beta)}$$

$$= \langle 1[r_N(\sigma,\sigma')=1] \rangle_\beta^{\otimes}$$

duplicated system's Gibbs average

$$\langle \cdot \rangle_\beta^{\otimes} = \sum_{\sigma,\sigma'} \rho_\beta(\sigma)\rho_\beta(\sigma')\,(\cdot) \qquad \rho_\beta(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z(\beta)}$$

We thus proved:

Cor. Replica symmetry breaking for $\beta > \beta_c$

$$\mathbb{E}\langle 1[r_N(\sigma,\sigma')=1] \rangle_\beta^{\otimes} = \begin{cases} 0 & \beta \leq \beta_c \\ 1 - \frac{\beta_c}{\beta} & \beta > \beta_c \end{cases}$$

Quantum glasses: why & how?

Transversal field models N qbits $\bigotimes_{j=1}^{N} \mathbb{C}^2$

$$\boxed{H = U(\sigma_1^z, \dots, \sigma_N^z) - b \sum_{j=1}^{N} \sigma_j^x}$$

Pauli matrices $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

U any spin glass landscape, e.g.

SK $\rightarrow$ QSK, REM $\rightarrow$ QREM

Other glass models

Heisenberg glass $\quad H = \frac{1}{\sqrt{N}} \sum_{j,k} g_{jk} \vec{\sigma}_j \cdot \vec{\sigma}_k$

SYK model $\quad H_q = \frac{i^{q/2}}{\sqrt{\binom{N}{q}}} \sum_{j_1 \dots j_q} g_{j_1 \dots j_q} \gamma_{j_1} \cdots \gamma_{j_q}$

(Majorana Fermions $\{\gamma_j, \gamma_k\} = 2\delta_{jk}$)

Questions:

Q1. Ground- & low-energy states

Q2. Stat Mech

$$\Phi(\beta, b) = \lim_{N \to \infty} \frac{1}{N} \ln \operatorname{tr} e^{-\beta H} = ?$$

...

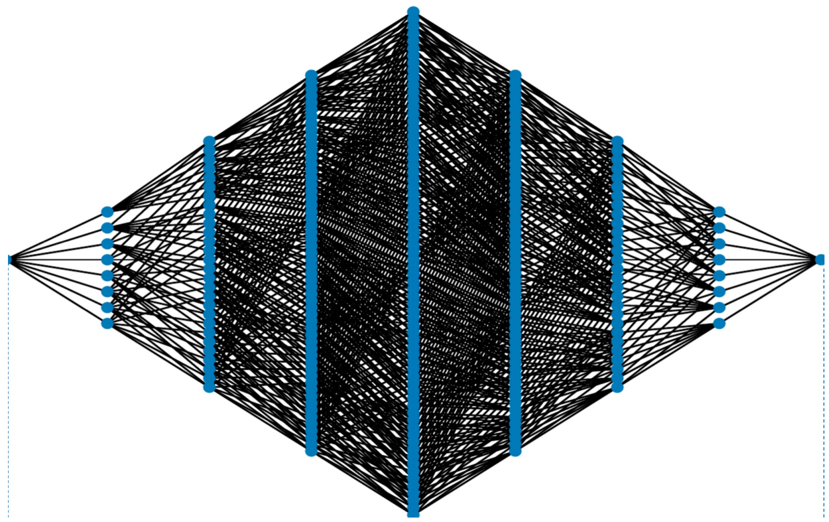
Facets of transversal field Hamiltonians

A. Random matrix – Anderson (de)localization

Canonical z-basis in $\bigotimes_{j=1}^{N} \mathbb{C}^2 \equiv \ell^2(Q_N)$

$$\langle \underline{\sigma} | H | \psi \rangle = U(\underline{\sigma}) \langle \underline{\sigma} | \psi \rangle - b \sum_{j=1}^{N} \langle \sigma_1, \ldots, -\sigma_j, \ldots \sigma_N | \psi \rangle$$

Adjacency matrix $T = \sum_{j=1}^{N} \sigma_j^x$ on hypercube $Q_N = \{-1,1\}^N$



Spectral theory exercise(s)

1. Eigenvalues & normalized eigenvectors of T on $\ell^2(Q_N)$

Indexed by subsets $A \subset \{1, \ldots, N\}$

$$f_A(\underline{\sigma}) = \frac{1}{\sqrt{2^N}} \prod_{j \in A} \sigma_j \qquad T f_A = (2|A| - N) f_A$$

Degeneracy: $\binom{N}{|A|}$.

2. Show $\langle \sigma' | e^{tT} | \sigma \rangle = \cosh(t)^N \tanh(t)^{d(\sigma,\sigma')}$ with

$d(\sigma,\sigma') = \sum_{j=1}^{N} 1[\sigma_j \neq \sigma_j']$ and $\operatorname{tr} e^{tT} = (2\cosh t)^N$.

B. Stat Mechanics

→ more in next lecture

C. Testing ground for adiabatic computing

→ more in seperate lecture

D. Math biology

Simple organism char. by genotype $\underline{\sigma} \in \{-1,1\}^N$

Mean number of genotype $\underline{\sigma}$: $n(\underline{\sigma})$

Evolution equation:

$$\boxed{\frac{d}{dt} n(\underline{\sigma},t) = \sum_{\underline{\sigma}'} \langle \underline{\sigma} | H | \underline{\sigma}' \rangle n(\underline{\sigma}',t) - n(\underline{\sigma},t) J(t)}$$

Transition by mutation & selection $\langle \sigma | H | \sigma' \rangle$

Simple model $H = b(T - N) + U_{REM}$

rough fitness landscape

Death, re due to overpopulation

$$J(t) = J_0 \sum_{\underline{\sigma}} n(\underline{\sigma},t)$$

Q: relative number of genotypes?

$$r(\underline{\sigma},t) := n(\underline{\sigma},t)\, \exp\left(\int_0^t J(s)\, ds\right)$$

$$\frac{d}{dt} r(\sigma,t) = \sum_{\underline{\sigma}'} \langle \sigma | H | \sigma' \rangle\, n(\underline{\sigma}',t)$$

$$\Rightarrow \quad r(\sigma,t) = \langle \sigma | e^{tH} | r(\cdot,0)\rangle = \sum_j e^{tE_j} \langle \sigma | \psi_j \rangle \langle \psi_j | r(\cdot,0)\rangle$$

$$\approx e^{tE_0}\, \psi_0(\underline{\sigma}) \langle \psi_0 | r(\cdot,0)\rangle$$

Summary: largest eigenvalue $E_0 > E_1 \geq \dots$ of H with non-negative (Perron-Frobenius!) eigenvector ψ_0 dominates behavior.

2 regimes: - ψ_0 sharply localized in one entry $\hat{\underline{\sigma}}$
- ψ_0 — delocalized

$\rightarrow$ more in last lecture.

II. Statistical mechanics of spin glasses

Fate of the spin glass transition in transversal fields?

$$\Phi(\beta,b) = \lim_{N\to\infty} \frac{1}{N} \ln \operatorname{tr} e^{-\beta H} \qquad H = U - bT$$

Example QREM

Recall: $b = 0$ $\quad \Phi(\beta,0) = \begin{cases} \frac{\beta_c^2}{2} + \frac{\beta^2}{2} & \beta \leq \beta_c \\ \beta\beta_c & \beta > \beta_c \end{cases}$

$b = \infty$: from exercise $\frac{1}{N} \ln \operatorname{tr} e^{-\beta bT} = \ln 2 \cosh \beta b = \Phi_P(\beta)$

Lemma: Self-averaging property

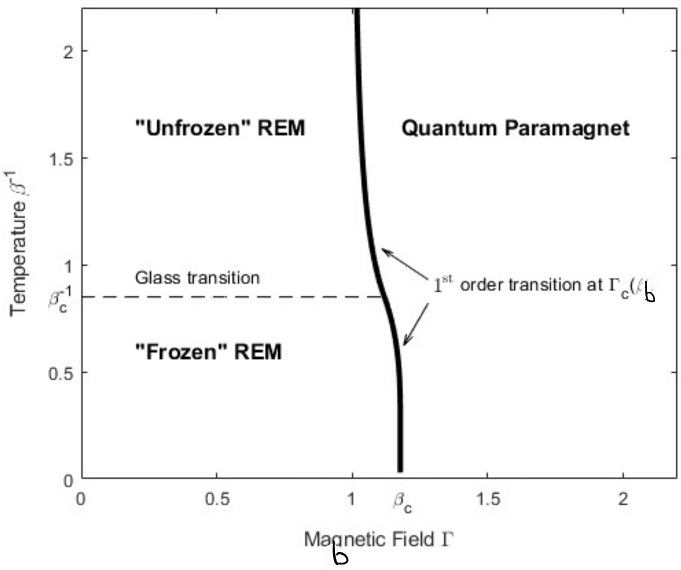
$$\mathbb{P}\left(\left| \Phi_N(\beta,b) - \mathbb{E}\Phi_N(\beta,b) \right| > \frac{\lambda}{\sqrt{N}} \beta \right) \leq 2 e^{-\frac{2}{\pi^2}\lambda^2}.$$

Proof: Gaussian concentration – exercise!

Thm Goldschmidt's formula $\quad$ Manai/W. JSP 180 (2020)

Almost surely: $\quad \Phi(\beta,b) = \max\{\Phi(\beta,0), \Phi_P(\beta)\}$

First order transition connecting to Quantum phase transition at $T=0$, $b=\beta_c$.



Proof: Upper & lower bds

Lower bound: Gibbs variational principle

$$\boxed{\ln \operatorname{tr} e^{-\beta H} = - \inf_{\varrho} \left\{ \beta \operatorname{tr} H\varrho + \operatorname{tr} \varrho \ln \varrho \right\}}$$

Density matrices ϱ on Hilbertspace, ie $\varrho \geq 0$, $\operatorname{tr}\varrho = 1$.

Note: equality for Gibbs state $\varrho = \dfrac{e^{-\beta H}}{\operatorname{tr} e^{-\beta H}}$

Application: $H = U - bT$

① $\varrho_{REM} = \dfrac{e^{-\beta U}}{\operatorname{tr} e^{-\beta U}}$:

$$N \Phi_N(\beta, b) \geq - \left[\beta \operatorname{tr} U \varrho_{REM} + \operatorname{tr} \varrho_{REM} \ln \varrho_{REM} + \beta \operatorname{tr} T \varrho_{REM} \right]$$

$$= N \Phi_N(\beta, 0) + \beta \sum_\sigma \underbrace{\langle \sigma | T | \sigma \rangle}_{=0} \varrho_{REM}(\sigma) = N \Phi_N(\beta, 0)$$

② $\varrho_P = \dfrac{e^{-\beta bT}}{\operatorname{tr} e^{-\beta bT}}$

$$N \Phi_N(\beta, b) = - \left[\beta \operatorname{tr} (-bT \varrho_P) + \operatorname{tr} \varrho_P \ln \varrho_P + \beta \operatorname{tr} U \varrho_P \right]$$

$$= N \Phi_P(\beta) + \beta \sum_\sigma \sqrt{N}\, g(\underline{\sigma}) \underbrace{\langle \underline{\sigma} | \varrho_P | \underline{\sigma} \rangle}_{= 2^{-N}}$$

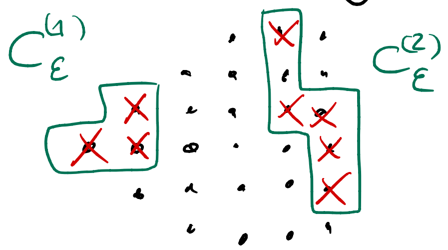
LLN / CLT: $\dfrac{1}{2^N} \sum_\sigma g(\underline{\sigma}) = \mathcal{O}\left(\dfrac{1}{\sqrt{2^N}} \right)$

Summary: $\liminf_{N \to \infty} \Phi_N(\beta, b) \geq \max \left\{ \Phi(\beta, 0), \Phi_P(\beta) \right\}$ a.s.

Upper bound:

Stochastic geometry

$\mathcal{X}_\varepsilon := \{\underline{\sigma} \mid U(\underline{\sigma}) \le -\varepsilon N\}, \ \varepsilon > 0.$



Decomposition into maximally gap-connected components

$$\mathcal{X}_\varepsilon = \biguplus_\alpha C_\varepsilon^{(\alpha)}$$

- $\operatorname{dist}(C_\varepsilon^{(\alpha)}, C_\varepsilon^{(\beta)}) > 2$

Probabilistic lemma: For all $\varepsilon > 0$, $N \in \mathbb{N}$, there is a subset $\Omega_{\varepsilon,N}$ of REM realizations s.t.

1. $\mathbb{P}(\Omega_{\varepsilon,N}) \ge 1 - e^{-c_\varepsilon N}$ with some $c_\varepsilon > 0$
2. on $\Omega_{\varepsilon,N}$: $\max_\alpha |C_\varepsilon^{(\alpha)}| < K_\varepsilon = \frac{4\ln 2}{\varepsilon^2}$

Proof idea: $\Omega_{\varepsilon,N} := \bigcap_{\sigma \in Q_N} \{|B_{4K_\varepsilon}(\sigma) \cap \mathcal{X}_\varepsilon| < K_\varepsilon\}$

- gap-connected component $C_\varepsilon^{(\alpha)} \ni \sigma$ with $|\quad| < K_\varepsilon$ is contained in ball of radius $2(K_\varepsilon - 1) < 4K_\varepsilon - 2$
- $\mathbb{P}(\Omega_{\varepsilon,N}^c) \overset{\text{union bd}}{\le} \sum_{\sigma \in Q_N} \mathbb{P}\{|B_{4K_\varepsilon}(\sigma) \cap \mathcal{X}_\varepsilon| \ge K_\varepsilon\}$

$$\le \sum_\sigma \sum_{j=K_\varepsilon}^{|B_{4K_\varepsilon}|} \underbrace{\mathbb{P}(|B_{4K_\varepsilon}(\sigma) \cap \mathcal{X}_\varepsilon| = j)}_{\le \binom{|B_{4K_\varepsilon}|}{j} e^{-Nj\frac{\varepsilon^2}{2}}} \le 2^N \sum_{j=K_\varepsilon}^{\infty} \frac{|B_{4K_\varepsilon}|^j}{j!} e^{-Nj\frac{\varepsilon^2}{2}} \le 2^N \frac{|B_{4\varepsilon}|^{K_\varepsilon}}{K_\varepsilon!} e^{-K_\varepsilon N \frac{\varepsilon^2}{2}} e^{|B_{4K_\varepsilon}| e^{-N\frac{\varepsilon^2}{2}}}$$

Geometry of B_r balls on hypercube

$$|B_r| = \sum_{j=0}^{r} \binom{N}{j} \le \sum_{j=0}^{r} \frac{N^j}{j!} \le e N^r$$

Only polynomial growth in N! $\square$

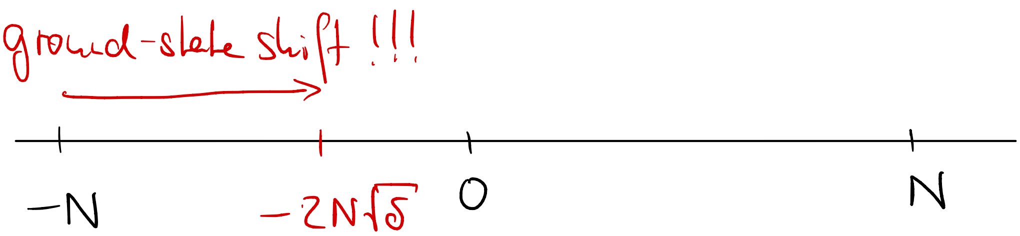
Spectral geometry on hypercube

Excursion Spectral lift due to confinement

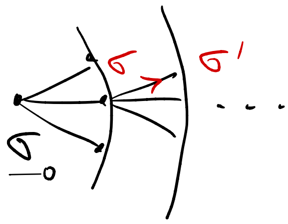
(Dirichlet) restriction of $T_{B_{\delta N}}$ to ball $B_{\delta N}$

Lemma Friedman / Tillich '05

For any $\delta \in (0, \frac{1}{2})$: $\quad \| T_{B_{\delta N}} \| \le 2N\sqrt{\delta(1-\delta)} + o(N)$



Proof: Center of ball



Write $T_{B_{\delta N}} = A + A^{\dagger}$ with $\langle \sigma' | A | \sigma \rangle = \begin{cases} 1 & \sigma' > \sigma \\ 0 & \text{else} \end{cases}$

$$\| T_{B_{\delta N}} \| \le \| A \| + \| A^{\dagger} \| = 2 \| A \| \le 2\sqrt{\| A^{\dagger} A \|}$$

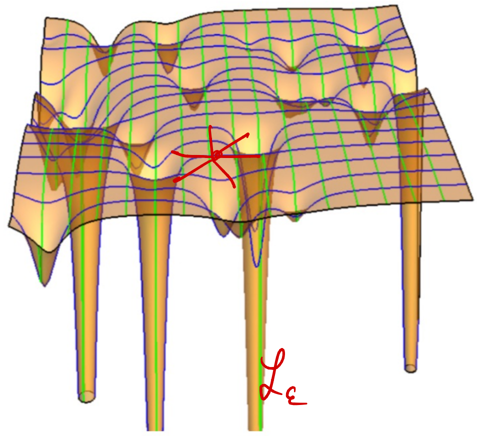
$$\| A^{\dagger} A \| \le \max_{\sigma} \sum_{\sigma'} | \langle \sigma' | A^{\dagger} A | \sigma \rangle | = N\delta \times N(1-\delta) + o(N^2)$$

wlog $\underline{\sigma}_0 = (\downarrow \downarrow \downarrow \dots \downarrow)$

outer sphere: $\underbrace{\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow}_{\delta N} \; \underbrace{\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow}_{(1-\delta)N}$ $\qquad \square$

Back to the proof of Goldschmidt's formula:

Write $H = U_{\mathcal{Y}_{\mathcal{L}_\varepsilon}} \oplus H_{\mathcal{Y}^c_{\mathcal{L}_\varepsilon}} - b\, A_{\mathcal{Y}_{\mathcal{L}_\varepsilon}}$



$$\langle \sigma' | A_{\mathcal{Y}_{\mathcal{L}_\varepsilon}} | \sigma \rangle = \begin{cases} 1 & \sigma \text{ or } \sigma' \in \mathcal{L}_\varepsilon \text{ and } d(\sigma,\sigma') = 1 \\ 0 & \text{else} \end{cases}$$

Lemma: $\| A_{\mathcal{Y}_{\mathcal{L}_\varepsilon}} \| \leq \sqrt{2N \max_\alpha |C_\varepsilon^{(\alpha)}|}$

Proof: By construction: $A_{\mathcal{Y}_{\mathcal{L}_\varepsilon}} = \bigoplus_\alpha A_{C_\varepsilon^{(\alpha)}}$

$$\Rightarrow \quad \| A_{\mathcal{Y}_{\mathcal{L}_\varepsilon}} \| \leq \max_\alpha \| A_{C_\varepsilon^\alpha} \|$$

$$\| A_{C_\varepsilon^{(\alpha)}} \|^2 \underset{\text{Frobenius estimate}}{\leq} \sum_{\sigma,\sigma'} |\langle \sigma' | A_{C_\varepsilon^{(\alpha)}} | \sigma \rangle|^2 \underset{\substack{\text{either } \sigma \in C_\varepsilon^{(\alpha)} \\ \text{or } \sigma \in C_\varepsilon^{(\alpha)}}}{\leq} 2N\, |C_\varepsilon^{(\alpha)}| \qquad \square$$

Note: On $\Omega_{N,\varepsilon}$: $\| A_{\mathcal{Y}_{\mathcal{L}_\varepsilon}} \| \leq \sqrt{2N K_\varepsilon} = O(\sqrt{N})$!

(16)

$$\boxed{\text{Golden-Thompson inequality: } \quad \mathrm{tr}\, e^{A+B} \le \mathrm{tr}\, e^{A} e^{B}}$$

Application:

$$\mathrm{tr}\, e^{-\beta H} = \mathrm{tr}\, e^{-\beta\, \mathcal{U}_{\mathcal{Y}_\varepsilon} \oplus H_{\mathcal{Y}_\varepsilon^c} + \beta b A_\varepsilon} \le \mathrm{tr}\, e^{-\beta\, \mathcal{U}_{\mathcal{Y}_\varepsilon} \oplus H_{\mathcal{Y}_\varepsilon^c}}\, e^{\beta b A_{\mathcal{Y}_\varepsilon}}$$

$$\le e^{\beta b \|A_{\mathcal{Y}_\varepsilon}\|} \left(\mathrm{tr}_{\ell^2(\mathcal{Y}_\varepsilon)}\, e^{-\beta \mathcal{U}} + \mathrm{tr}_{\ell^2(\mathcal{Y}_\varepsilon^c)}\, e^{-\beta H_{\mathcal{Y}_\varepsilon^c}} \right)$$

$$\le e^{O(\sqrt{N})} \left(\mathrm{tr}\, e^{-\beta \mathcal{U}} + e^{\beta N \varepsilon} \underbrace{\mathrm{tr}_{\ell^2(\mathcal{Y}_\varepsilon^c)}\, e^{\beta b T_{\mathcal{Y}_\varepsilon^c}}}_{\le\, \mathrm{tr}\, e^{\beta b T}} \right)$$

Note: Dyson series $\quad e^{-\beta(\mathcal{U} - bT)} =$

$$= \sum_{n=0}^{\infty} (\beta b)^n \int_0^1 ds_n \cdots \int_0^{s_2} ds_1\, e^{-\beta(1-s_n)\mathcal{U}}\, T\, e^{-\beta(s_n - s_{n-1})\mathcal{U}}\, T \cdots T\, e^{-\beta s_1 \mathcal{U}}$$

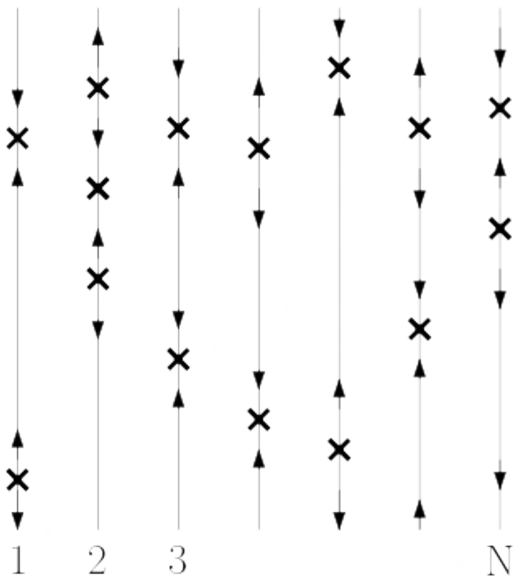
Matrix elements $\langle \sigma' | T | \sigma \rangle \ge 0$ ⇒ positivity preserving semigroup

Path integral interpretation as RW on Q_N

$$\boxed{\langle \sigma' | e^{-\beta(\mathcal{U} - bT)} | \sigma \rangle = \int \mu_{\sigma',\sigma,\beta b}(d\omega)\, \exp\left(-\beta \int_0^1 \mathcal{U}(\omega(s))\, ds\right)}$$

Poisson process with intensity $b\beta$ on each of N copies of $[0,1]$

spin-flip at ×



Summary of upper bd: On $\Omega_{N,\varepsilon}$

$$\limsup_{N \to \infty} \Phi_N(\beta, b) \le \max\{\Phi(\beta, 0), \Phi_p(\beta)\} \quad \square$$

17

Hierarchical Caricatures: Generelized Random Energy Model (GREM)

Derrida '85, ..., Ruelle '87

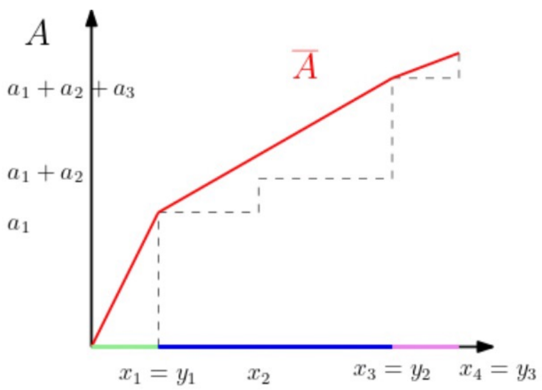
Apriori decomposition of spins in groups

$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 \ldots \boldsymbol{\sigma}_n \qquad \boldsymbol{\sigma}_j \in \{-1,1\}^{(x_j - x_{j-1})N}$

GREM as hierarchical REMs on decomposed hypercube

$$U_{\mathrm{GREM}}(\boldsymbol{\sigma}) = \sqrt{a_1}\, U^{(1)}_{\mathrm{REM}}(\boldsymbol{\sigma}_1) + \sqrt{a_2}\, U^{(2)}_{\mathrm{REM}}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) + \cdots + \sqrt{a_n}\, U^{(n)}_{\mathrm{REM}}(\boldsymbol{\sigma}_1 \ldots \boldsymbol{\sigma}_n)$$

with independent REMs $U^{(j)}_{\mathrm{REM}}(\boldsymbol{\sigma}_1 \ldots \boldsymbol{\sigma}_j)$, $j = 1, \ldots, n$.



Probability distribution function A on $[0,1]$.

Concave hull $\overline{A}$ with right-derivative $\overline{a}$

Freezing of 1st group in concave hull $\overline{A}$ at $\beta_1 = \sqrt{\frac{2\ln 2}{\overline{a}(0)}}$

... **2nd group** ... $\beta_2 = \sqrt{\frac{2\ln 2}{\overline{a}(y_1)}}$

... 3rd group ... $\beta_3 = \sqrt{\frac{2\ln 2}{\overline{a}(y_2)}}$

e.g. 3-step replica symmetry breaking

Gardner-Derrida '86, Capocaccia-Cassandro-Picco '87, ..., Bovier-Kurkova '06

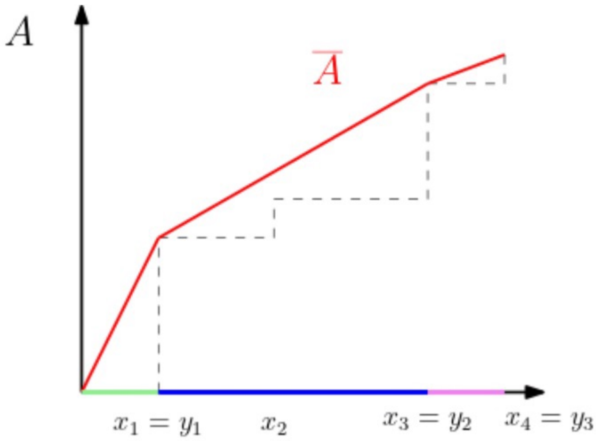
Theorem (Manai-W. '20) $\quad \Phi^{\mathrm{QGREM}}(\beta, b) = \sup_{x\in[0,1]} \left[\varphi^{\mathrm{GREM}}(\beta; x) + (1-x)\ln 2\cosh(\beta b)\right]$

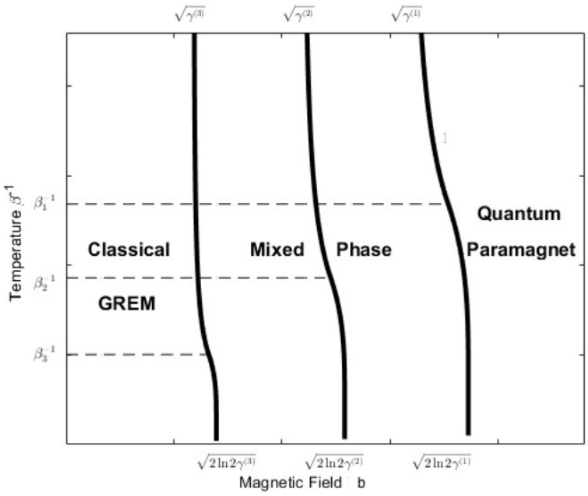
- Classical GREM pressure $\varphi^{\mathrm{GREM}}(\beta; x)$ of the first fraction $[0,x] \subset [0,1]$ of spins.

Gardner-Derrida '86, Capocaccia-Cassandro-Picco '87

For details, see: PMP 3 (2022).

Second order **spin glass transition(s)** plus second/first order **(transversal) magnetic transition(s)** (depending on the structure of $\overline{A}$):



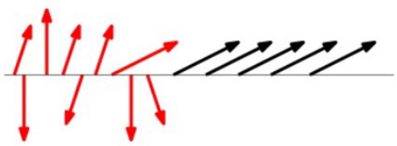


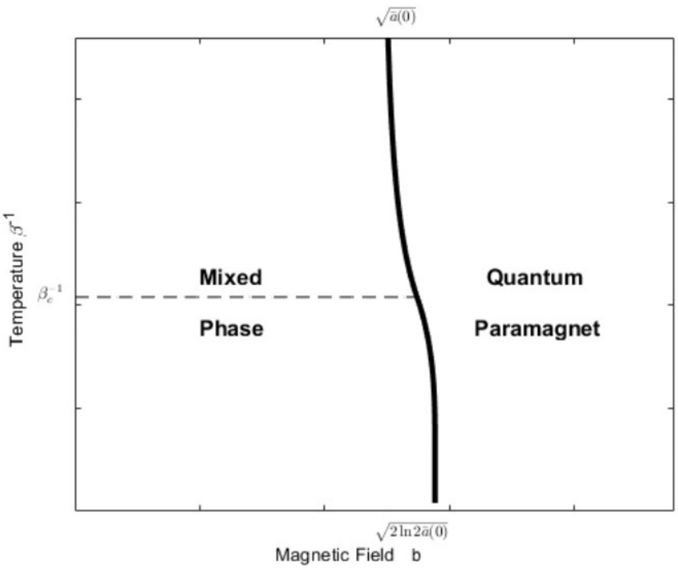
3-step replica symmetry breaking (RSB)

Quantum erasure of RSB:

groups of spins decide whether to stay in spin glass order or flip jointly with the transversal field

Emergence of intermediate regimes!





Smooth $\overline{A}$ with $\lim_{x\to 1} \overline{a}(x) = 0$

On replica symmetry breaking in quantum glasses

A. Quantum random energy model

Similar calculation to p 4: $H = J\,U - bT$

$$\frac{1}{\beta}\frac{d}{dJ}\mathbb{E}\,\Phi_N(\beta,b)\Big|_{J=1} = -\,\mathbb{E}\left\langle \frac{U}{N}\right\rangle_\beta$$

$$= -\frac{1}{\sqrt{N}}\sum_{\underline{\sigma}} \mathbb{E}\left[g(\underline{\sigma})\,\langle\underline{\sigma}|\,\varrho_\beta\,|\underline{\sigma}\rangle\right]$$

$\langle\cdot\rangle_\beta = \operatorname{tr}\varrho_\beta(\cdot)$

$\varrho_\beta = e^{-\beta H}/\operatorname{tr}e^{-\beta H}$

$$= -\frac{1}{\sqrt{N}}\sum_{\underline{\sigma}}\mathbb{E}\Bigg[\underbrace{\frac{\partial}{\partial g(\underline{\sigma})}\langle\underline{\sigma}|\,\varrho_\beta\,|\underline{\sigma}\rangle}\Bigg]$$

Gaussian int. by parts

$$= \int_0^1 ds\,\langle\underline{\sigma}|e^{-\beta(1-s)H}|\underline{\sigma}\rangle\langle\underline{\sigma}|e^{-\beta s H}|\underline{\sigma}\rangle(-\beta\sqrt{N})$$

$$-\,\langle\underline{\sigma}|e^{-\beta H}|\underline{\sigma}\rangle^2(-\beta\sqrt{N})$$

$P_{\underline{\sigma}} = |\underline{\sigma}\rangle\langle\underline{\sigma}|$

$$= \beta\left(\mathbb{E}\sum_{\underline{\sigma}}\langle P_{\underline{\sigma}};P_{\underline{\sigma}}\rangle_\beta - \mathbb{E}\left[\left\langle 1[r(\underline{\sigma},\underline{\sigma}')=1]\right\rangle_\beta^{\otimes 2}\right]\right)$$

Duhamel correlation of $P_{\underline{\sigma}} = |\underline{\sigma}\rangle\langle\underline{\sigma}|$

$$\langle A;B\rangle_\beta := \int_0^1 ds\,\operatorname{tr}e^{-(1-s)\beta H}A\,e^{-s\beta H}B\,/\operatorname{tr}e^{-\beta H}.$$

Note: For $\langle A\rangle_\beta$ with $\varrho_\beta = e^{-\beta H}/\operatorname{tr}e^{-\beta H}$, $H := H_0 - \frac{\lambda}{\beta}B$:

one has: $\frac{\partial}{\partial\lambda}\langle A\rangle_\beta = \langle A;B\rangle_\beta - \langle A\rangle_\beta\langle B\rangle_\beta$

Properties of Duhamel correlations

- $0 \le \langle A\rangle_\rho^2 \le \langle A;A\rangle_\rho \le \langle A^2\rangle_\rho$
- Falk-Bruch inequality

$$\boxed{\langle A;A\rangle_\rho \ge \langle A^2\rangle_\rho\, \Phi\left(\frac{1}{4\langle A^2\rangle_\rho}\langle [A,[\beta H,A]]\rangle_\rho\right)}$$

with $\Phi(r\tanh r) = \frac{\tanh r}{r}$

$\phi(r) \ge \frac{1}{r}(1-e^{-r})$

Here $\sum_\sigma \langle P_\sigma^2\rangle_\rho = \sum_\sigma \langle P_\sigma\rangle = 1$

Expect for $b<b_c(\beta)$: $\quad \sum_\sigma \langle P_\sigma;P_\sigma\rangle_\rho \simeq 1$

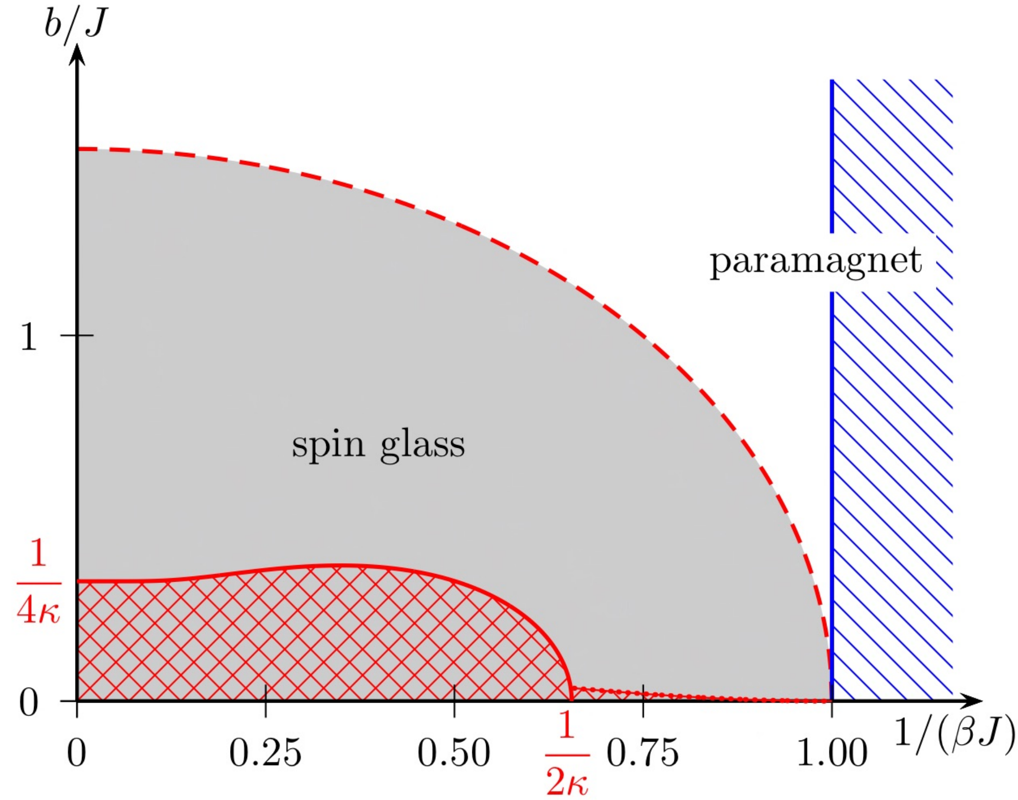
With more work: Manai/W. '21 (unpublished)

Thm Replica symmetry breaking in QREM

$$\mathbb{E}\Big\langle 1[r(\sigma,\sigma')=1]\rangle_2^{\otimes}\Big\rangle = \begin{cases} 1-\frac{\beta_c}{\beta} & \beta>\beta_c \wedge b<b_c(\beta) \\ 0 & \text{else} \end{cases}$$

B. Quantum SK model $\quad H = U_{SK} - bT$

$$U_{SK} = -\frac{J}{\sqrt{N}} \sum_{j<k} g_{jk}\, \sigma_j^z \sigma_k^z \,, \qquad J > 0.$$



Physics predictions
Fedorov/Shender '86
Yamamoto/Ishii '87
⋮
Mukherjee/Rajak/Chakrabarti '18
⋮

Quantum Parisi formula ?

cp. $b=0$

$$\Phi(\beta,0) = \inf_{\substack{x:[0,1]\circlearrowleft \\ \text{monotone}}} \mathcal{P}[x], \qquad \mathcal{P}[x] = \ln 2 + f_x(0,0) - \frac{\beta^2}{2}\int_0^1 q\, x(q)\, dq$$

with f_x unique solution of

$$\frac{\partial f}{\partial q} + \frac{1}{2}\left[\frac{\partial^2 f}{\partial y^2} + x(q)\left(\frac{\partial f}{\partial y}\right)^2\right] = 0 \,, \qquad f_x(1,y) = \ln\cosh \beta y$$

As an ∞-dim. limit of vector spin glasses

Adhikari/Brennecke '21 (no explicit info from that)

Adhikari / Brennecke '21 :

Theorem 2.1. *For any $\beta > 0$, the quenched free energy $N^{-1}\mathbb{E}\log\mathcal{Z}_N$ satisfies*

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log\mathcal{Z}_N = \lim_{d\to\infty}\sup_{\rho\in\Gamma^d}\left[\inf_{\pi_\rho\in\Pi_\rho,\,\lambda\in\mathcal{L}_s^d}\mathcal{F}(\rho,\pi_\rho,\lambda)\right].$$

$$\Gamma^d = \operatorname{clos}\left(\operatorname{conv}\left\{\sum_{k,l=1}^{d}\langle e_k,\sigma_1\rangle_2\langle\sigma_1\, e_l\rangle_2|e_k\rangle\langle e_l| \in \mathcal{J}_1 : \sigma_1\in\Omega\right\}\right),$$

$$\mathcal{L}_s^d = \left\{A\in\mathcal{L}_s : \langle e_k, A\, e_l\rangle_2 = 0,\ \text{for}\ \forall\,(k,l)\notin\{1,\dots,d\}^2\right\}.$$

With these preparations, consider a self-overlap $\rho\in\Gamma$, a discrete path $\pi_\rho\in\Pi_\rho$ and the processes X_j as above, and let $\lambda\in\mathcal{L}_s$. We then define the random variable Y_r by

$$Y_r = \log\int_\Omega d\mathcal{P}(\sigma_1)\exp\left(\beta\sum_{j=1}^{r}\langle\sigma_1,X_j\rangle_2 + \operatorname{tr}\lambda|\sigma_1\rangle\langle\sigma_1|\right) \tag{2.8}$$

and the random variables Y_j, for $j=0,\dots,r-1$, inductively through

$$Y_j = \frac{1}{m_j}\log\mathbb{E}_{j+1}e^{m_jY_{j+1}}. \tag{2.9}$$

Here, $\mathbb{E}_{j+1}$ denotes the expectation w.r.t. the process X_{j+1} only. Finally, setting

$$\Phi(\pi_\rho,\lambda) = Y_0, \tag{2.10}$$

which is non-random, the generalized Parisi functional $\mathcal{F}$ is defined by

$$\mathcal{F}(\rho,\pi_\rho,\lambda) = \Phi(\pi_\rho,\lambda) + \frac{\beta^2}{2}\int_0^1 dt\,\|\pi_\rho(t)\|_{HS}^2 - \frac{\beta^2}{2}\|\rho\|_{HS}^2 - \operatorname{tr}\lambda\rho. \tag{2.11}$$

Replica order parameter

$$R := \frac{1}{N}\sum_j \sigma_j^z \otimes \sigma_j^z$$

$$\langle R^2 \rangle^{\otimes} = \frac{2}{N(N-1)} \sum_{j<k} \langle \sigma_j^z \sigma_k^z \rangle^2 + O\left(\frac{1}{N}\right)$$

Edwards-Anderson order parameter

$$q_{EA} := \mathbb{E}\,\langle R^2 \rangle^{\otimes} = \mathbb{E}\left[\langle \sigma_1^z \sigma_2^z \rangle^2\right] + O\left(\frac{1}{N}\right)$$

Theorem Persistence of RSB for QSK

① $q_{EA} > 0$ in red region — Leschke/Manai/Ruder/W'21

② $q_{EA} = 0$ in blue region — Leschke/Rothlauf/Ruder/Spitzer'21

Proof of ① – after Bray/Moore'80, Aizenman/Lebowitz/Ruelle'87

$$u = \frac{-1}{N-1}\,\mathbb{E}\langle U_{SK} \rangle = + \frac{\sqrt{N}\,J}{2}\,\mathbb{E}\left[g_{12} \langle \sigma_1^z \sigma_2^z \rangle\right]$$

Gaussian int by parts – exercise!

$$= + \frac{\beta J^2}{2}\left(\mathbb{E}\,\langle \sigma_1^z\sigma_2^z ; \sigma_1^z \sigma_2^z \rangle_\beta - \mathbb{E}\left[\langle \sigma_1^z \sigma_2^z \rangle^2\right]\right)$$

$$\Rightarrow \quad q_{EA} = \mathbb{E}\,\langle A ; A \rangle - \frac{2}{\beta J^2} u \qquad A = \sigma_1^z \sigma_2^z$$

Classically: $\text{r.h.s} = 1 - \frac{2}{\beta J^2} u \geq 1 - \frac{2\,u_{\beta=\infty}}{\beta J^2} > 0$

β large enough

Quantum with the help of Falk-Bruch (cf. p.18):

$$\langle A; A\rangle_\rho \geq \Phi\left(\frac{1}{4}\langle [A,[\rho H, A]]\rangle_\rho\right)$$

Exercise: $[A,[\beta H, A]] = 4\beta b(\sigma_1^x + \sigma_2^x)$

Differential inequality: $\quad H = U - \sum_{j=1}^{N} b_j \sigma_j^x$

$$\frac{\partial}{\partial b_1}\langle \sigma_1^x\rangle \underset{\text{Exercise}}{=} \beta\left(\langle \sigma_1^x; \sigma_1^x\rangle - \langle \sigma_1^x\rangle^2\right)$$

$$\leq \beta\left(\langle (\sigma_1^x)^2\rangle - \langle \sigma_1^x\rangle^2\right) = \beta\left(1 - \langle \sigma_1^x\rangle^2\right)$$

Integration $\Longrightarrow$ $\quad \langle \sigma_1^x\rangle \leq \operatorname{th}(\beta b)$.

with $\langle \sigma_1^x\rangle_{b_1=0} = 0$

Summary Falk-Bruch estimate

$$\boxed{q_{EA} \geq \Phi(2\beta b \operatorname{th} \beta b) - \frac{u_{\beta=\infty}}{\beta J^2}}$$

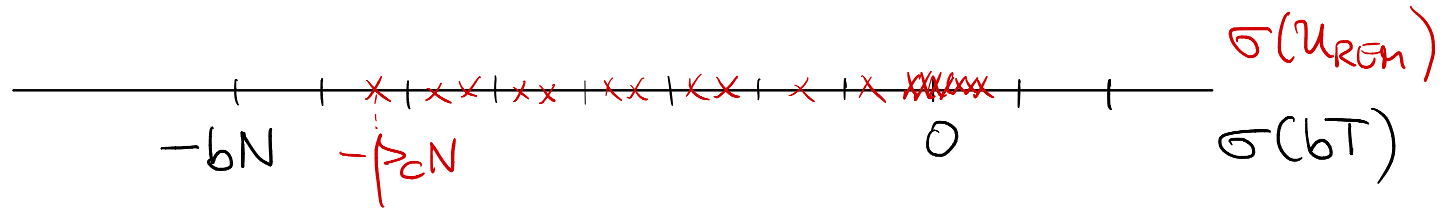
This gives fat red region with $u_{\beta=\infty} \approx J \cdot 0{,}763\ldots =: J\kappa$ (Parisi number κ)

To connect to $\beta = 1$ one has to work harder.

Details: PRL 127: 207204 (2021)

III. A glance at Spectral theory for QREM

Low energy spectrum of QREM: $H = U_{REM} - bT$



Thm: Manai / W' arXiv: 2202.00334

① Quantum paramagnetic phase $b > \beta_c$

For any $\eta, \tau > 0$ and all suff. large N on an event with prob. exponentially close to one all eigenvalues below $-(\beta_c+\eta)N$ are in union of intervals of size $O(N^{-1/2+\tau})$ at

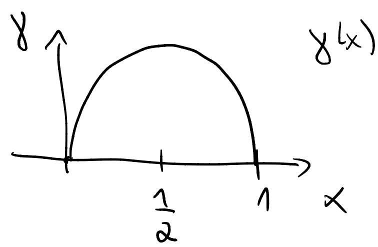
$$(2n-N)b + \frac{N}{(2n-N)b}$$

Number: $\binom{N}{n}$.

Delocalization of eigenvectors:

$$\|\psi\|_\infty^2 \le \frac{1}{2^N} \exp\left(N\gamma\left(\frac{\beta_c+\eta}{2b}\right) + o(N)\right)$$

γ Binary entropy:



$\gamma(x) = -x\ln x - (1-x)\ln(1-x)$

② Spin glass phase $b < \beta_c$

For $\delta > 0$ small and N large enough on an event of prob. exponentially close to one all eigenvalues below $-(\beta_c - \delta)N$ are in 1-1 correspondence to large deviations from

$$\mathcal{L}_{\beta_c - \delta} = \{ \sigma \mid U(\sigma) < (-\beta_c + \delta) N \}$$

More precisely they are given by

$$U(\sigma) + \frac{b^2 N}{U(\sigma)} + O(N^{-1/4}).$$

The corresponding eigenvectors are localized near σ:

- close to extremum: for all $K \in \mathbb{N}$, $\sigma' \in S_K(\sigma)$

$$|\psi(\sigma')| \leq O(N^{-K})$$

$$\sum_{\sigma' \notin B_K(\sigma)} |\psi(\sigma')|^2 \leq O(N^{-(K+1)})$$

- far from extremum: for any $\alpha \in (0,1)$ at some c_α:

$$\sum_{\sigma' \notin B_{\alpha N}(\sigma)} |\psi(\sigma')|^2 \leq e^{-c_\alpha N}.$$

Proof of 1 :

I. Hypercontractivity of Laplacian on hypercube

$$\boxed{\langle \sigma' | e^{tT} | \sigma \rangle = \cosh(t)^N \tanh(t)^{d(\sigma,\sigma')}}$$

from exercise p. 8

$$\| e^{tT} \|_{2\to\infty} = \sup_{\|\psi\|_2=1} \sup_{\sigma} |\langle \sigma | e^{tT} \psi \rangle|$$

$$\leq \sup_{\sigma} \sqrt{\langle \sigma | e^{2tT} | \sigma \rangle} = \cosh(2t)^{\frac{N}{2}}$$

Consequence for delocalization of ψ s.t.

$$H \psi_E = E \psi_E \, , \quad H = \mathcal{U} - bT \qquad E < 0$$

$$|\psi_E(\sigma)|^2 \leq \langle \sigma | 1_{(-\infty,E]}(H) | \sigma \rangle \leq \inf_{t>0} e^{tE} \langle \sigma | e^{-tH} | \sigma \rangle$$

$$\leq \inf_{t>0} e^{t(E + \|\mathcal{U}\|_\infty)} \underbrace{\langle \sigma | e^{tbT} | \sigma \rangle}_{= \cosh(tb)^N}$$

$$= 2^{-N} e^{N\gamma\left(\frac{E + \|\mathcal{U}\|_\infty}{N}\right)} .$$

II. Spectral concentration

$$Q_\varepsilon = 1 - P_\varepsilon = 1_{(-\varepsilon N, \varepsilon N)}(T) \qquad \varepsilon \in (0,1).$$

Chernoff estimate: $\displaystyle\sum_{n=0}^{(N-a)/2} \binom{N}{n} \leq 2^N e^{-a^2/2N}, \quad a \in (0,N)$

$$\Rightarrow \dim P_\varepsilon = \sum_{|k-\frac{N}{2}| > \frac{\varepsilon N}{2}} \binom{N}{k} \leq 2^{N+1} e^{-\varepsilon^2 N/2}.$$

Lemma $\{W(\sigma)\}_{\sigma \in Q_N}$ independent, mean zero $\mathbb{E}W(\sigma) = 0$ with variance $\mathbb{E}[W(\sigma)^2] \leq 1$ and bounded by M_N with $M_N^2 \dim P_\varepsilon / 2^N < 1$. Then for all $\lambda > 0$:

$$\mathbb{P}\left(\left| \|P_\varepsilon W P_\varepsilon\| - \mathbb{E}\|P_\varepsilon W P_\varepsilon\| \right| > \lambda M_N \sqrt{\frac{\dim P_\varepsilon}{2^N}} \right) \leq C e^{-c\lambda^2}$$

Moreover:

$$\mathbb{E}(\|P_\varepsilon W P_\varepsilon\|) \leq 2N \sqrt{\frac{\dim P_\varepsilon}{2^N}}$$

Sketch of proof:

1. Talagrand concentration $F(W) := \|P_\varepsilon W P_\varepsilon\|$

   Convexity: $F(\alpha W + (1-\alpha) W') \leq \alpha F(W) + (1-\alpha) F(W')$

   triangle inequ.

Lipshitz continuity: Pick $\psi \in P_\varepsilon \ell^2(Q_N)$, $\|\psi\|=1$ s.t.

$\|P_\varepsilon (W-W')P_\varepsilon\| = \langle \psi | W-W' | \psi \rangle$

$$\Rightarrow \quad |F(W)-F(W')| \le \|P_\varepsilon(W-W')P_\varepsilon\| = \langle \psi | W-W' | \psi\rangle$$

$$= \sum_\sigma (W(\sigma)-W'(\sigma))|\psi(\sigma)|^2 \le \Big(\sum_\sigma |W(\sigma)-W'(\sigma)|^2\Big)^{1/2} \underbrace{\|\psi\|_4^2}$$

$$\le \|W-W'\|_2 \; \|\psi\|_\infty$$

$$\le \|W-W'\|_2 \sqrt{\langle \sigma | P_\varepsilon | \sigma\rangle}$$

$$= \|W-W'\|_2 \sqrt{\frac{\dim P_\varepsilon}{2N}} .$$

$$= \Big(\sum_\sigma |\psi(\sigma)|^4\Big)^{1/2}$$

$$\le \|\psi\|_2 \cdot \|\psi\|_\infty$$

2. Upper bdd: method of moments

$$\mathbb{E}\|P_\varepsilon W P_\varepsilon\| \le \Big(\mathbb{E}\, \mathrm{tr}(P_\varepsilon W P_\varepsilon)^{2N}\Big)^{1/2N} \le \ldots \le 2N \sqrt{\frac{\dim P_\varepsilon}{2N}}$$

Schatten norm bd + Jensen

some tedious calculations + estimates.

Application: $\|U\|_\infty \le \beta_c N$

- $\| P_\varepsilon U P_\varepsilon \| \le C N^{3/2} e^{-\frac{\varepsilon^2 N}{4}}$
- $\| P_\varepsilon (U^2 - N) P_\varepsilon \| \le C N^2 e^{-\frac{\varepsilon^2 N}{4}}$
- $\| P_\varepsilon (U^4 - cN^2) P_\varepsilon \| \le C N^8 e^{-\frac{\varepsilon^2 N}{4}}$

III. Krein-Feshbach-Schur

Proposition $P + Q = 1$ $\quad H \equiv \begin{pmatrix} PHP & PHQ \\ QHP & QHQ \end{pmatrix}$

For all $E < \inf \sigma(QHQ)$ with $R(E) = [Q(H-E)Q]^{-1}$

1. $E \in \sigma(H)$ iff $0 \in \sigma(PHP - E - PHR(E)HP)$
2. $H\psi = E\psi$ with $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ iff

$$(PHP - E - PHR(E)HP)\psi_1 = 0$$
$$\psi_2 = -R(E)QHP\psi_1$$

Proof. Exercise!

Application :

- $$Q_\varepsilon H Q_\varepsilon \geq -\varepsilon b N Q_\varepsilon + Q_\varepsilon U Q_\varepsilon$$
$$\geq -N(\beta_c + \varepsilon b) Q_\varepsilon$$

Choose ε small enough s.t. $E \leq -N(\beta_c + \varepsilon b + \delta)$

- $$R(E) + \frac{Q_\varepsilon}{E} = R(E) QHQ \frac{Q}{E}$$

$$P_\varepsilon H Q_\varepsilon R(E) Q_\varepsilon H P_\varepsilon + P_\varepsilon \frac{N}{E} = PUR(E)UP + P\frac{N}{E}$$
$$= P \frac{N - UQU}{E} P + PURQ(U - bT)QUP\frac{1}{E}$$
$$= P\frac{N-U^2}{E}P + \frac{PUPUP}{E} + PURQUQP\frac{1}{E}$$
$$- bPURQTUP\frac{1}{E}$$

all in norm negligible

$$\frac{1}{|E|}\|R(E)\| \leq \frac{C}{\delta N^2}$$

gain $\sqrt{N}$!

$$\|UP\| \leq \left(N + N^2 e^{-\frac{\varepsilon^2 N}{4}}\right)^{1/2}$$
$$\|QT\| \leq \varepsilon N$$

Choose $\varepsilon = O\left(\sqrt{\frac{\ln N}{N}}\right)$

Summary: $E \in \sigma(H)$ iff

$$0 \in \sigma\left(-bTP_\varepsilon - E + \frac{N}{E} + O\left(N^{-\frac{1}{2}+0}\right)\right) \quad \square$$