Integrand Reduction Techniques for Multi-Loop Amplitudes

Simon Badger (NBIA & Discovery Center)

11$^{\text{th}}$ September 2012

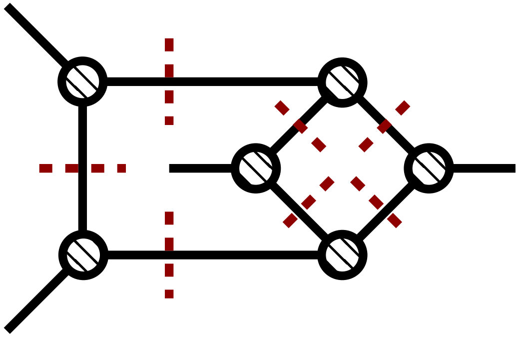
Frontiers in Perturbative Quantum Field Theory, Bielefeld University

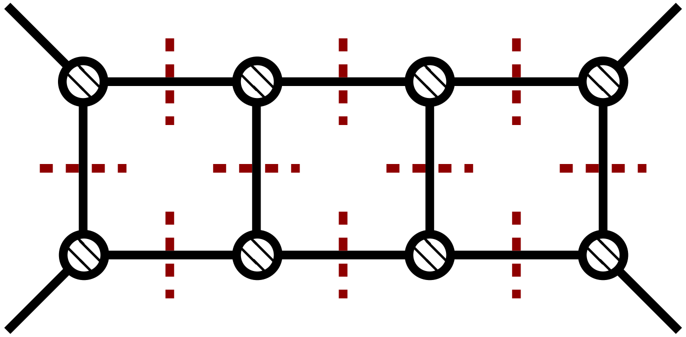
Outline

- Integrand reduction and generalized unitarity: going beyond one-loop
- Multi-loop integral coefficients via computational algebraic geometry
- Two-loop hepta-cuts : planar and non-planar

  [SB, Frellesvig, Zhang arXiv:1202.2019, JHEP 1204:055 (2012)]
- Three-loop maximal cuts : triple box

  [SB, Frellesvig, Zhang arXiv:1207:2976, JHEP 1208:065 (2012)]





Background

- One-loop techniques: [Ossola,Papadopoulos,Pittau (2006)]

  [Ellis,Giele,Kunszt,Melnikov (2007-2008)]

  [Bern,Dixon,Dunbar,Kosower (1994)][Britto,Cachazo,Feng (2004)]

$\Rightarrow$ Automation of NLO predictions for the LHC phenomenology

- NNLO predictions in QCD would be extremely valuable!

  Experimental precision will likely reach $\sim 1\%$ for a large number of processes

- Recent progress in extensions to two-loops:
  - OPP reduction at two-loops [Mastrolia, Ossola arXiv:1107.6041]

    [Mastrolia, Mirabella, Ossola, Peraro arXiv:1205.7087]

    [Kleiss, Malamos, Papadopoulos, Verheyen arXiv:1206.4180]
  - Maximal cuts via contour integration [Kosower, Larsen arXiv:1108.1180]

    [Larsen arXiv:1205.0297], [Larsen, Caron-Huot arXiv:1205.0801]

    [Johansson, Kosower, Larsen arXiv:1208.1754]

Background

- Feynman diagrams and integration-by-parts reduction

  current state-of-the-art for QCD corrections

  - $2 \to 2$ scattering amplitudes:
    - massless QCD [Anastasiou, Glover, Tejeda-Yeomans, Oleari (2000-2002)] [Bern, Dixon, Kosower (2000)][Bern, De-Frietas ,Dixon (2002)]
    - $pp \to W + j / e^+e^- \to 3j$ [Garland, Gerhmann, Glover, Koukoutsakis, Remiddi (2002)]
    - $pp \to H + 1j$ [Gerhmann, Jaquier, Glover, Koukoutsakis (2011)]
  - Full NNLO predicitions for $2 \to 2$ processes
    - $e^+e^- \to 3j$ [Gehrmann-De Ridder, Gehrmann, Glover, Heinrich (2007)]
    - $q\bar{q} \to t\bar{t}$ [Bernreuther, Czakon, Mitov (2012)]
- On-shell methods for higher multiplicity at two loops?

Background

- Maximal cut and techniques and leading singularity methods well established in super-symmetric theories

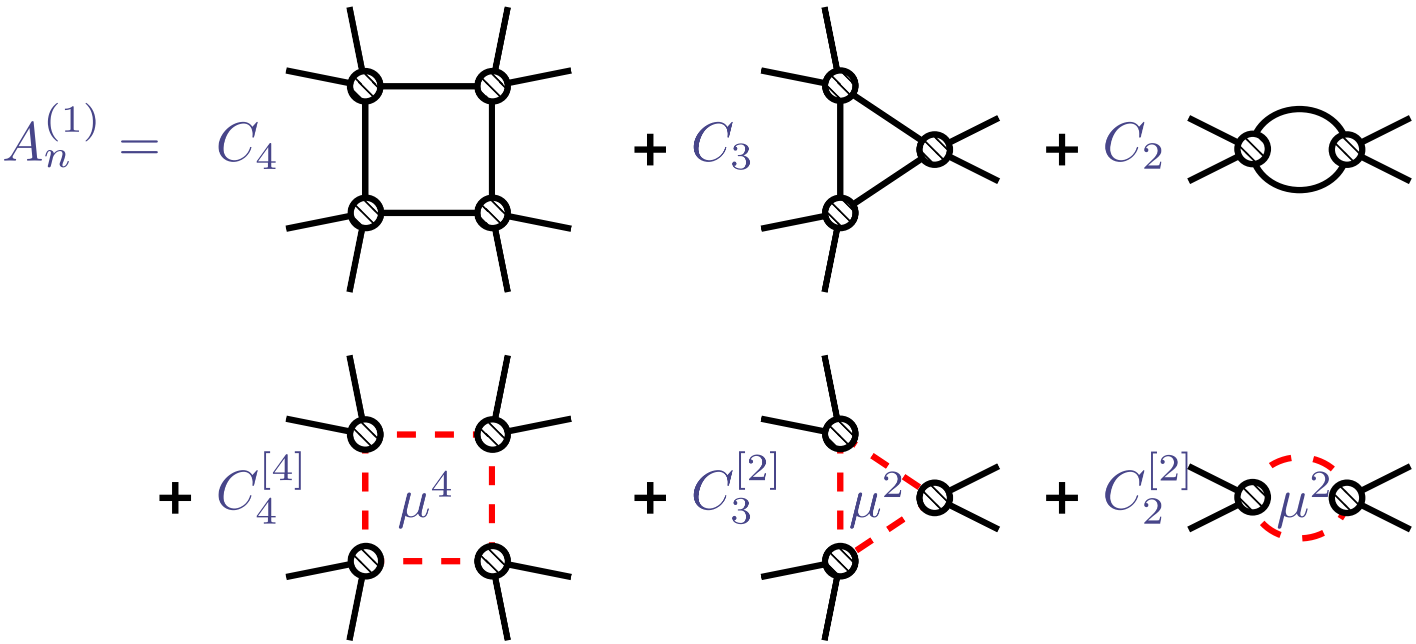
	$2 \to 2$	$2 \to 3$	$2 \to 4$
2-loop	$\mathcal{N} \leq 4$	$\mathcal{N} = 4$	$\mathcal{N} = 4$
3-loop	$\mathcal{N} = 4$	$\mathcal{N} = 4$	
4-loop	$\mathcal{N} = 4$		
5-loop	$\mathcal{N} = 4$		

Bern, Dixon, Kosower, Carrasco, Johansson, Cachzao, Buchbinder, Vergu, Spradlin, Volovich, Wen, Roiban, Drummond, Henn, Korchemsky, Sokatchev, Plefka, Alday, Schuster, Eden, Helsop, Smirnov, . . .

- Additional symmetries make amplitudes simpler, e.g. dual conformal symmetry
- Would nice if some of this applied to QCD. . .

One-Loop Overview

- Scalar integral ≤ 4-point functions form a basis with rational coefficients



- Integrand representation (OPP) : $\Delta_4(k \cdot \omega) = C_4 + \tilde{C}_4 k \cdot \omega$
- 2 solutions to $\{l_i^2 = 0\}$:

$$2C_4 = \Delta_4(k^{(1)} \cdot \omega) + \Delta_4(k^{(2)} \cdot \omega)$$

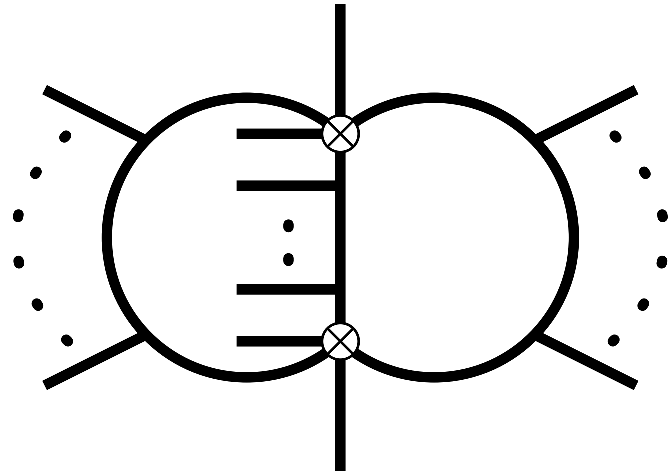
Two-Loop Integral Bases

- Complete basis of scalar integrals unknown
- Progress in understanding the planar case

[See Gluza's Talk]

[Gluza, Kosower, Kajda arXiv:1009.0472]

[Schabinger arXiv:1111.4220]



- No longer just scalar integrals, also tensor integrals in basis

A Two-Loop Integrand Basis

- Integrand is polynomial in irreducible scalar products (ISPs)
  spanned by indep. ext. moms. : $\{p_1, \ldots, p_k\}$ and spurious vecs. : $\{\omega_1, \ldots, \omega_j\}$.
- Gram matrix gives (non-linear) constraints on the polynomial form.

$$G \begin{pmatrix} v_1 \ldots v_n \\ r_1 \ldots r_n \end{pmatrix}, G_{ij} = v_i \cdot r_j$$

- Important to identify spurious terms which integrate to zero.

$$A_n^{(2)} = \int \int \frac{d^D k_1}{(4\pi)^{D/2}} \frac{d^D k_2}{(4\pi)^{D/2}} \sum_{p=3}^{11} \sum_{T_p \in \text{topologies}} \frac{\Delta_{p,T_p}(\{k_i \cdot p_j, k_i \cdot \omega_j\}, \epsilon)}{\prod_{i=1}^{p} l_i(k_1, k_2)}$$

A Two-Loop Integrand Basis

- Integrand is polynomial in irreducible scalar products (ISPs)
  spanned by indep. ext. moms. : $\{p_1, \ldots, p_k\}$ and spurious vecs. : $\{\omega_1, \ldots, \omega_j\}$.
- Gram matrix gives (non-linear) constraints on the polynomial form.

$$G \begin{pmatrix} v_1 \ldots v_n \\ r_1 \ldots r_n \end{pmatrix}, G_{ij} = v_i \cdot r_j$$

- Important to identify spurious terms which integrate to zero.

$$A_n^{(2)} = \int\int \frac{d^D k_1}{(4\pi)^{D/2}} \frac{d^D k_2}{(4\pi)^{D/2}} \sum_{p=3}^{11} \sum_{T_p \in \text{topologies}} \frac{\Delta_{p,T_p}(\{k_i \cdot p_j, k_i \cdot \omega_j\}, \epsilon)}{\prod_{i=1}^{p} l_i(k_1, k_2)}$$

sum from 3 to 11 propagators (8 in 4-D)

A Two-Loop Integrand Basis

- Integrand is polynomial in irreducible scalar products (ISPs)
  spanned by indep. ext. moms. : $\{p_1, \ldots, p_k\}$ and spurious vecs. : $\{\omega_1, \ldots, \omega_j\}$.
- Gram matrix gives (non-linear) constraints on the polynomial form.

$$G\begin{pmatrix} v_1 \ldots v_n \\ r_1 \ldots r_n \end{pmatrix}, G_{ij} = v_i \cdot r_j$$

- Important to identify spurious terms which integrate to zero.

$$A_n^{(2)} = \int\int \frac{d^D k_1}{(4\pi)^{D/2}} \frac{d^D k_2}{(4\pi)^{D/2}} \sum_{p=3}^{11} \sum_{T_p \in \text{topologies}} \frac{\Delta_{p,T_p}(\{k_i \cdot p_j, k_i \cdot \omega_j\}, \epsilon)}{\prod_{i=1}^{p} l_i(k_1, k_2)}$$

sum over all topologies e.g. planar and non-planar

A Two-Loop Integrand Basis

- Integrand is polynomial in irreducible scalar products (ISPs)
  spanned by indep. ext. moms. : $\{p_1, \ldots, p_k\}$ and spurious vecs. : $\{\omega_1, \ldots, \omega_j\}$.
- Gram matrix gives (non-linear) constraints on the polynomial form.

$$G\begin{pmatrix} v_1 \ldots v_n \\ r_1 \ldots r_n \end{pmatrix}, G_{ij} = v_i \cdot r_j$$

- Important to identify spurious terms which integrate to zero.

$$A_n^{(2)} = \int\int \frac{d^D k_1}{(4\pi)^{D/2}} \frac{d^D k_2}{(4\pi)^{D/2}} \sum_{p=3}^{11} \sum_{T_p \in \text{topologies}} \frac{\Delta_{p,T_p}(\{k_i \cdot p_j, k_i \cdot \omega_j\}, \epsilon)}{\prod_{i=1}^{p} l_i(k_1, k_2)}$$

Integrand parametrised as coeff $\times$ ISP

A Two-Loop Integrand Basis

- Integrand is polynomial in irreducible scalar products (ISPs)
  spanned by indep. ext. moms. : $\{p_1, \ldots, p_k\}$ and spurious vecs. : $\{\omega_1, \ldots, \omega_j\}$.
- Gram matrix gives (non-linear) constraints on the polynomial form.

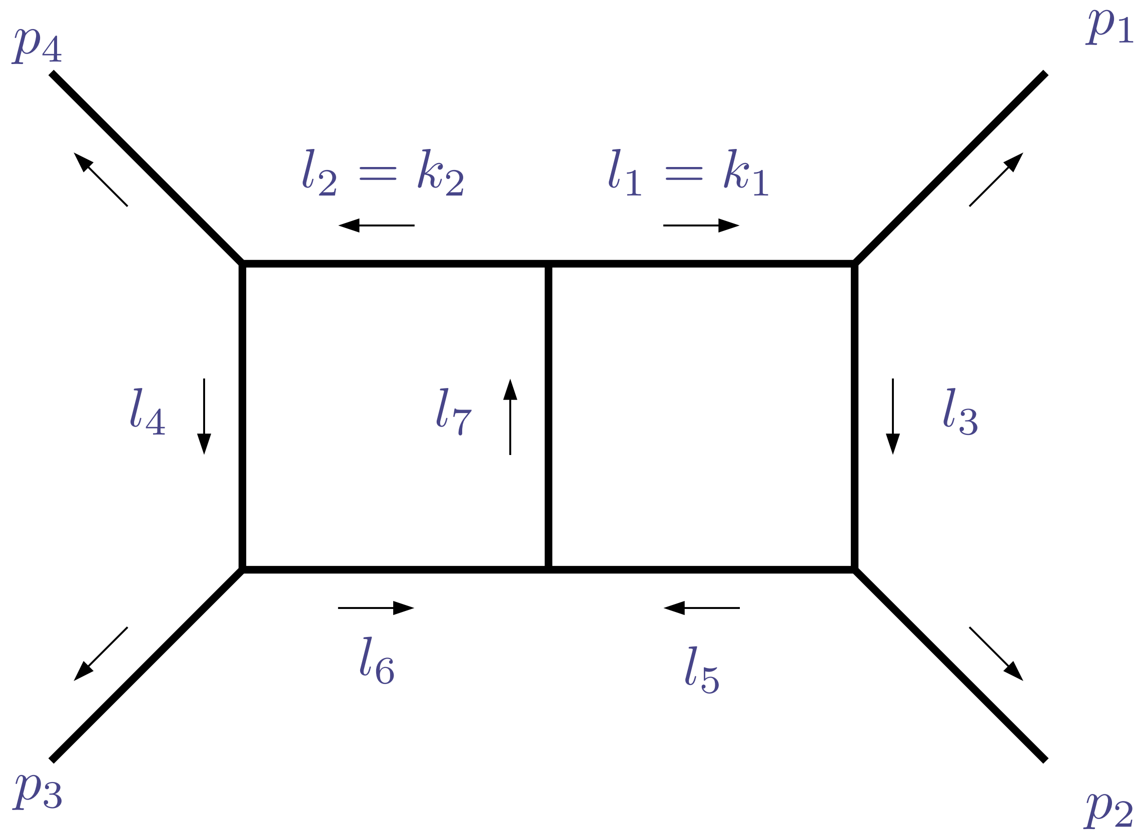
$$G \begin{pmatrix} v_1 \ldots v_n \\ r_1 \ldots r_n \end{pmatrix}, G_{ij} = v_i \cdot r_j$$

- Important to identify spurious terms which integrate to zero.

$$A_n^{(2)} = \int\int \frac{d^D k_1}{(4\pi)^{D/2}} \frac{d^D k_2}{(4\pi)^{D/2}} \sum_{p=3}^{11} \sum_{T_p \in \text{topologies}} \frac{\Delta_{p,T_p}(\{k_i \cdot p_j, k_i \cdot \omega_j\}, \epsilon)}{\prod_{i=1}^{p} l_i(k_1, k_2)}$$

Set of propagators for given topology

Example : Planar Double Box



RSPs :

$2k_1 \cdot p_1 = -(l_1 - p_1)^2 + l_1^2$

$2k_1 \cdot p_1 = -l_3^2 + l_1^2$

- ISPs $= \{k_1 \cdot p_4, k_2 \cdot p_1, k_1 \cdot \omega, k_2 \cdot \omega\}$
- $\Delta = c_0 + c_1 k_1 \cdot p_4 + \cdots + c_{16} k_1 \cdot \omega + \ldots$
- Everything will be simplifed to 4 dimensions from now on. . .

Generalized Unitarity Cuts

- On a solution to $\{l_i^2 = 0\}$ the integrand factorizes onto a product of tree-level amplitudes
- Algorithm to fit a generic integrand:
  - Parametrize the full set of on-shell solutions, $l_i^{(s)}(\tau_1, \ldots, \tau_p)$
  - Identify the ISPs on this solution:

$$k_i \cdot p_j = f_{ij}(\tau_1, \ldots, \tau_p)$$

  - Construct and solve the resulting linear system:

$$\Delta^{(s)}(\tau_1, \ldots, \tau_p) = \sum d_a \tau_1 \ldots \tau_p$$

$$\boxed{\mathbf{M} \cdot \vec{c} = \vec{d}}$$

Integrand Reduction

- Take a top-down approach to fitting each Δ
- Subtract previously determined poles, e.g.

$$\Delta_{6;\text{tri}|\text{box}} = \prod_{i=1}^{5} A_i^{(0)} - \frac{\Delta_{7;\text{box}|\text{box}}}{(k_1 - p_1)^2} = \sum_{i,j} d_{ij}\tau_i\tau_j$$

- Fitting can be done numerically or analytically
- Total number of topologies is still very large. . . .
- Towards automation:
  Solving the non-linear integrand constraints using algebraic geometry

  [Zhang arXiv:1205.5705]
- Public Mathematica code `BasisDet` [http://www.nbi.dk/~zhang/BasisDet.html]

An Algorithm for the Integrand Basis

- $B = \{v_1, v_2, v_3, v_4\}$, $[G_4]_{ij} = v_i \cdot v_j$, $P = \{l_1^2, \ldots, l_p^2\}$
- Gram matrix $[G_4]_{ij} = v_i \cdot v_j$. to re-write scalar products:

$$a \cdot b = (a \cdot v_1 \ a \cdot v_2 \ a \cdot v_3 \ a \cdot v_4) \ G_4^{-1} \begin{pmatrix} b \cdot v_1 \\ b \cdot v_2 \\ b \cdot v_3 \\ b \cdot v_4 \end{pmatrix} \tag{1}$$

- Re-write P using (**??**) $\Rightarrow$ set of equations for the scalar products.
- $\{P_i = 0\}$ has linear parts (RSPs) non-linear parts : ISP constraints $= I$
- Construct general ISP polynomial using renormalization constraints $= R$
- Remove I from R $(R/I) \Rightarrow$ Integrand Basis $= \Delta(ISPs)$.
  - Carried out using Gröbner bases and polynomial division

Solving the On-Shell Contraints

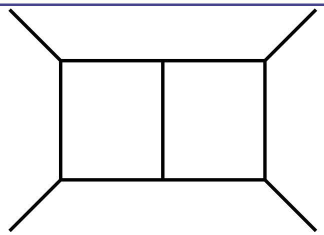
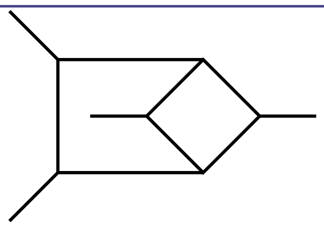
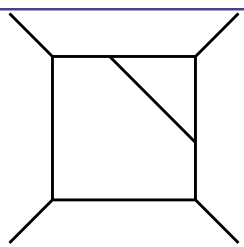
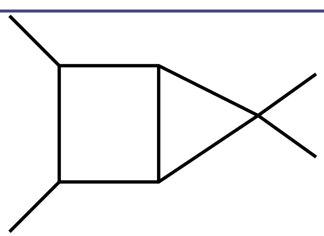
- *primary decomposition of ideals* to identify all on-shell solutions

  [Lasker-Noether theorem (1905,1921)]

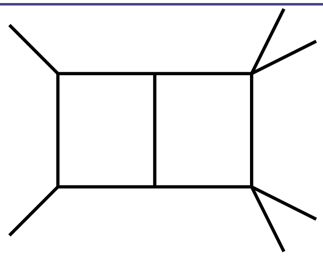
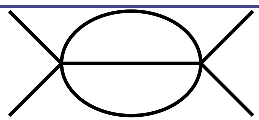
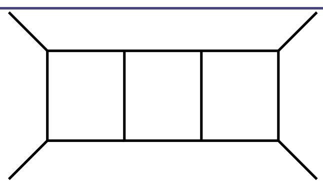
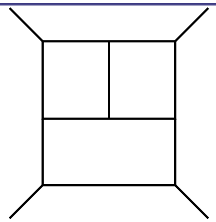
- Decompose $Z(I) \sim \{I = 0\}$ into a finite number of irreducible components
  - e.g. consider $I = \{x^2 - y^2\}$
    $I = \{x + y\} \cup \{x - y\} \Rightarrow Z(I) = \{x + y = 0\} \cup \{x - y = 0\}$
- Prime decomposition "factorizes" all solutions to identify different branches.
- Available in the public `Macaulay2` program [http://www.math.uiuc.edu/Macaulay2/]

All of this applies to higher loops as well!

A Few Examples

Topology	ISPs (non-spurious+spurious)	$\lvert\Delta\rvert$(non-sp.+sp.)	#branches(dimension)
	2+2	$32(16+16)$	$6(1)$
	2+2	$38(19+19)$	$8(1)$
	2+1	$20(10+10)$	$2(2)$
	1+4	$69(18+51)$	$4(2)$

A Few Examples

Topology	ISPs (non-spurious+spurious)	$\|\Delta\|$(non-sp.+sp.)	#branches(dimension)
	2+2	$32(16+16)$	$4(1)$
	2+6	$42(12+30)$	$1(5)$
	4+3	$398(199+199)$	$14(2)$
	5+3	$584(292+292)$	$12(2)+4(3)$

Further reduction to MI's via IBPs

- The integrand representation contains hundreds of integrals
- From this form we can apply further identities from conventional IBPs

  [Tkachov, Chetyrkin (1980)]

- Doubled propagators can be allowed at this stage
- Public codes : [AIR, Anastasiou, Lazopoulos (2004)]

  [FIRE, Smirnov ,Smirnov (2008)]

  [Reduze2, Studerus, von Manteuffel (2009-2011)]

$$A_n^{(2)} = \int\int \frac{d^4k_1}{(4\pi)^2}\frac{d^4k_2}{(4\pi)^2}\sum_{p=3}^{8}\sum_{T_p\in\text{topologies}}\frac{\Delta_{p,T_p}(\{k_i\cdot p_j, k_i\cdot\omega_j\})}{\prod_{i=1}^{p} l_i(k_1,k_2)}$$

Further reduction to MI's via IBPs

- The integrand representation contains hundreds of integrals
- From this form we can apply further identities from conventional IBPs

[Tkachov, Chetyrkin (1980)]

- Doubled propagators can be allowed at this stage
- Public codes :

[AIR, Anastasiou, Lazopoulos (2004)]

[FIRE, Smirnov ,Smirnov (2008)]

[Reduze2, Studerus, von Manteuffel (2009-2011)]

$$A_n^{(2)} = \int \int \frac{d^4 k_1}{(4\pi)^2} \frac{d^4 k_2}{(4\pi)^2} \sum_{p=3}^{8} \sum_{T_p \in \text{topologies}} \frac{c_{\vec{T}_p} \cdot \vec{B}_{T_p}}{\prod_{i=1}^{p} l_i(k_1, k_2)}$$

Further reduction to MI's via IBPs

- The integrand representation contains hundreds of integrals
- From this form we can apply further identities from conventional IBPs

[Tkachov, Chetyrkin (1980)]

- Doubled propagators can be allowed at this stage
- Public codes :

[AIR, Anastasiou, Lazopoulos (2004)]

[FIRE, Smirnov ,Smirnov (2008)]

[Reduze2, Studerus, von Manteuffel (2009-2011)]

$$A_n^{(2)} = \int\int \frac{d^4k_1}{(4\pi)^2}\frac{d^4k_2}{(4\pi)^2}\frac{\vec{C}\cdot\vec{B}}{\prod_{i=1}^n l_i(k_1,k_2)}$$

Further reduction to MI's via IBPs

- The integrand representation contains hundreds of integrals
- From this form we can apply further identities from conventional IBPs

[Tkachov, Chetyrkin (1980)]

- Doubled propagators can be allowed at this stage
- Public codes :

[AIR, Anastasiou, Lazopoulos (2004)]

[FIRE, Smirnov ,Smirnov (2008)]

[Reduze2, Studerus, von Manteuffel (2009-2011)]

$$A_n^{(2)} = \int \int \frac{d^4 k_1}{(4\pi)^2} \frac{d^4 k_2}{(4\pi)^2} \frac{\vec{C} \cdot \vec{B}}{\prod_{i=1}^n l_i(k_1, k_2)}$$

$$A_n^{(2)} = \vec{C} \cdot M_{IBP} \cdot \int \int \frac{d^4 k_1}{(4\pi)^2} \frac{d^4 k_2}{(4\pi)^2} \frac{\vec{B}'}{\prod_{i=1}^n l_i(k_1, k_2)}$$

Further reduction to MI's via IBPs

- The integrand representation contains hundreds of integrals
- From this form we can apply further identities from conventional IBPs

[Tkachov, Chetyrkin (1980)]

- Doubled propagators can be allowed at this stage
- Public codes :

[AIR, Anastasiou, Lazopoulos (2004)]

[FIRE, Smirnov ,Smirnov (2008)]

[Reduze2, Studerus, von Manteuffel (2009-2011)]

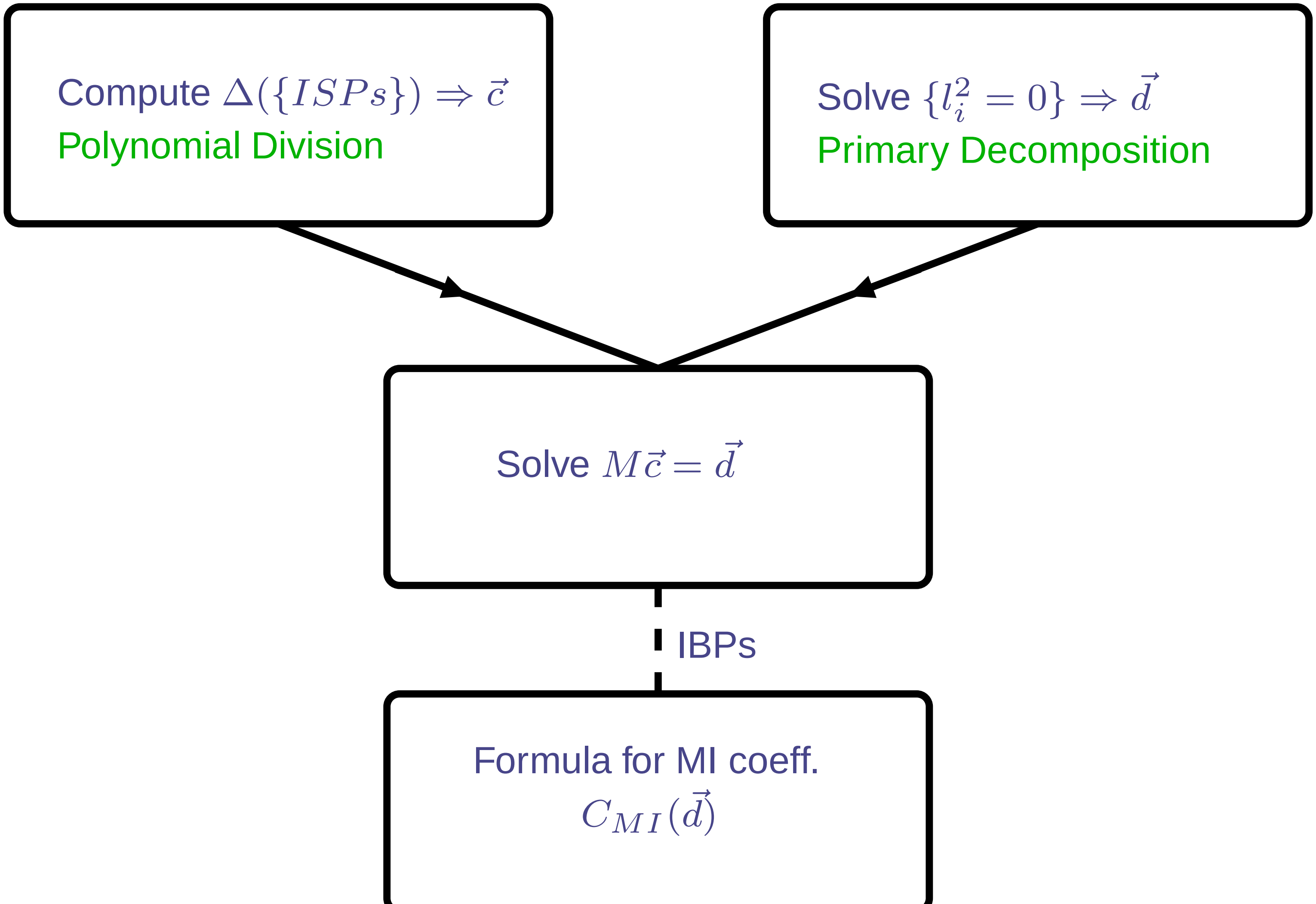
$$A_n^{(2)} = \int\int \frac{d^4k_1}{(4\pi)^2}\frac{d^4k_2}{(4\pi)^2}\frac{\vec{C}\cdot\vec{B}}{\prod_{i=1}^n l_i(k_1,k_2)}$$

solution to system of IBPs :

$\int\int \vec{B} = M_{IBP}\cdot\int\int\vec{B}'$

$$A_n^{(2)} = \vec{C}\cdot M_{IBP}\cdot\int\int \frac{d^4k_1}{(4\pi)^2}\frac{d^4k_2}{(4\pi)^2}\frac{\vec{B}'}{\prod_{i=1}^n l_i(k_1,k_2)}$$

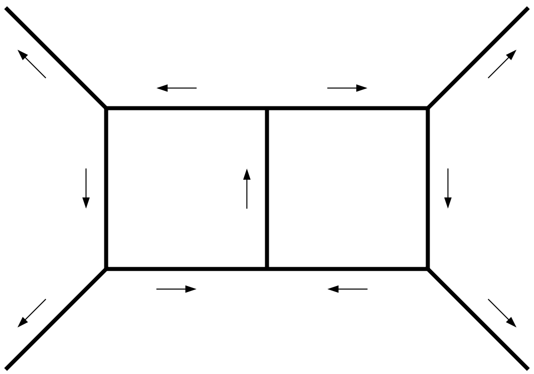
Integrand Reduction Procedure



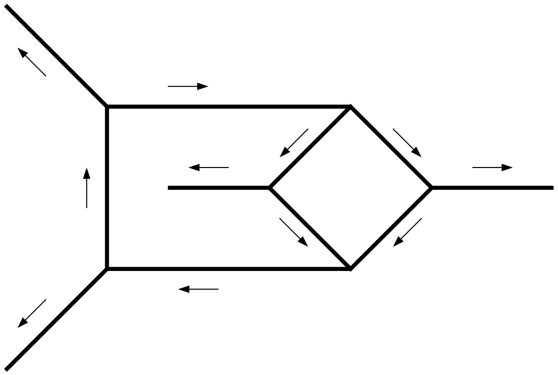
Applications and Tests

- Two-loop Hepta-cuts: planar and non-planar [SB, Frellesvig, Zhang arXiv:1202.2019]
- IBPs with `FIRE` [AV Smirnov,VA Smirnov]
- General analytic formulae for the MI coefficients
- Check $gg \to gg$ scattering with adjoint fermions and scalars

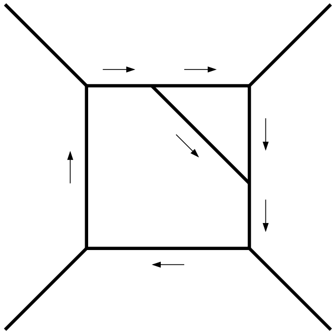
[Full agreement with Bern, De-Freitas, Dixon (2002)]



38×32 system, 2 MIs



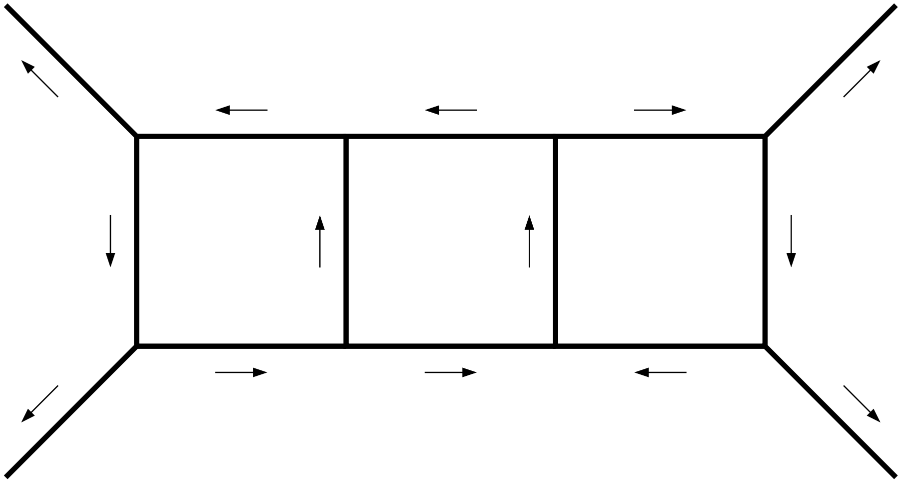
48×38 system, 2 MIs



20×20 system, no MIs

Application at Three Loops

- Planar triple box [SB, Frellesvig, Zhang arXiv:1207.2796]
- IBPs with `Reduze2` [Studerus, von Manteuffel]
- General analytic formulae for the MI coefficients
- 14 branches of the on-shell solutions
- New results valid in non-supersymmetric theories (QCD)



622×398 system, 3 MIs(!)

Outlook

- A few small steps towards automated multi-loop amplitudes
- Computational algebraic geometry for integrand reduction
  - efficent solutions to constraint equations
  - generalizes easily to D-dimensional systems
- We didn't address the evaluation of the Master Integrals
- IBPs with many scales are hard:
  - massive amplitudes, higher multiplicity