Random matrices and the epsilon-regime at finite density

Gernot Akemann

Lecture 2

Blockcourse on Aspects of QCD at Finite Density

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  example $N_f = 2$:

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- today: **QCD-Dirac operator spectrum**

  - Dirac Operator Spectrum and Chiral Symmetry
  - Spectral Densities from $\varepsilon\chi$PT
  - Random Matrix Theory
  - Phenomenology away from $\varepsilon\chi$PT: Phase diagram

Dirac Operator Spectrum and Chiral Symmetry

Properties of the QCD Dirac Spectrum

- in a box V we have $\boxed{D\psi_k = i\lambda_k \psi_k}$
- chiral symmetry $\{D, \gamma_5\} = 0$ implies $\gamma_5 D\psi_k = -D\gamma_5\psi_k = i\lambda_k\gamma_5\psi_k$
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- What to the Dirac eigenvalues have to do with XSB?

Dirac Eigenvalues and XSB for $\mu = 0$: Banks-Casher Relation

- define the mass-dependent condensate $\Sigma(m)$ or **resolvent**:

$$\Sigma(m) \equiv \frac{1}{VN_f}\partial_m \log \mathcal{Z}_{\mathbf{QCD}} = -\langle\bar{\Psi}\Psi\rangle_{\mathbf{QCD}} = \frac{1}{V}\langle \mathrm{Tr}\frac{1}{D+m}\rangle_{\mathbf{QCD}}$$

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- eigenvalues $\lambda_k \approx 0$ **dense and important for XSB:**

  spacing $\Delta\lambda = \frac{1}{\rho_D(0)} = \frac{\pi}{V\Sigma}$, much closer than $\sim 1/L$

The Microscopic Spectral density

- one-loop correction to mean density at $\mu = 0$: [Stern,Smilga 93]

$$\boxed{\rho_D(\lambda \approx 0) = \frac{1}{\pi} V\Sigma \ + \ (N_f^2 - 4)\frac{\Sigma^2}{32\pi F^4 N_f}|\lambda| \ + \ \ldots}$$

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- define **rescaled microscopic density**

$$\boxed{\rho_S(x) \equiv \frac{1}{V\Sigma}\rho_D(\lambda = \frac{x}{V\Sigma})}$$

in terms of **rescaled eigenvalues** (and masses) $\boxed{x \equiv \lambda V \Sigma}$

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- **How to compute the microcopic density?**

- in $\varepsilon\chi$PT we know $\mathcal{Z}_{\varepsilon\chi\mathrm{PT}}$ and thus $\Sigma(m) \rightarrow$ invert the integral eq.?

- problem: the density in the integrand itself depends on m

Spectral Densities from $\varepsilon\chi$PT

Generating Function for Dirac Eigenvalue Density at $\mu = 0$

- **auxiliary flavour** with mass m_{aux} to generate the resolvent
  - BUT: auxiliary flavour unwanted in QCD average
  - add **bosonic quark** with mass m_b($\in \mathbb{C}$): $N_f \to N_f + 1; 1$

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- denote $\mathcal{Z}^{N_f} = \langle \prod_{f=1}^{N_f} \det[D + m_f] \rangle_{quench}$
- $G(m) \equiv \langle \text{Tr} \frac{1}{D+m} \rangle_{\mathbf{QCD}} =$
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- we can now invert $G(m) = \int d\lambda \rho_D(\lambda; \{m_f\}) \frac{1}{i\lambda + m}$ by

$$\boxed{\rho_D(Y; \{m_f\}) = \lim_{\varphi \to 0} \frac{1}{2\pi} [G(-iY + \varphi) - G(-iY - \varphi)]}$$

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- need **supergroup integrals** $U(N_f + 1; 1)$ (same XSB pattern(!))

  example $U(1;1)$: $\boxed{G(x) = x(I_\nu(x) K_\nu(x) + I_{\nu+1} K_{\nu-1}(x)) + \frac{\nu}{x}}$

$$\boxed{\Rightarrow \rho_D(x) = \frac{x}{2} [J_\nu(x)^2 - J_{\nu-1}(x) J_{\nu+1}(x)]}$$ [Damgaard et al. 98]

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  $\boxed{\lim_{\kappa\to0}\frac{\kappa^2}{\pi(|z|^2+\kappa^2)^2} = \delta^{(2)}(z)}$
- generating function for spectral density $\mathbb{C}$ **has** $\pm\mu$ **from sources**
  $\left\langle\frac{\det[(D(\mu)+m_a)(D(\mu)^\dagger+m_a^*)]}{\det[(D(\mu)+m_a)(D(\mu)^\dagger+m_a^*)+\kappa^2]}\prod_{f=1}^{N_f}\det[D(\mu)+m_f]\right\rangle$
- μ-dependent spectral density in $\varepsilon\chi$PTwith baryonic μ

More Spectral Correlation functions from $\varepsilon\chi$PT?

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- alternative way: use replica methods (differ for $\mu = 0$ and $\mu \neq 0$)

Random Matrix Theory

Random Matrix Theory of QCD for $\mu = 0$

• in RMT computation of ALL spectral densities and individual eigenvalue distributions possible and much simpler

$$\mathcal{Z}_{\nu,\ RMT} \equiv \int dW\ \Pi_{f=1}^{N_f} \det \begin{bmatrix} m_f & iW \\ iW^\dagger & m_f \end{bmatrix} e^{-N\Sigma^2 \mathrm{Tr} WW^\dagger}$$

[Shuryak, Verbaarschot 93]

- block matrix has **same global symmetry** as QCD-Dirac operator D:

  $W_{ij} \in \mathbb{C}$: $N \times (N + \nu)$ Gaussian random variables (chGUE)

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- different $\varepsilon\chi$PT-χSB, classes: $W_{ij} \in \mathbb{R},\ \mathbb{H}$: RMT still solvable

Large-N limit of RMT and Equivalence to $\varepsilon\chi$PT

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- *step 2.* ψ^4 $\longrightarrow$ **Hubbard-Stratonovich**: extra $\int dQ_{N_f \times N_f}$ $\longrightarrow$ do $\int d\psi$:

$$\mathcal{Z}_{\nu,RMT} \sim \int dQ \det[Q^\dagger]^\nu \det\left[Q^\dagger Q\right]^N e^{-N \mathrm{Tr}\left(Q^\dagger Q \,-\, M(Q+Q^\dagger)\right)}$$

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- *step 3.* **Saddle Point** $N \to \infty$:
  parametrise $Q = UR$, at SP: $R \sim 1\!\!|$, + scale masses Nm_f

$$\boxed{\mathcal{Z}_{\nu,RMT} \rightarrow \int dU_{U(N_f)} \det[U^\dagger]^\nu e^{-N \mathrm{Tr} M(U+U^\dagger)}}$$

  0-dimensional σ-model

RMT with $\mu \neq 0$ and Generating Functions

$$\mathcal{Z}_{MM} \equiv \int dW_1 dW_2 \, \Pi_{f=1}^{N_f} \det \begin{bmatrix} m_f & iW_1 + \mu_f W_2 \\ iW_1^\dagger + \mu_f W_2^\dagger & m_f \end{bmatrix} e^{-N \mathrm{Tr} W_j W_j^\dagger}$$

[Stephanov 96, Osborn 04]

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  [Splittorff, Verbaarschot; GA, Osborn, Splittorff, Damgaard]

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[Stephanov 96, Osborn 04]

- agrees with $\varepsilon\chi$PTin the large-N limit, for baryon and isospin $\mu \neq 0$
  [Splittorff, Verbaarschot; GA, Osborn, Splittorff, Damgaard]
- **generating functions** for all k-point densities:
  $U(N_F + k + k^*; k + k^*)$**–supergroup integral of $\varepsilon\chi$PT**
  agree with $\varepsilon\chi$PT, so all individual eigenvalues do [F. Basile, G.A. 07]

  proof: <u>Superbosonisation Theorem</u> (also [Littelmann,Sommers,Zirnbauer])

$$\int d\Psi f(\Sigma_k \Psi_k \times \Psi_k) \sim \int_{U(N_F+k|k)} dU_0\ \mathrm{sdet}[U_0]^N f(U_0)$$

Phenomenology away from $\varepsilon\chi$PT: Phase diagram

On the Phasediagram of QCD

- consider a **temperature dependent phenomenological RMT**

  [Halasz et al. 98] (link to $\varepsilon\chi$PT lost!)

- replace $D(\mu;T) \longrightarrow \begin{pmatrix} 0 & W + (\mu + iT\sigma_3)1\!\!| \\ -W^\dagger + (\mu + iT\sigma_3)1\!\!| & 0 \end{pmatrix}$

  - with Gaussian average over W

  - incorporates "effect of the lowest Matsubara frequency $\pm\pi T$"

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- **effective Lagrangian in terms of auxiliary field** Q (neglect ν)

$$\boxed{\mathcal{L} = N\Sigma QQ^\dagger - \frac{N}{2}\ln[|Q+M|^2 - (\mu+iT)][|Q+M|^2 - (\mu-iT)]}$$

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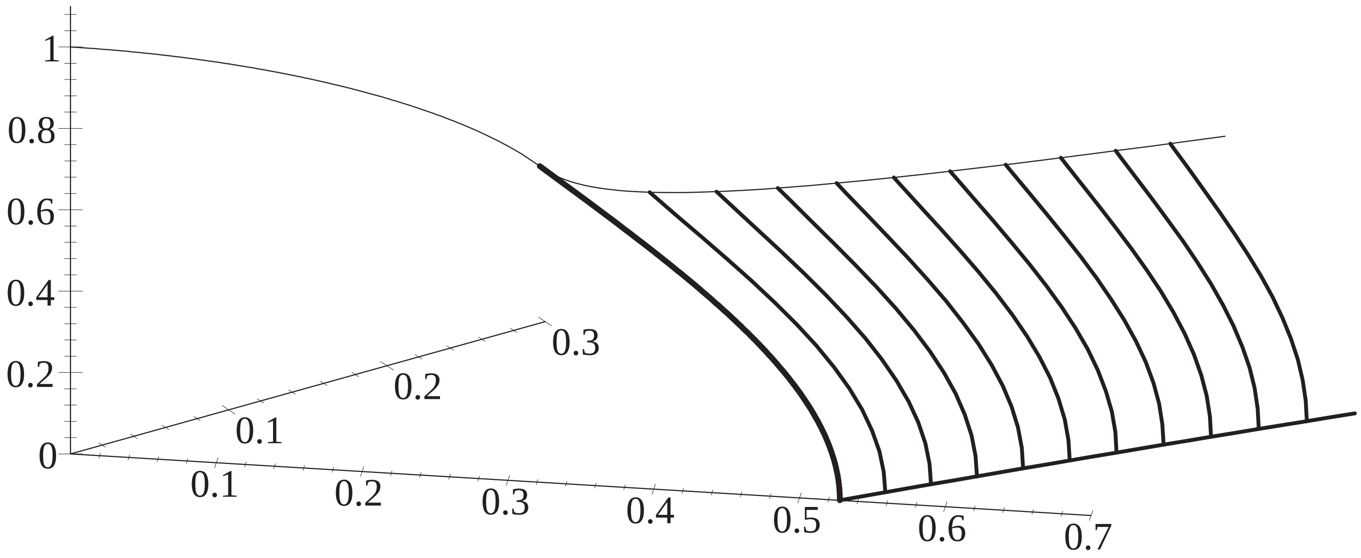
- Phase boundary $\phi^4 - 2\left(\mu^2 - T^2 + \frac{1}{2}\right)\phi^2 + (\mu^2+T^2)^2 + \mu^2 - T^2 = 0$

  for m=0 with tri-critical point $\boxed{(T_3, \mu_3) = (\frac{1}{2}\sqrt{\sqrt{2}+1}, \frac{1}{2}\sqrt{\sqrt{2}-1})}$

RMT Phasediagram



from [Halasz et al. 98]

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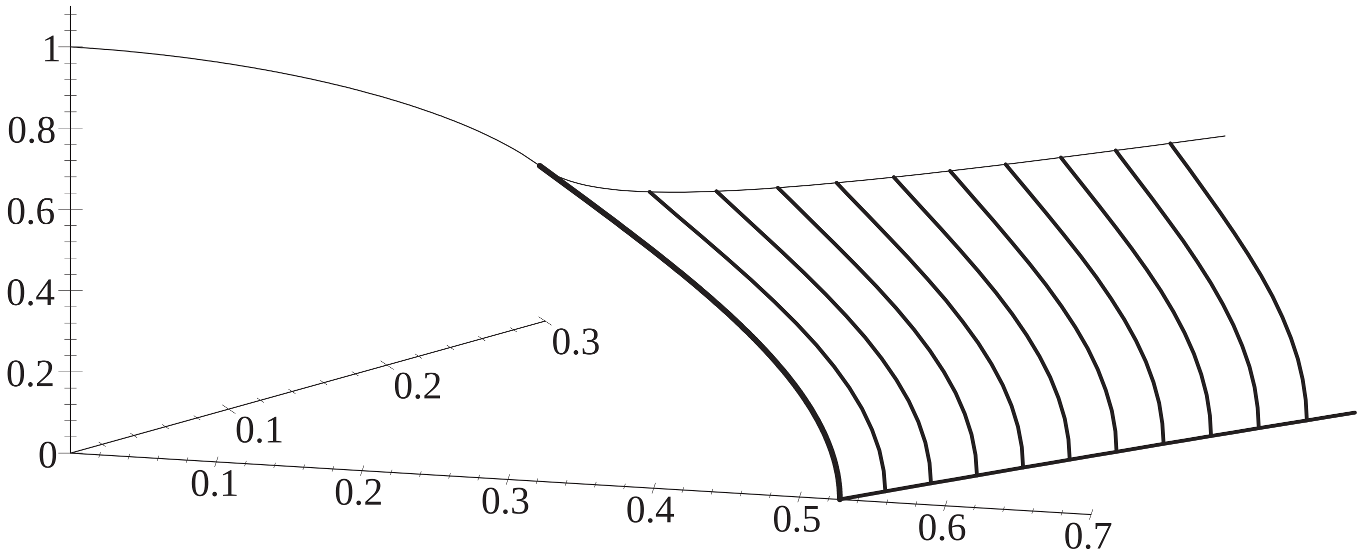

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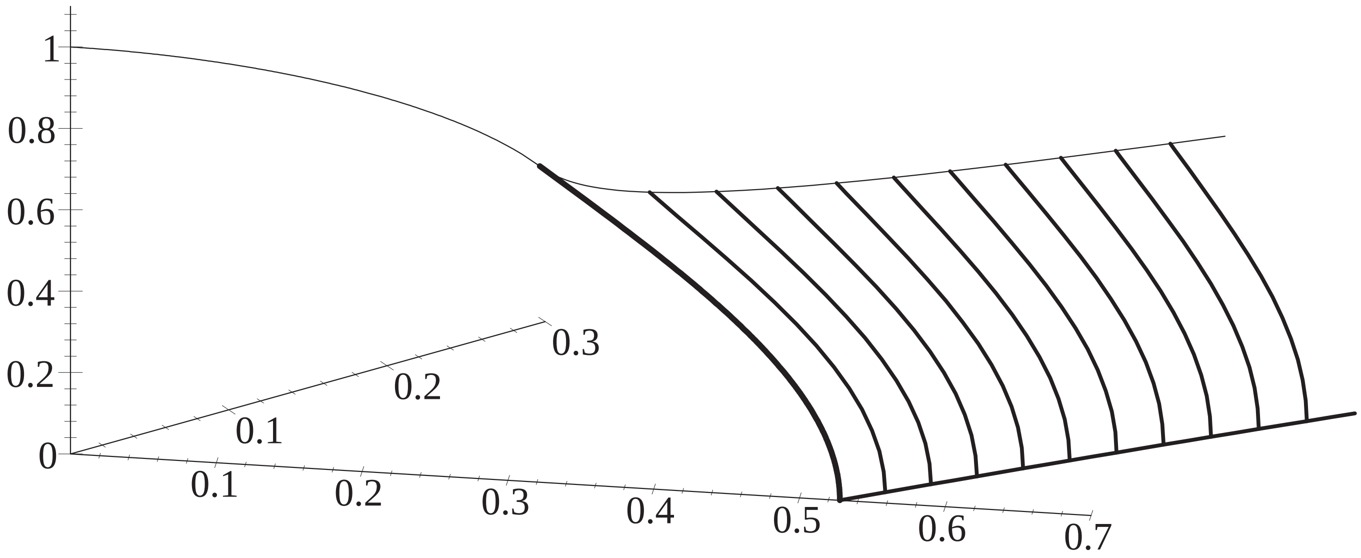

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- effect of **two chemical potentials** $\mu_1 \neq \mu_2$ or **two mases** $m_1 \neq m_2$

  split into two lines [Klein, Toublan, Verbaarschot; GA, Fyodorov, Vernizzi]

Exercise 2

- Check the leading order contribution $O(\varepsilon^0)$ in the chiral Lagrangian and compute the subleading terms to order $O(\varepsilon^2)$ for $\mu = 0$
- Can you derive the value of the temperature T_3 of the tricritical point in physical units, given T_c?

More literature

- J.J.M. Verbaarschot, T. Wettig ”Random Matrix Theory and Chiral Symmetry in QCD”, hep-ph/0003017 = Ann.Rev.Nucl.Part.Sci. 50 (2000) 343-410
- J.J.M. Verbaarschot, QCD, Chiral Random Matrix Theory and Integrability, Lectures given at the Les Houches Summer School, arXiv:hep-th/0502029
- P.H. Damgaard, J.C. Osborn, D. Toublan, J.J.M. Verbaarschot, The Microscopic Spectral Density of the QCD Dirac Operator, Nucl.Phys. B547 (1999) 305-328 = arXiv:hep-th/9811212v1
- G. Akemann, Matrix Models and QCD with Chemical Potential, Int.J.Mod.Phys.A22:1077-1122,2007 = arXiv:hep-th/0701175