Random Matrix Theory and the epsilon-regime at finite density

Gernot Akemann

Lecture 3

Blockcourse on Aspects of QCD at Finite Density

Bielefeld, 19.-23. September 2011

Summary of Lecture 2

- we found the **relation Dirac Spectrum** $\leftrightarrow$ **XSB:** $\Sigma \sim \rho_D(0)$

Summary of Lecture 2

- we found the **relation Dirac Spectrum** $\leftrightarrow$ **XSB:** $\Sigma \sim \rho_D(0)$
- we constructed **generating functions** (resolvents) for $\rho_D(0)$ in $\varepsilon\chi$PT

  $\rightarrow$ supergroup integrals $U(N_f + k; k)$

  $\rightarrow$ for baryonic μ this IS μ-depentent in $\varepsilon\chi$PT

Summary of Lecture 2

- we found the **relation Dirac Spectrum** $\leftrightarrow$ **XSB:** $\Sigma \sim \rho_D(0)$
- we constructed **generating functions** (resolvents) for $\rho_D(0)$ in $\varepsilon\chi$PT

  $\rightarrow$ supergroup integrals $U(N_f + k; k)$

  $\rightarrow$ for baryonic μ this IS μ-depentent in $\varepsilon\chi$PT
- we **introduced RMT** with or without μ,

  showed $\mathcal{Z}$'s **equivalent to group integral** part of $\varepsilon\chi$PTfor fixed ν

Summary of Lecture 2

- we found the **relation Dirac Spectrum** $\leftrightarrow$ **XSB:** $\Sigma \sim \rho_D(0)$
- we constructed **generating functions** (resolvents) for $\rho_D(0)$ in $\varepsilon\chi$PT

  $\rightarrow$ supergroup integrals $U(N_f + k; k)$

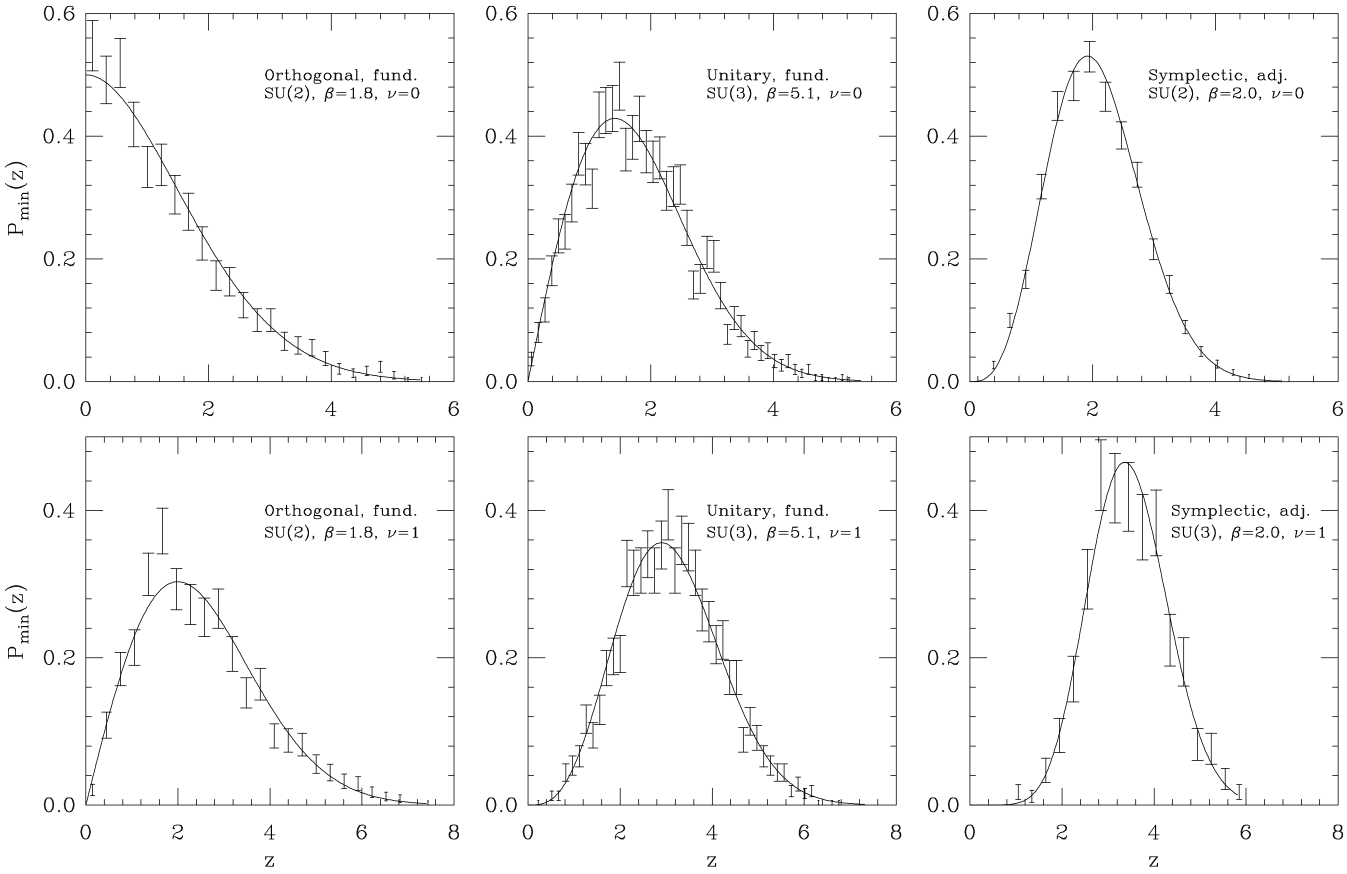
  $\rightarrow$ for baryonic μ this IS μ-depentent in $\varepsilon\chi$PT
- we **introduced RMT** with or without μ,

  showed $\mathcal{Z}$'s **equivalent to group integral** part of $\varepsilon\chi$PTfor fixed ν
- today: solution of **all Dirac eigenvalue correlation functions**

  from Random Matrix Theory

Example IV: Lattice vs. smallest real Dirac eigenvalues

- 1st Dirac-eigenvalue vs. Lattice with chiral fermions [Edwards et at. 98]:
- columns: SU(2) — SU(3) — SU(2) adj. with different χSB patters
  rows: $\nu = 0$ (top) and $\nu = 1$ (bottom)



Random Matrix Theory for $\mu = 0$ (chGUE or Wishart-Laguerre)

$$\mathcal{Z}_\nu \equiv \int \prod_{i,j} d\Re e(W_{ij}) d\Im m(W_{ij}) \; \Pi_{f=1}^{N_f} \det \begin{bmatrix} m_f & iW \\ iW^\dagger & m_f \end{bmatrix} e^{-N\Sigma^2 \mathrm{Tr} W_j W_j^\dagger}$$

- matrix elements are **independent Gaussian random variables**, here dressed with N_f determinants

Random Matrix Theory for $\mu = 0$ (chGUE or Wishart-Laguerre)

$$\mathcal{Z}_\nu \equiv \int \prod_{i,j} d\Re e(W_{ij}) d\Im m(W_{ij}) \; \Pi_{f=1}^{N_f} \det \begin{bmatrix} m_f & iW \\ iW^\dagger & m_f \end{bmatrix} e^{-N\Sigma^2 \mathrm{Tr} W_j W_j^\dagger}$$

- matrix elements are **independent Gaussian random variables**, here dressed with N_f determinants
- diagonalise $[D + m_f 1\|]$ using $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det[D] \det[A - BD^{-1}C]$

  $\Rightarrow \; \det[D + m] = m^\nu \det[WW^\dagger + m^2]$

Random Matrix Theory for $\mu = 0$ (chGUE or Wishart-Laguerre)

$$\mathcal{Z}_\nu \equiv \int \prod_{i,j} d\Re e(W_{ij}) d\Im m(W_{ij}) \, \Pi_{f=1}^{N_f} \det \begin{bmatrix} m_f & iW \\ iW^\dagger & m_f \end{bmatrix} e^{-N\Sigma^2 \mathrm{Tr} W_j W_j^\dagger}$$

- matrix elements are **independent Gaussian random variables**, here dressed with N_f determinants
- diagonalise $[D + m_f 1\!\!1]$ using $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det[D] \det[A - BD^{-1}C]$

  $\Rightarrow \; \det[D + m] = m^\nu \det[WW^\dagger + m^2]$
- $WW^\dagger = U \lambda U^\dagger$ **positive definite Hermitian matrix**, $\lambda =$diag$(\lambda_k \geq 0)$ (or singular values y_k of $W = uyv$ with $\sqrt{\lambda_k} = y_k$ Dirac "eigenvalues")
- $\boxed{\mathcal{Z}_\nu = \int du\, dv \int_0^\infty \prod_{k=1}^N [d\lambda_k \lambda_k^\nu e^{-N\Sigma^2 \lambda_k} \prod_{f=1}^{N_f} m_f^\nu (\lambda_k + m_f^2)] \, |\Delta_N(\lambda)|^2}$
- Vandermonde det $\Delta_N(\lambda) = \prod_{k>l}^N (\lambda_k - \lambda_l)$

RMT Eigenvalue Picture

- **Jacobian from the diagonalisation** has 2 parts:

  1) $\prod_k^N \lambda_k^\nu$ from rectangular matrices: changes weight,

  - repells eigenvalues from origin

  2) $|\Delta_N(\lambda)|^2$ Vandermonde determinant squared:

  $\prod_{k>l}(\lambda_k - \lambda_l)^2 = \exp[2\sum_{k>l} \log|\lambda_k - \lambda_l|]$

RMT Eigenvalue Picture

- **Jacobian from the diagonalisation** has 2 parts:

  1) $\prod_k^N \lambda_k^\nu$ from rectangular matrices: changes weight,

  - repells eigenvalues from origin

  2) $|\Delta_N(\lambda)|^2$ Vandermonde determinant squared:

  $\prod_{k>l}(\lambda_k - \lambda_l)^2 = \exp[2\sum_{k>l}\log|\lambda_k - \lambda_l|]$

  - eigenvalues now strongly coupled: log-gas in 2-dim,

  repulsion of eigenvalues $\neq$ independent random variables

- this repulsion (of some operator replaced by a random matrix) is a **universal feature in many phenomena in nature**

RMT Eigenvalue Picture

- **Jacobian from the diagonalisation** has 2 parts:

  1) $\prod_k^N \lambda_k^\nu$ from rectangular matrices: changes weight,

  - repells eigenvalues from origin

  2) $|\Delta_N(\lambda)|^2$ Vandermonde determinant squared:

  $\prod_{k>l}(\lambda_k - \lambda_l)^2 = \exp[2\sum_{k>l}\log|\lambda_k - \lambda_l|]$

  - eigenvalues now strongly coupled: log-gas in 2-dim,

  repulsion of eigenvalues $\neq$ independent random variables

- this repulsion (of some operator replaced by a random matrix) is a **universal feature in many phenomena in nature**

- first consequence:

  flavour-topology duality trivial, sending $m_f \to 0$ increases $\nu + 1$

  (after normalising) - true at finite-N & $N \to \infty$ mapping to $\varepsilon\chi$PT)

Correlation Functions I: Individual Eigenvalues

- distribution of the **smallest = 1st eigenvalue** $p_1(y)$:

  - from **gap probability** (cumulative distribution) = probability all $> \lambda$:

$$E(\lambda) \equiv \frac{1}{\mathcal{Z}} \int_{\lambda}^{\infty} \prod_{k=1}^{N} d\lambda_k \lambda_k^{\nu} e^{-N\Sigma^2 \lambda_k} \prod_{f=1}^{N_f} (\lambda_k + m_f^2) \, |\Delta_N(\lambda)|^2$$

$-\partial_\lambda E(\lambda) =$ probability that 1 eigenvalue at λ and all others $\geq \lambda$

Correlation Functions I: Individual Eigenvalues

- distribution of the **smallest = 1st eigenvalue** $p_1(y)$:

  - from **gap probability** (cumulative distribution) = probability all $> \lambda$:

$$E(\lambda) \equiv \frac{1}{\mathcal{Z}} \int_{\lambda}^{\infty} \prod_{k=1}^{N} d\lambda_k \lambda_k^{\nu} e^{-N\Sigma^2 \lambda_k} \prod_{f=1}^{N_f} (\lambda_k + m_f^2) \, |\Delta_N(\lambda)|^2$$

  $-\partial_\lambda E(\lambda) =$ probability that 1 eigenvalue at λ and all others $\geq \lambda$

- **Exercise 1:**

  Derive the distribution of the 1st eigenvalue in the large-N limit:

  1) for quenched $N_f = 0$ with $\nu = 0$ and $\nu = 1$ (hint: $p_1^{(\nu=0)}(x) = \frac{1}{2} x e^{-x^2/4}$ with $x = 2N\Sigma y$ holds even at finite N)

  2) derive $E(x)$ in the large-N limit for arbitrary $N_f > 0$ at $\nu = 0$

- distribution of the **smallest = 1st eigenvalue** $p_1(y)$:

  - from **gap probability** (cumulative distribution) = probability all $> \lambda$:

$$E(\lambda) \equiv \frac{1}{\mathcal{Z}} \int_{\lambda}^{\infty} \prod_{k=1}^{N} d\lambda_k \lambda_k^{\nu} e^{-N\Sigma^2 \lambda_k} \prod_{f=1}^{N_f} (\lambda_k + m_f^2) \, |\Delta_N(\lambda)|^2$$

  $-\partial_\lambda E(\lambda) =$ probability that 1 eigenvalue at λ and all others $\geq \lambda$

- **Exercise 1:**

  Derive the distribution of the 1st eigenvalue in the large-N limit:

  1) for quenched $N_f = 0$ with $\nu = 0$ and $\nu = 1$ (hint: $p_1^{(\nu=0)}(x) = \frac{1}{2} x e^{-x^2/4}$ with $x = 2N\Sigma y$ holds even at finite N)

  2) derive $E(x)$ in the large-N limit for arbitrary $N_f > 0$ at $\nu = 0$

- distributions of the **2nd, 3rd etc. eigenvalue also known** [Damgaard, Nishigaki 00]

Correlation Functions II: Spectral Densities

- recall $\mathcal{Z}_\nu = \int \prod_{k=1}^{N} dy_k \mathcal{P}_{jpdf}$

  with integrand (joint probability distribution function), here using Dirac eigenvalues $y_k = \sqrt{\lambda_k}$

$$\boxed{\mathcal{P}_{jpdf} \equiv \prod_{k=1}^{N} [y_k^{2\nu+1} e^{-N\Sigma^2 y_k^2} \prod_{f=1}^{N_f} m_f^\nu (y_k^2 + m_f^2)] \, |\Delta_N(y^2)|^2}$$

Correlation Functions II: Spectral Densities

- recall $\mathcal{Z}_\nu = \int \prod_{k=1}^{N} dy_k \mathcal{P}_{jpdf}$

  with integrand (joint probability distribution function), here using Dirac eigenvalues $y_k = \sqrt{\lambda_k}$

$$\boxed{\mathcal{P}_{jpdf} \equiv \prod_{k=1}^{N} [y_k^{2\nu+1} e^{-N\Sigma^2 y_k^2} \prod_{f=1}^{N_f} m_f^\nu (y_k^2 + m_f^2)] \, |\Delta_N(y^2)|^2}$$

- 2 definitions of **spectral k-point functions**

  1) $\boxed{\rho_k(x_1, \ldots, x_k) \sim \langle \prod_{j=1}^{k} \mathrm{Tr}\delta(D - x_j) \rangle_{RMT}}$

  2) $\boxed{\rho_k(y_1, \ldots, y_k) \equiv \frac{N!}{\mathcal{Z}(N-k)!} \int dy_{k+1} \ldots dy_N \mathcal{P}_{jpdf}}$

- recall $\mathcal{Z}_\nu = \int \prod_{k=1}^N dy_k \mathcal{P}_{jpdf}$

  with integrand (joint probability distribution function), here using Dirac eigenvalues $y_k = \sqrt{\lambda_k}$

$$\boxed{\mathcal{P}_{jpdf} \equiv \prod_{k=1}^N [y_k^{2\nu+1} e^{-N\Sigma^2 y_k^2} \prod_{f=1}^{N_f} m_f^\nu (y_k^2 + m_f^2)]\, |\Delta_N(y^2)|^2}$$

- 2 definitions of **spectral k-point functions**

  1) $\boxed{\rho_k(x_1, \ldots, x_k) \sim \langle \prod_{j=1}^k \text{Tr}\delta(D - x_j) \rangle_{RMT}}$

  2) $\boxed{\rho_k(y_1, \ldots, y_k) \equiv \frac{N!}{\mathcal{Z}(N-k)!} \int dy_{k+1} \ldots dy_N \mathcal{P}_{jpdf}}$

- **Exercise 2:** they are not the same, why?

  We will use the second definition in the following

Correlation Functions II: Spectral Densities

- recall $\mathcal{Z}_\nu = \int \prod_{k=1}^{N} dy_k \mathcal{P}_{jpdf}$

  with integrand (joint probability distribution function), here using Dirac eigenvalues $y_k = \sqrt{\lambda_k}$

$$\boxed{\mathcal{P}_{jpdf} \equiv \prod_{k=1}^{N} [y_k^{2\nu+1} e^{-N\Sigma^2 y_k^2} \prod_{f=1}^{N_f} m_f^\nu (y_k^2 + m_f^2)] \, |\Delta_N(y^2)|^2}$$

- 2 definitions of **spectral k-point functions**

  1) $\boxed{\rho_k(x_1, \ldots, x_k) \sim \langle \prod_{j=1}^{k} \text{Tr}\delta(D - x_j) \rangle_{RMT}}$

  2) $\boxed{\rho_k(y_1, \ldots, y_k) \equiv \frac{N!}{\mathcal{Z}(N-k)!} \int dy_{k+1} \ldots dy_N \mathcal{P}_{jpdf}}$

- **Exercise 2:** they are not the same, why?

  We will use the second definition in the following

- definition **weight function** for $\mu = 0$

$$\boxed{w(y) \equiv y^{2\nu+1} e^{-N\Sigma^2 y^2}} \geq 0$$

RMT for baryonic and isospin $\mu \neq 0$

$$\mathcal{Z}_\nu \equiv \int dW_1 dW_2 \, \Pi_{f=1}^{N_f} \det \begin{bmatrix} m_f & iW_1 + \mu_f W_2 \\ iW_1^\dagger + \mu_f W_2^\dagger & m_f \end{bmatrix} e^{-N \mathrm{Tr} W_j W_j^\dagger}$$

- **baryonic:** $\mu_f = \mu \, \forall f$: Schur decomposition of 2 off diagonal blocks

$$\boxed{\mathcal{Z}_\nu \sim \int_{\mathbb{C}} \prod_k^N [dz_k^2 \, w(z_k;\mu) \prod_f^{N_f} m_f^\nu (z_k^2 + m_f^2)] \, |\Delta_N(z^2)|^2}$$ [Osborn 04]

complex eigenvalue representation with weight

$$w(z;\mu) = |z|^{2\nu+2} K_\nu \left(\frac{N(1+\mu^2)}{2\mu^2} |z|^2 \right) \exp[\frac{N(1-\mu^2)}{4\mu^2}(z^2 + z^{*\,2})] \geq 0$$

RMT for baryonic and isospin $\mu \neq 0$

$$\mathcal{Z}_\nu \equiv \int dW_1 dW_2 \, \Pi_{f=1}^{N_f} \det \begin{bmatrix} m_f & iW_1 + \mu_f W_2 \\ iW_1^\dagger + \mu_f W_2^\dagger & m_f \end{bmatrix} \mathrm{e}^{-N \mathrm{Tr} W_j W_j^\dagger}$$

- **baryonic:** $\mu_f = \mu \, \forall f$: Schur decomposition of 2 off diagonal blocks

  $$\boxed{\mathcal{Z}_\nu \sim \int_{\mathbb{C}} \prod_k^N [dz_k^2 \, w(z_k;\mu) \prod_f^{N_f} m_f^\nu (z_k^2 + m_f^2)] \, |\Delta_N(z^2)|^2}$$ [Osborn 04]

  complex eigenvalue representation with weight

  $$w(z;\mu) = |z|^{2\nu+2} K_\nu \left(\frac{N(1+\mu^2)}{2\mu^2} |z|^2 \right) \exp[\frac{N(1-\mu^2)}{4\mu^2}(z^2 + z^{*\,2})] \geq 0$$

- the structure is **very similar** to the RMT at $\mu = 0$, can be solved with same methods despite $\prod_f^{N_f}(z^2 + m_f^2)$ being **not real positiv**

RMT for baryonic and isospin $\mu \neq 0$

$$\mathcal{Z}_\nu \equiv \int dW_1 dW_2\, \Pi_{f=1}^{N_f} \det \begin{bmatrix} m_f & iW_1 + \mu_f W_2 \\ iW_1^\dagger + \mu_f W_2^\dagger & m_f \end{bmatrix} \mathrm{e}^{-N \mathrm{Tr} W_j W_j^\dagger}$$

- **baryonic:** $\mu_f = \mu\ \forall f$: Schur decomposition of 2 off diagonal blocks

  $$\boxed{\mathcal{Z}_\nu \sim \int_{\mathbb{C}} \prod_k^N [dz_k^2\, w(z_k;\mu) \prod_f^{N_f} m_f^\nu (z_k^2 + m_f^2)]\ |\Delta_N(z^2)|^2}$$ [Osborn 04]

  complex eigenvalue representation with weight

  $$w(z;\mu) = |z|^{2\nu+2} K_\nu \left(\frac{N(1+\mu^2)}{2\mu^2}|z|^2\right) \exp[\frac{N(1-\mu^2)}{4\mu^2}(z^2 + z^{*\,2})] \geq 0$$

- the structure is **very similar** to the RMT at $\mu = 0$, can be solved with same methods despite $\prod_f^{N_f}(z^2 + m_f^2)$ being **not real positiv**

- **isospin = phase quenched QCD** with $2N_f$ flavours $\mu_f = \pm\mu$:

  $$\boxed{\mathcal{Z}_\nu \sim \int_{\mathbb{C}} \prod_k^N [dz_k^2 w(z_k;\mu) \prod_f^{N_f} m_f^\nu \left|z_k^2 + m_f^2\right|^2]\ |\Delta_N(z^2)|^2}$$ solvable

Main Result All Density Correlations

- Theorem does $N-k$ integrations in k-point function ($\mu = 0, \neq 0$)

$$\boxed{\rho_k(z_1,\ldots,z_k) = \prod_l^k w(z_l)\,\det_{1,\ldots,k}[K_N(z_j,z_l^*)]}$$

$k \times k$ determinant - only large-N kernel $K_N(z_j, z_l^*)$

Main Result All Density Correlations

- Theorem does $N-k$ integrations in k-point function ($\mu=0,\neq 0$)

  $$\boxed{\rho_k(z_1,\ldots,z_k)=\prod_l^k w(z_l)\,\det_{1,\ldots,k}[K_N(z_j,z_l^*)]}$$

  $k\times k$ determinant - only large-N kernel $K_N(z_j,z_l^*)$

- **example $N_f=0$, $\mu=0$:**

  $$\rho_1(\lambda)=w(\lambda)K_N(\lambda,\lambda)=\lambda^\nu e^{-N\Sigma^2\lambda}\sum_{j=0}^{N-1}L_j^\nu(\lambda^2)^2/h_j$$

Main Result All Density Correlations

- Theorem does $N-k$ integrations in k-point function ($\mu = 0, \neq 0$)

  $$\boxed{\rho_k(z_1, \ldots, z_k) = \prod_l^k w(z_l) \det_{1,\ldots,k}[K_N(z_j, z_l^*)]}$$

  $k \times k$ determinant - only large-N kernel $K_N(z_j, z_l^*)$

- **example $N_f = 0$, $\mu = 0$:**

  $$\rho_1(\lambda) = w(\lambda) K_N(\lambda, \lambda) = \lambda^\nu e^{-N\Sigma^2\lambda} \sum_{j=0}^{N-1} L_j^\nu(\lambda^2)^2 / h_j$$

- **Christoffel-Darboux formula** large-N limit easy ($\mu = 0$):

  $$K_N(x, y) = \sum_{j=0}^{N} P_j(x) P_j(y) / h_j \sim \frac{P_N(x)P_{N-1}(y) - P_N(y)P_{N-1}(x)}{x-y}$$

Main Result All Density Correlations

- Theorem does $N-k$ integrations in k-point function ($\mu = 0, \neq 0$)

$$\boxed{\rho_k(z_1,\ldots,z_k) = \prod_l^k w(z_l)\,\mathrm{det}_{1,\ldots,k}[K_N(z_j,z_l^*)]}$$

$k \times k$ determinant - only large-N kernel $K_N(z_j, z_l^*)$

- **example $N_f = 0$, $\mu = 0$:**

$$\rho_1(\lambda) = w(\lambda)K_N(\lambda,\lambda) = \lambda^\nu e^{-N\Sigma^2\lambda}\sum_{j=0}^{N-1} L_j^\nu(\lambda^2)^2/h_j$$

- **Christoffel-Darboux formula** large-N limit easy ($\mu = 0$):

$$K_N(x,y) = \sum_{j=0}^{N} P_j(x)P_j(y)/h_j \sim \frac{P_N(x)P_{N-1}(y)-P_N(y)P_{N-1}(x)}{x-y}$$

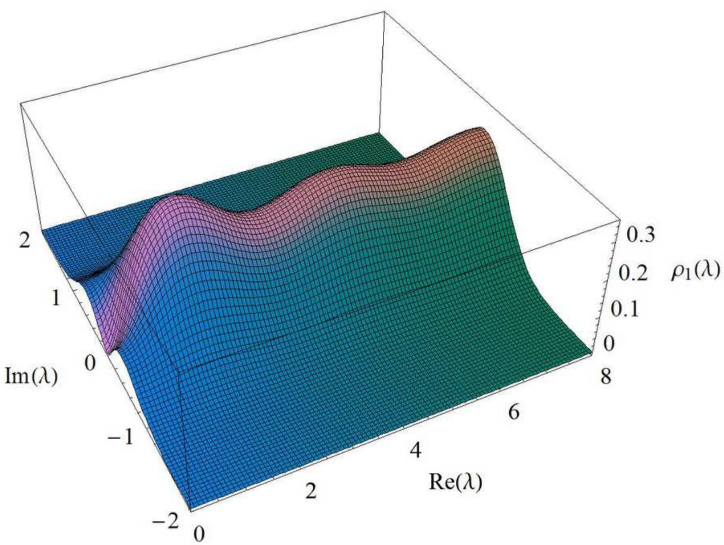
- **Bessel-asymptotic** of Laguerre polynomials gives

$$\boxed{\lim_{N\to\infty}\rho(x=\lambda N\Sigma) \sim \frac{x}{2}\left(J_\nu(x)^2 - J_{\nu-1}(x)J_{\nu+1}(x)\right)} \quad \mu = 0$$

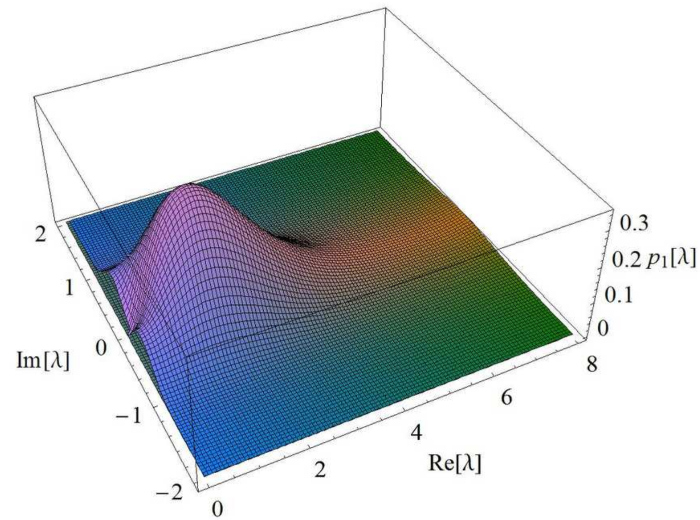
$$\boxed{\rho(\xi) = \frac{1}{\alpha^2}|\xi|^2 K_\nu\left(\frac{|\xi|^2}{2\alpha^2}\right)\mathrm{e}^{+\frac{1}{4\alpha^2}(\xi^2+\xi^{*\,2})}\int_0^1 dt\,t\,\mathrm{e}^{-t^2\alpha^2}J_\nu(\xi)J_\nu(t\xi^*)} \quad \mu \neq 0$$

Quenched Density & p_1 for Small (top) and Large μ_B (down)

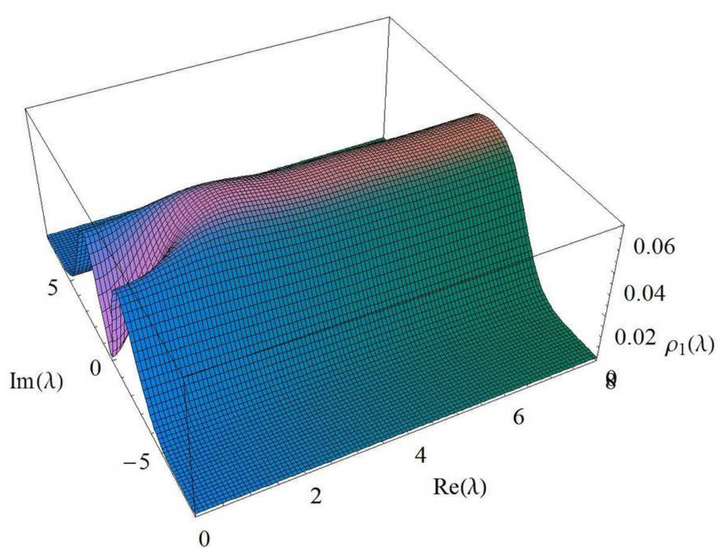
One−point density $\rho_1(\lambda)$, for $\nu = 0$ and $\mu = 0.1$ $(\alpha = 0.591)$



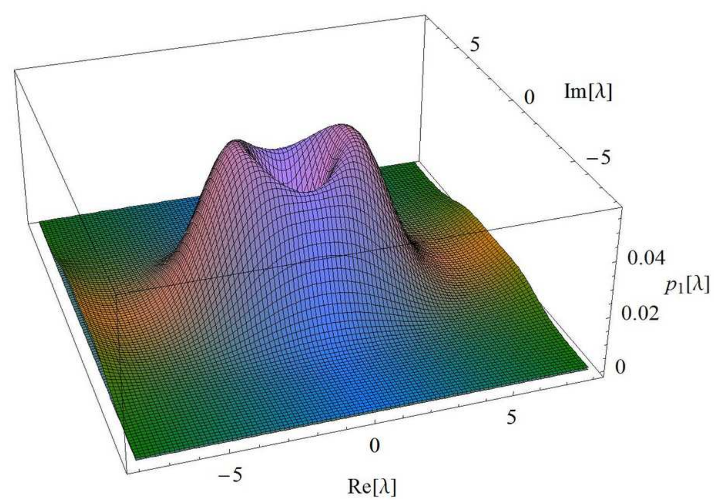
The distribution of the first eigenvalue $p_1(\lambda)$ for $\nu = 0$ and $\mu = 0.1$ $(\alpha = 0.591)$



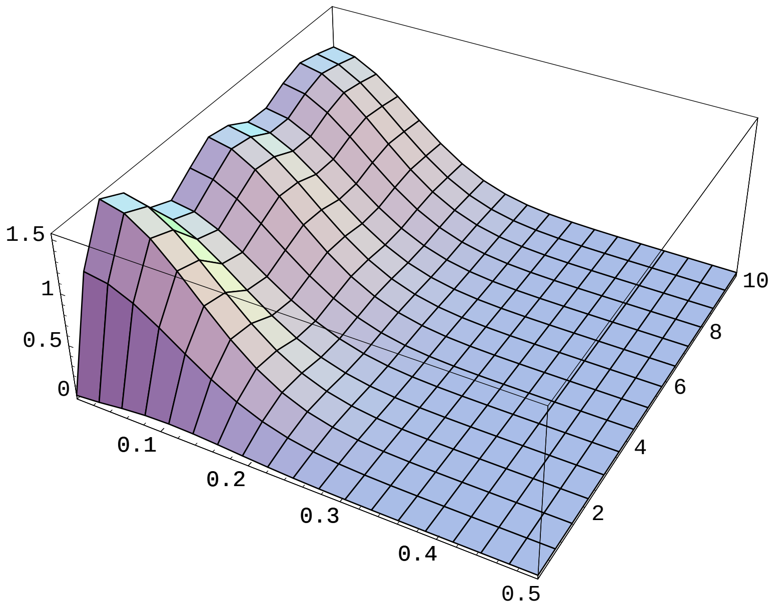
One−point density $\rho_1(\lambda)$, for $\nu = 0$ and $\mu = 0.3$ $(\alpha = 1.683)$

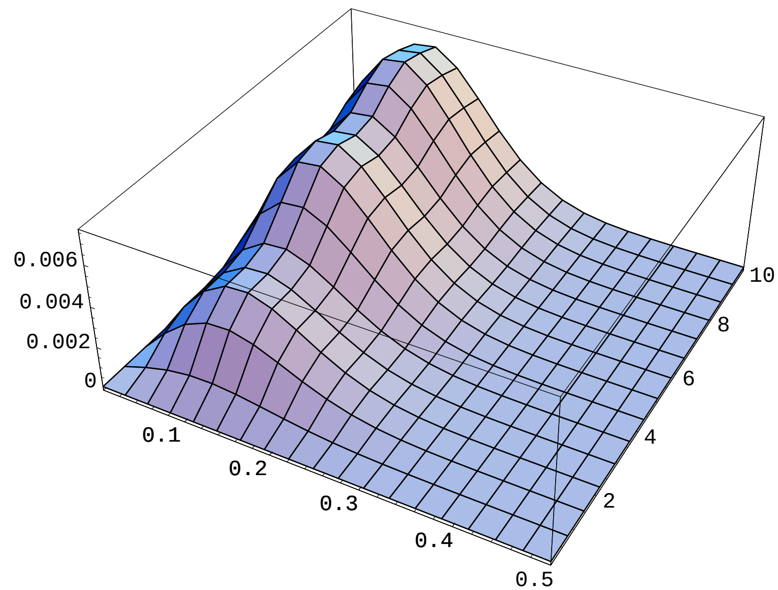


The distribution of the first eigenvalue $p_1(\lambda)$ for $\nu = 0$ and $\mu = 0.3$ $(\alpha = 1.683)$



Unquenched Complex Valued Density for $N_f = 1$





- Unquenched density of complex Dirac eigenvalues for $N_f = 1$: real- (left) & imaginary part (right) [Osborn, Splittorff, Verbaarschot, G.A. 05]
- verified for **small volumes** using reweighting [Wettig]

Relation to $\varepsilon\chi$PT: Densities

- in the large-N limit mapping to $\varepsilon\chi$PT we obtain

  \- **quenched** $\mu = 0, \neq$: limiting Kernel K_S

$$\boxed{K_S(x, y^*) \sim \mathcal{Z}_\nu^{(N_f=2)}(ix, iy^*)}$$

  \- **unquenched** $\mu = 0$:

$$\boxed{K_S(x, y) \sim \mathcal{Z}_\nu^{(N_f+2)}(\{m_f\}, ix, iy) \;/\; \mathcal{Z}_\nu^{(N_f)}(\{m_f\})}$$

  \- unquenched & phase quenched $\mu \neq 0$:

  slightly more complicated, with 1 and 2 flavour $\mathcal{Z}$'s as building blocks

  [GA, Damgaard 98; GA, Osborn, Splittorff, Verbaarschot 04]

The Orthogonal Polynomial Method

- the Vandermonde trick for arbitrary polynomials

$$\Delta_N(\lambda) = \prod_{k>l}(\lambda_k - \lambda_j) = \begin{vmatrix} 1 & \lambda_1 & \dots \\ 1 & \lambda_2 & \dots \\ \dots & & \end{vmatrix} = \begin{vmatrix} P_0(\lambda_1) & P_1(\lambda_1) & \dots \\ P_0(\lambda_2) & P_1(\lambda_2) & \dots \\ \dots & & \end{vmatrix}$$

The Orthogonal Polynomial Method

- the Vandermonde trick for arbitrary polynomials

$$\Delta_N(\lambda) = \prod_{k>l}(\lambda_k - \lambda_j) = \begin{vmatrix} 1 & \lambda_1 & \dots \\ 1 & \lambda_2 & \dots \\ \dots & & \end{vmatrix} = \begin{vmatrix} P_0(\lambda_1) & P_1(\lambda_1) & \dots \\ P_0(\lambda_2) & P_1(\lambda_2) & \dots \\ \dots & & \end{vmatrix}$$

- choosing **Orthogonal Polynomials** wrt correct weight useful

$$\int_{\mathbb{C}} d^2z w(z;\mu;m_f)P_k(z)P_l(z)^* = h_k\delta_{kl}$$

The Orthogonal Polynomial Method

- the Vandermonde trick for arbitrary polynomials

$$\Delta_N(\lambda) = \prod_{k>l}(\lambda_k - \lambda_j) = \begin{vmatrix} 1 & \lambda_1 & \dots \\ 1 & \lambda_2 & \dots \\ \dots & & \end{vmatrix} = \begin{vmatrix} P_0(\lambda_1) & P_1(\lambda_1) & \dots \\ P_0(\lambda_2) & P_1(\lambda_2) & \dots \\ \dots & & \end{vmatrix}$$

- choosing **Orthogonal Polynomials** wrt correct weight useful

$$\int_{\mathbb{C}} d^2 z w(z;\mu;m_f) P_k(z) P_l(z)^* = h_k \delta_{kl}$$

- for $\mu = 0$ weight $w(\lambda) = \lambda^\nu e^{-\lambda}$ Laguerre polynomials

- for $\mu \neq 0$ Laguerre polynomials on $\mathbb{C}$ (weight exp $\cdot$ K-Bessel)

e.g.[Bender, G.A. 10]

- for $\mu = 0$ weight $w(\lambda) = \lambda^\nu(\lambda^2 + m^2)e^{-\lambda}$ det of Laguerre's

- the Vandermonde trick for arbitrary polynomials

$$\Delta_N(\lambda) = \prod_{k>l}(\lambda_k - \lambda_j) = \begin{vmatrix} 1 & \lambda_1 & \dots \\ 1 & \lambda_2 & \dots \\ \dots & & \end{vmatrix} = \begin{vmatrix} P_0(\lambda_1) & P_1(\lambda_1) & \dots \\ P_0(\lambda_2) & P_1(\lambda_2) & \dots \\ \dots & & \end{vmatrix}$$

- choosing **Orthogonal Polynomials** wrt correct weight useful

$$\int_{\mathbb{C}} d^2z w(z;\mu;m_f) P_k(z) P_l(z)^* = h_k \delta_{kl}$$

- for $\mu = 0$ weight $w(\lambda) = \lambda^\nu e^{-\lambda}$ Laguerre polynomials

- for $\mu \neq 0$ Laguerre polynomials on $\mathbb{C}$ (weight exp $\cdot$ K-Bessel)

e.g.[Bender, G.A. 10]

- for $\mu = 0$ weight $w(\lambda) = \lambda^\nu (\lambda^2 + m^2) e^{-\lambda}$ det of Laguerre's

- **two tricks**

- inclusion of **mass terms into larger Vandermonde**

- write $|\Delta_N(z)|^2$ as determinant of Kernel

RMT Computation of Partition Functions

- **the partition function**

  - quenched: $\mathcal{Z}_{\nu,RMT} \sim \int_{\mathbb{C}} \prod_k^N dz_k^2 w(z_k;\mu) |\det[P_k]|^2 \sim N! \prod h_k$

RMT Computation of Partition Functions

- **the partition function**

  - quenched: $\mathcal{Z}_{\nu,RMT} \sim \int_{\mathbb{C}} \prod_k^N dz_k^2 w(z_k;\mu) |\det[P_k]|^2 \sim N! \prod h_k$

  - unquenched:

  $N_f = 1$: $\prod_k (\lambda_k + m^2) \Delta_N(\{\lambda\}) = \Delta_{N+1}(\{\lambda\})$ with $\lambda_{N+1} = -m^2$

- **the partition function**

  - quenched: $\mathcal{Z}_{\nu,RMT} \sim \int_{\mathbb{C}} \prod_k^N dz_k^2 w(z_k;\mu)|\det[P_k]|^2 \sim N! \prod h_k$

  - unquenched:

  $N_f = 1$: $\prod_k(\lambda_k + m^2)\Delta_N(\{\lambda\}) = \Delta_{N+1}(\{\lambda\})$ with $\lambda_{N+1} = -m^2$

  N_f $\boxed{\prod_f \prod_k(\lambda_k + m_f^2)\Delta_N(\{\lambda\}) = \Delta_{N+N_f}(\{\lambda\})/\Delta(\{m_f^2\})}$ with $\lambda_{N+f} = -m_f^2$

- **the partition function**

  - quenched: $\mathcal{Z}_{\nu,RMT} \sim \int_{\mathbb{C}} \prod_k^N dz_k^2 w(z_k;\mu)|\det[P_k]|^2 \sim N! \prod h_k$

  - unquenched:

  $N_f = 1$: $\prod_k(\lambda_k + m^2)\Delta_N(\{\lambda\}) = \Delta_{N+1}(\{\lambda\})$ with $\lambda_{N+1} = -m^2$

  N_f $\boxed{\prod_f \prod_k(\lambda_k + m_f^2)\Delta_N(\{\lambda\}) = \Delta_{N+N_f}(\{\lambda\})/\Delta(\{m_f^2\})}$ with $\lambda_{N+f} = -m_f^2$

  $\rightarrow$ replace Δ_N's by detP_k's and use orthogonality:

$$\boxed{\mathcal{Z}_{\nu,RMT}^{(N_f)} = \left\langle \prod_f \prod_k(\lambda_k + m_f^2)\right\rangle \sim \det[L_{N+j}^{(\nu)}(-m_f^2)]/\Delta(\{m_f^2\})}$$

  expectation value of characteristic polynomials

RMT Computation of Partition Functions

- **the partition function**

  - quenched: $\mathcal{Z}_{\nu,RMT} \sim \int_{\mathbb{C}} \prod_k^N dz_k^2 w(z_k;\mu)|\det[P_k]|^2 \sim N! \prod h_k$

  - unquenched:

  $N_f = 1$: $\prod_k (\lambda_k + m^2)\Delta_N(\{\lambda\}) = \Delta_{N+1}(\{\lambda\})$ with $\lambda_{N+1} = -m^2$

  N_f $\boxed{\prod_f \prod_k (\lambda_k + m_f^2)\Delta_N(\{\lambda\}) = \Delta_{N+N_f}(\{\lambda\})/\Delta(\{m_f^2\})}$ with $\lambda_{N+f} = -m_f^2$

  $\rightarrow$ replace Δ_N's by detP_k's and use orthogonality:

  $$\boxed{\mathcal{Z}_{\nu,RMT}^{(N_f)} = \left\langle \prod_f \prod_k (\lambda_k + m_f^2) \right\rangle \sim \det[L_{N+j}^{(\nu)}(-m_f^2)]/\Delta(\{m_f^2\})}$$

  expectation value of characteristic polynomials

- **microscopic limit** $L_{N+j}^{(\nu)}(-m_f^2) \rightarrow I_\nu(\hat{m}_f)$ gives again $\mathcal{Z}_{\varepsilon\chi\text{PT}}$

- the **same for** $\mu = 0$ **&** $\mu \neq 0$

Derivation for k-Point Densities

- the kernel of **orthonormal** polynomials satisfies

$$\boxed{K_N(z,v^*) = \sum_{j=0}^{N-1} P_j(z)P_j(v^*)/h_k}$$

1. $\int dz w(z) K_N(z,v^*) = 1\ ,\quad \int dz w(z) K_N(z,z^*) = N$
2. $\int dz w(z) K_N(u,z^*) K_N(z,v^*) = K_N(u,v^*)$

Derivation for k-Point Densities

- the kernel of **orthonormal** polynomials satisfies

$$\boxed{K_N(z,v^*) = \sum_{j=0}^{N-1} P_j(z)P_j(v^*)/h_k}$$

1. $\int dz w(z) K_N(z,v^*) = 1\ ,\quad \int dz w(z) K_N(z,z^*) = N$
2. $\int dz w(z) K_N(u,z^*) K_N(z,v^*) = K_N(u,v^*)$

- the integrand (jpdf) can be written as a single determinant

$$\boxed{|\Delta_N(z)|^2 = \det[P_k(z_l)]\det[P_k(z_l^*)] = \det_{N\times N}[K_N(z_l,z_k^*)]\prod_j h_j}$$

Derivation for k-Point Densities

- the kernel of **orthonormal** polynomials satisfies

$$\boxed{K_N(z,v^*) = \sum_{j=0}^{N-1} P_j(z)P_j(v^*)/h_k}$$

1. $\int dz w(z) K_N(z,v^*) = 1\ , \quad \int dz w(z) K_N(z,z^*) = N$
2. $\int dz w(z) K_N(u,z^*) K_N(z,v^*) = K_N(u,v^*)$

- the integrand (jpdf) can be written as a single determinant

$$\boxed{|\Delta_N(z)|^2 = \det[P_k(z_l)]\det[P_k(z_l^*)] = \det_{N\times N}[K_N(z_l,z_k^*)]\prod_j h_j}$$

- **Integration Theorem Dyson:**

$$\boxed{\int dz_N w(z_N)\det_{k\times k}[K_N(z_i,z_j^*)] = (N-k+1)\det_{(k-1)\times(k-1)}[K_N(z_i,z_j^*)]}$$

Derivation for k-Point Densities

- the kernel of **orthonormal** polynomials satisfies

$$\boxed{K_N(z,v^*) = \sum_{j=0}^{N-1} P_j(z)P_j(v^*)/h_k}$$

1. $\int dz w(z) K_N(z,v^*) = 1 \; , \quad \int dz w(z) K_N(z,z^*) = N$
2. $\int dz w(z) K_N(u,z^*) K_N(z,v^*) = K_N(u,v^*)$

- the integrand (jpdf) can be written as a single determinant

$$\boxed{|\Delta_N(z)|^2 = \det[P_k(z_l)] \det[P_k(z_l^*)] = \det_{N\times N}[K_N(z_l,z_k^*)] \prod_j h_j}$$

- **Integration Theorem Dyson:**

$$\boxed{\int dz_N w(z_N) \det_{k\times k}[K_N(z_i,z_j^*)] = (N-k+1) \det_{(k-1)\times(k-1)}[K_N(z_i,z_j^*)]}$$

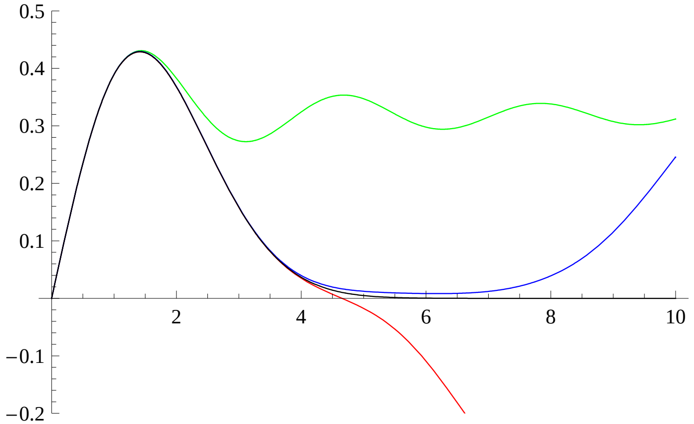
- **density correlation functions**

$$\boxed{\rho_k(z_1,\ldots,z_k) \sim \int dz_{k+1}\ldots dz_N \prod_j w(z_j) \det_{N\times N}[K_N] = \det_{k\times k}[K_N]}$$

Relation to $\varepsilon\chi$PT: First Eigenvalue?

- Knowing all **density $k$point correlation functions** should allow to determine all **individual eigenvalue distribution**?!
- or vice versa: $\rho(\lambda) = \sum_k p_k(\lambda)$
- **example gap:** $\boxed{E(r) = \sum_k \int \cdot\cdot \int \rho_k} = 1 - \int_0^r \rho(z) + \int \int_0^r \rho(z,u) + \ldots$

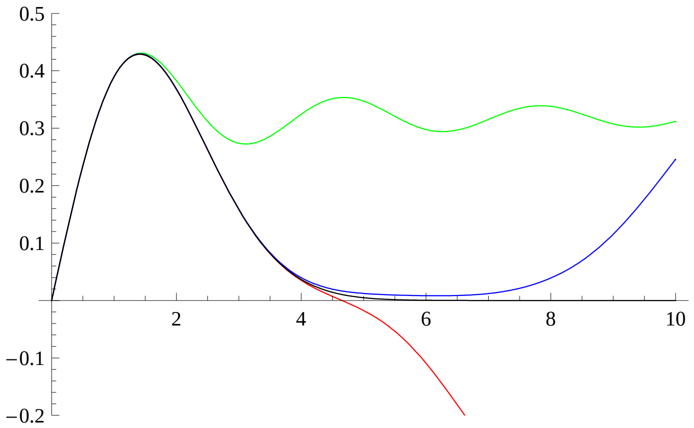
rapid convergence: on $\mathbb{R}$ [Damgaard, G.A.] and $\mathbb{C}$ [ABSW]



Relation to $\varepsilon\chi$PT: First Eigenvalue?

- Knowing all **density $k$point correlation functions** should allow to determine all **individual eigenvalue distribution**?!
- or vice versa: $\rho(\lambda) = \sum_k p_k(\lambda)$
- **example gap:** $\boxed{E(r) = \sum_k \int \cdots \int \rho_k} = 1 - \int_0^r \rho(z) + \int \int_0^r \rho(z,u) + \ldots$

  rapid convergence: on $\mathbb{R}$ [Damgaard, G.A.] and $\mathbb{C}$ [ABSW]



- Compute several k-point densities from $\varepsilon\chi$PT $\rightarrow$ reconstruct 1st eigenvalue
- for **complex eigenvalues** a more direct computation of $p_1(r)$ is difficult also from RMT

Correlation Functions III: Average Phase factor

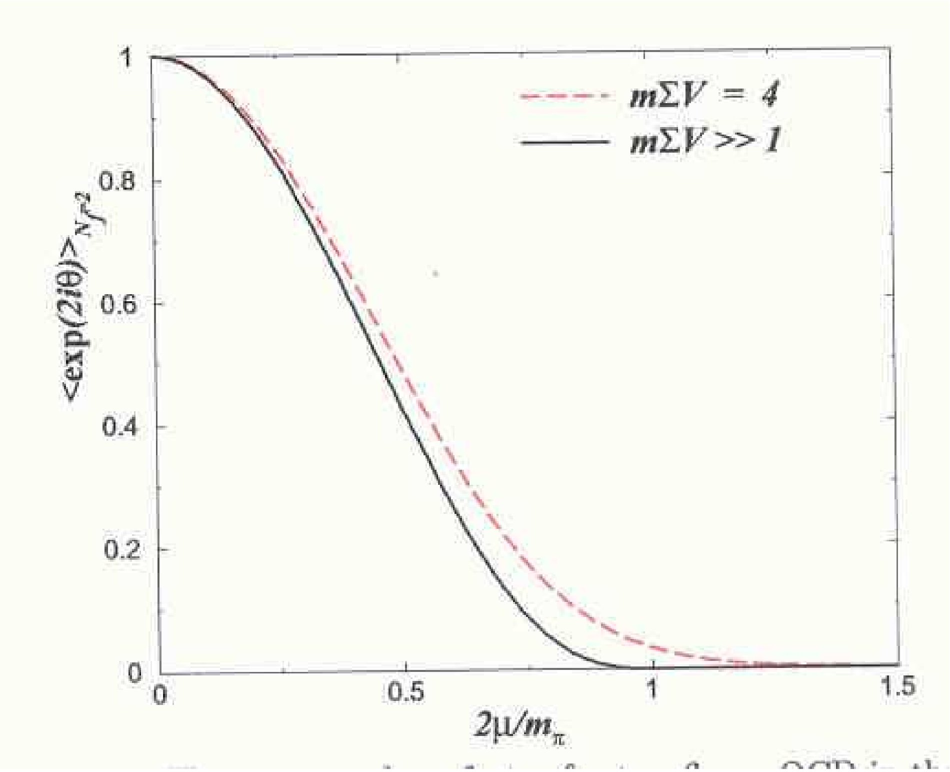
- define the **average phase factor** [Splittorff, Verbaarschot 06]

$$\boxed{\left\langle e^{2i\alpha}\right\rangle_{1+1^*} = \left\langle \frac{\det[D(\mu)+m]}{\det[D(\mu)+m]^*}\right\rangle_{1+1^*} = \frac{\mathcal{Z}_\nu^{N_f=1+1}}{\mathcal{Z}_\nu^{N_f=1+1^*}}} = \frac{\langle e^{2i\alpha}\left|\det[D(\mu)+m]\right|^2\rangle_{quench}}{\langle\left|\det[D(\mu)+m]\right|^2\rangle_{quench}}$$

- define the **average phase factor** [Splittorff, Verbaarschot 06]

$$\boxed{\left\langle e^{2i\alpha}\right\rangle_{1+1^*} = \left\langle \frac{\det[D(\mu)+m]}{\det[D(\mu)+m]^*}\right\rangle_{1+1^*} = \frac{\mathcal{Z}_\nu^{N_f=1+1}}{\mathcal{Z}_\nu^{N_f=1+1^*}}} = \frac{\langle e^{2i\alpha}\left|\det[D(\mu)+m]\right|^2\rangle_{quench}}{\langle\left|\det[D(\mu)+m]\right|^2\rangle_{quench}}$$

- both partition functions know: $\boxed{\left\langle e^{2i\alpha}\right\rangle_{1+1^*} = \frac{I_0^2(m)-I_1^2(m)}{2e^{2\mu^2}\int_0^1 dt\, t I_0^2(mt)}}$



Making Life Easy for the Lattice: imaginary μ

- F **without complex** Dirac eigenvalues:

  – Two-Matrix Model with 2 sets of Dirac's with **imaginary** μ
  $\boxed{D + \mathbf{i}\mu_{1,2}\gamma_0}$

  – correlations functions of **2 sets of real eigenvalues** $\{x\}_k, \{y\}_\ell$

  – unquenched simulations possible $N_f = N_1 + N_2$

  – analytic results obtained for "**partial quenching**"

Making Life Easy for the Lattice: imaginary μ

- F **without complex** Dirac eigenvalues:

  – Two-Matrix Model with 2 sets of Dirac's with **imaginary** μ

  $\boxed{D + \mathbf{i}\mu_{1,2}\gamma_0}$

  – correlations functions of **2 sets of real eigenvalues** $\{x\}_k, \{y\}_\ell$

  – unquenched simulations possible $N_f = N_1 + N_2$

  – analytic results obtained for "**partial quenching**"

$$\boxed{\mathcal{Z}_\nu \sim \int d\Phi d\Psi \; e^{-N\mathrm{Tr}\left(\Phi^\dagger\Phi + \Psi^\dagger\Psi\right)} \prod_{f=1}^{N_1} \det[D(i\mu_1)] \prod_{f=1}^{N_2} \det[D(i\mu_2)]}$$

with $D(i\mu_{1,2}) = \begin{pmatrix} m_f & i\Phi + i\mu_{1,2}\Psi \\ i\Phi^\dagger + i\mu_{1,2}\Psi^\dagger & m_f \end{pmatrix}$

Φ and Ψ are complex rectangular matrices of size $N \times (N + \nu)$,

- F **without complex** Dirac eigenvalues:

  – Two-Matrix Model with 2 sets of Dirac's with **imaginary** μ $\boxed{D + \mathbf{i}\mu_{1,2}\gamma_0}$

  – correlations functions of **2 sets of real eigenvalues** $\{x\}_k, \{y\}_\ell$

  – unquenched simulations possible $N_f = N_1 + N_2$

  – analytic results obtained for "**partial quenching**"

$$\boxed{\mathcal{Z}_\nu \sim \int d\Phi d\Psi\; e^{-N\mathrm{Tr}\left(\Phi^\dagger\Phi+\Psi^\dagger\Psi\right)} \prod_{f=1}^{N_1} \det[D(i\mu_1)] \prod_{f=1}^{N_2} \det[D(i\mu_2)]}$$

with $D(i\mu_{1,2}) = \begin{pmatrix} m_f & i\Phi + i\mu_{1,2}\Psi \\ i\Phi^\dagger + i\mu_{1,2}\Psi^\dagger & m_f \end{pmatrix}$

Φ and Ψ are complex rectangular matrices of size $N \times (N+\nu)$,

- obtained densities $\rho_{k,\ell}(\{x\}_k, \{y\}_\ell)$ and some of their individual eigenvalues $P_{k,\ell}$ [G.A., Damgaard, Osborn, Splittorff 06; G.A., Damgaard]

Summary: Easy RMT

- my favorite tool:

  orthogonal polynomials (real and complex eigenvalues):

- go to eigenvalues basis & express jpdf as det of OP or K_N

- **Integration Theorem** $\Rightarrow$ all ρ_k & all p_k

- Large-N limit of OP: **universal** for non-Gauß' weight

  – asymptotic behavior of class OP known: Laguerre $\rightarrow$ Bessel

- **all 3 RMT's for χSB classes $+\,\mu$ now solved for ρ_k:**

  W matrix elements in $\mathbb{R}, \mathbb{C}, \mathbb{H}$

  [Osborn 04, + Splittorff, Verbaarschot, GA; GA 05; Phillips, Sommers, GA.09+10]

- other ways of solving RMT possible: susy, replicas, ...

Summary: Easy RMT

- my favorite tool:

  orthogonal polynomials (real and complex eigenvalues):

- go to eigenvalues basis & express jpdf as det of OP or K_N
- **Integration Theorem** $\Rightarrow$ all ρ_k & all p_k
- Large-N limit of OP: **universal** for non-Gauß' weight

  – asymptotic behavior of class OP known: Laguerre $\rightarrow$ Bessel

- **all 3 RMT's for χSB classes $+\mu$ now solved for ρ_k:**

  W matrix elements in $\mathbb{R}, \mathbb{C}, \mathbb{H}$

  [Osborn 04, + Splittorff, Verbaarschot, GA; GA 05; Phillips, Sommers, GA.09+10]

- other ways of solving RMT possible: susy, replicas, ...

◦ NEW: corrections from finite lattice spacing: **Wilson chiral RMT**

The effect of lattice spacing: Wilson χPT

- **Symanzik's O(a) improvement:**

  close to the continuum discrete lattice quantum field theory can be described by an effective, *continuum* field theory

  $S_{eff} = S_0 + aS_1 + a^2 S_2 + \dots$

- applies to the effective field theory: Wilson χPT[Sharpe, Singleton 98] :

$$\begin{aligned} S &= \frac{1}{2} V\Sigma \mathrm{Tr}(U + U^\dagger) - a^2 V W_8 \mathrm{Tr}(U^2 + U^\dagger\, 2) \\ &\quad - a^2 V W_6 [\mathrm{Tr}(U + U^\dagger)]^2 - a^2 V W_7 [\mathrm{Tr}(U - U^\dagger)]^2 \end{aligned}$$

  = static part of action to O(a^2) with counting rule $a^2 V$

  & **new LEC:** $W_6,\ W_7,\ W_8$ (not all indep. for $N_f = 2$)

- same strategy: ε-regime (with our without RMT approach)

  - compute density and smallest eigenvalue LEC's

  - determine LEC from fit to Lattice data

Wilson $\varepsilon\chi$PT & Wilson RMT

- Wilson $\varepsilon\chi$PT with above S_{eff}, or

- Wilson RMT $D_W = \begin{bmatrix} aA & iW \\ iW\dagger & aB \end{bmatrix}$ [GA, Nagao 11]

  with **additional Hermitian matrices** A, B, Gaussian distribution

- $\boxed{D_W^\dagger = \gamma_5 D_W \gamma_5 \neq D_W}$ non-Hermitian operator

  $\Rightarrow$ define **Hermitian operator** $D_5 \equiv \gamma_5 D_W$

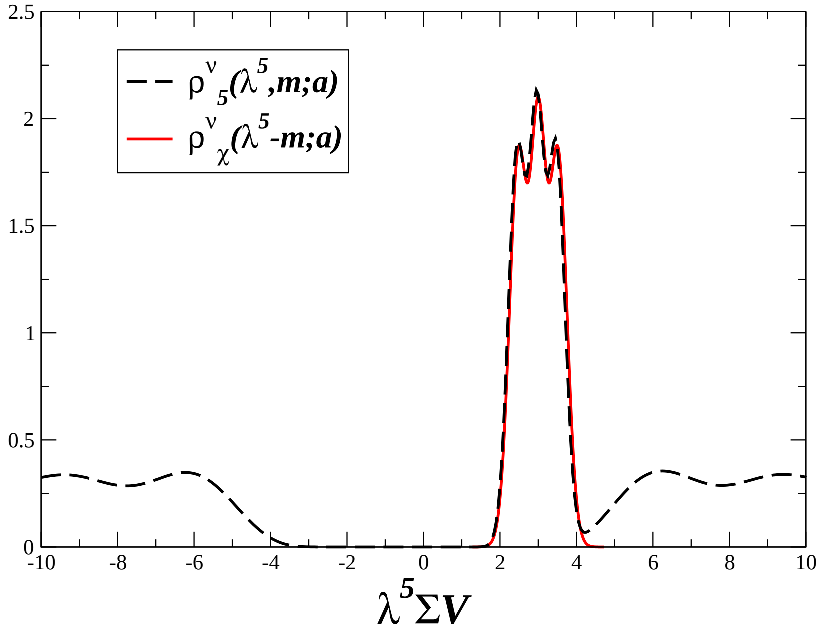
- problems at $a \neq 0$:

  - D_W non-Hermitian

  - concept on **topology no longer well defined,** $\exists$ several related densities of D_5, real modes of D_W, chirality density

  - ν **zero modes get smeared out** and decouple from rest of spectrum

First Wilson results



details:

[Damgaard, Splittorff, Verbaarschot 10, + G.A. 11]

- **partition function** at equal mass (same from $\varepsilon\chi$PT or RMT):

$\boxed{Z_\nu^{(N_f)} = \det[Z^{(N_f)}_{\nu+j-i}]}$ determinant of single flavour as before:

$$\underline{Z_\nu^{(N_f=1)} = \int_{-\pi}^{\pi} d\theta e^{i\nu\theta} \exp[mV\Sigma\cos(\theta) - 2a^2VW_8]} \to I_\nu(\hat{m})$$

- for $W_6 = W_7 = 0$ (switch on: Gaussian integrals over Z)

More literature

- *The Oxford Handbook of Random Matrix Theory*, Oxford University Press 2011, 952 pages, Editors: G.A., J. Baik, P. Di Francesco,

  e.g. chapter 32: J. Verbaarschot, ”Applications of Random Matrix Theory to QCD”, arXiv:0910.4134v1 [hep-th]

- G. A., P.H. Damgaard, K. Splittorff, J.J.M. Verbaarschot arXiv:1012.0752 ”Spectrum of the Wilson Dirac Operator at Finite Lattice Spacings”

- G.A., T. Nagao, Random Matrix Theory for the Hermitian Wilson Dirac Operator and the chGUE-GUE Transition, arXiv:1108.3035v1 [math-ph]