Random matrices and the epsilon-regime at finite density

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Blockcourse on Aspects of QCD at Finite Density

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- **Road map:**
  QCD $\longrightarrow$ low energy: effective QFT = chiral perturbation theory (XPT)
  $\longrightarrow$ epsilon-regime of XPT $\Leftrightarrow$ Random Matrix Theory (RMT)

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- **effect of chemical potential** for baryonic, isospin, imaginary μ
- **What can we compute - despite sign problem?**
  - partition function
  - spectral density, distribution of smallest eigenvalue, average phase
  as functions of: Σ, F, χSB pattern, gauge field topology = ν zero modes of D (index Theorem)

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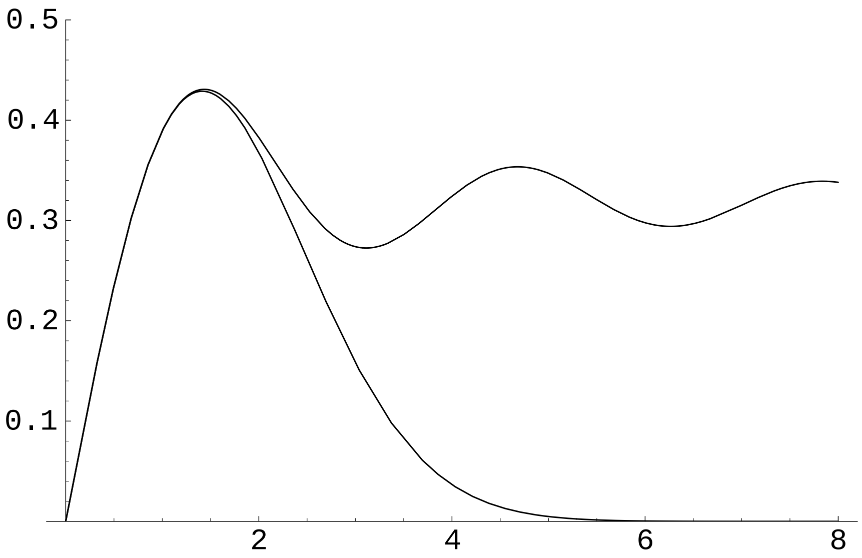
- **What cannot be computed?**
  - actual value of Σ or F in **QCD**
  - chiral symmetry broken or not: $\beta(N_f)$

Example I: Lattice vs spectrum of real Dirac eigenvalues



density $\rho_D(\lambda) \equiv \langle \mathrm{Tr}\,\delta(D-\lambda) \rangle_{\mathbf{QCD}}$

$\lim \frac{1}{V\Sigma}\rho(x) = \frac{x}{2}[J_0(x)^2 + J_1(x)^2]$

1st eigenvalue $p_1(x) = \frac{x}{2}e^{-x^2/4}$

at $\nu = 0$, with $\lambda V \Sigma = x$

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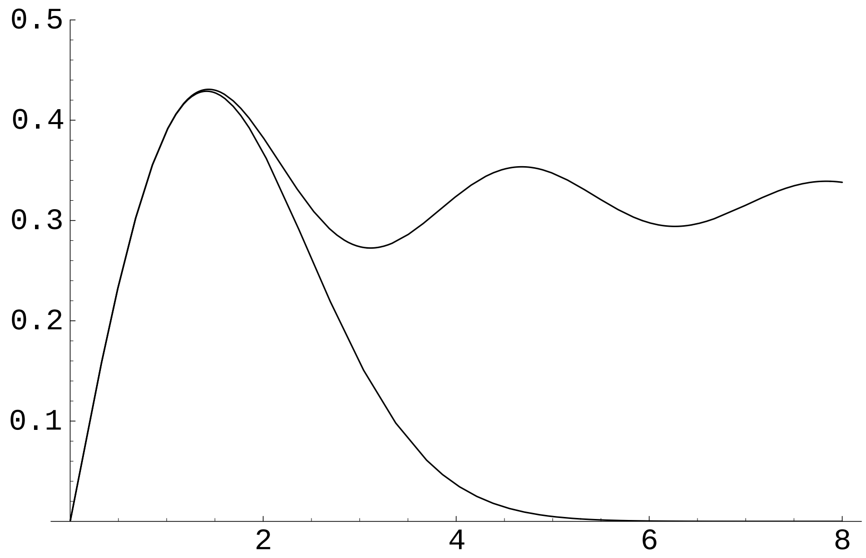

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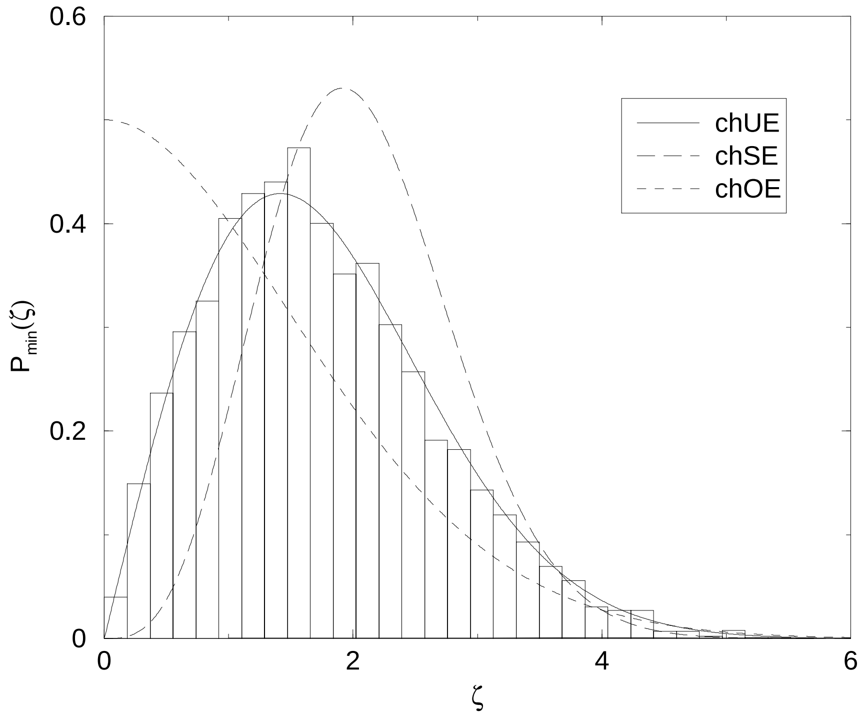

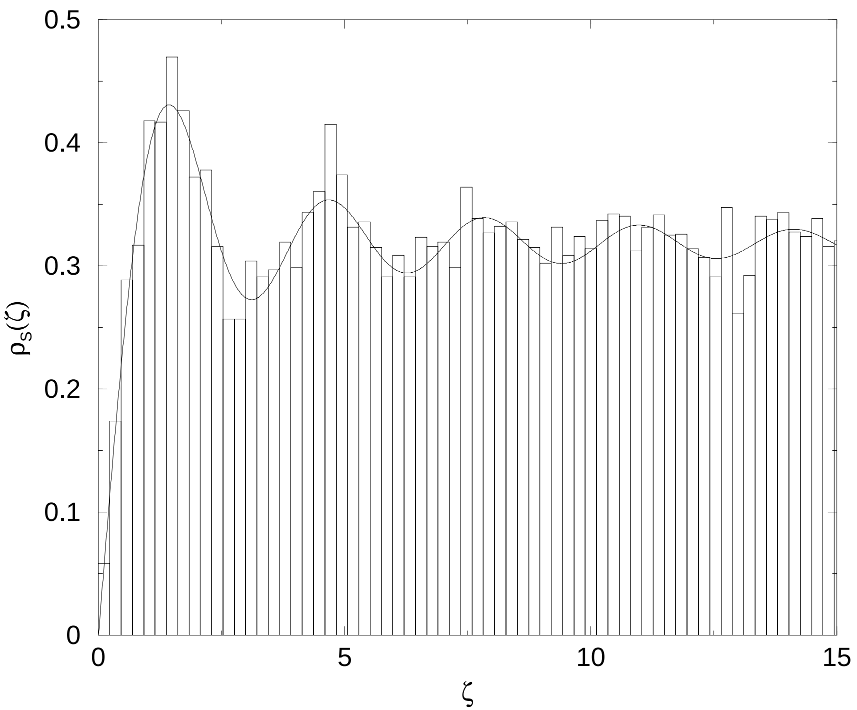

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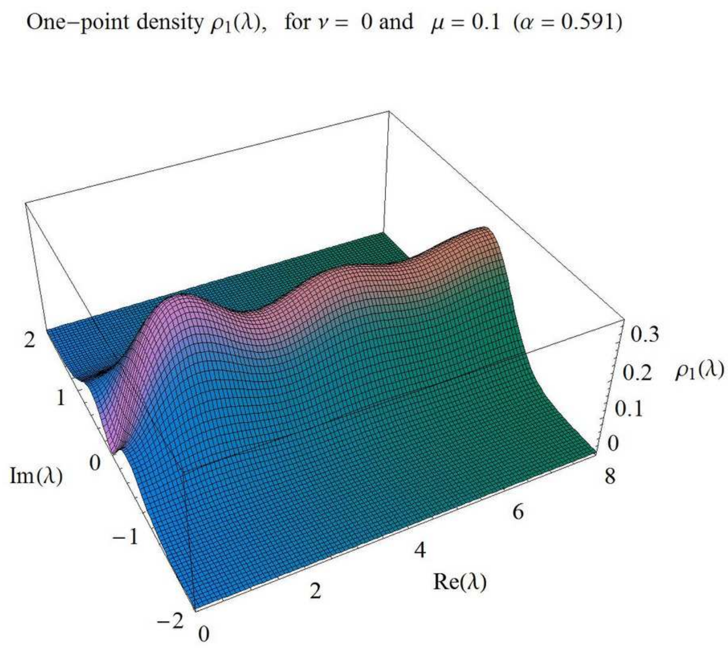

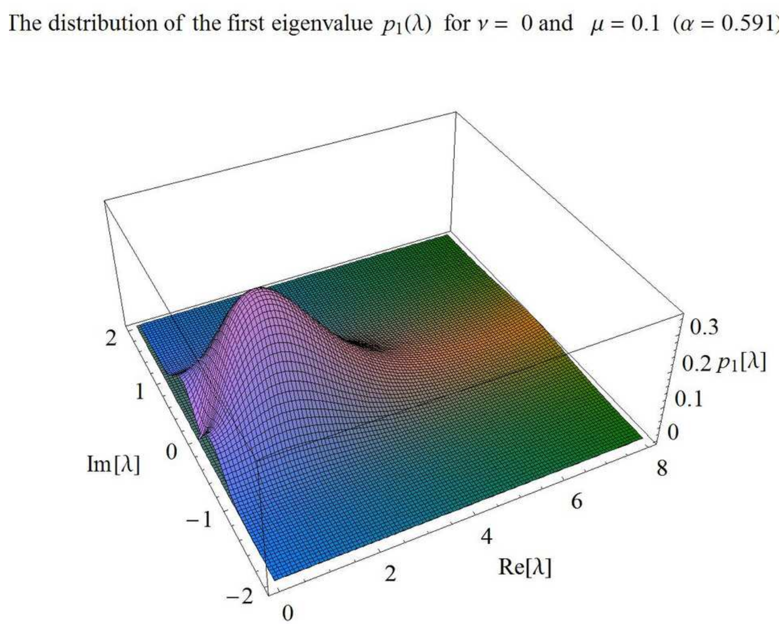

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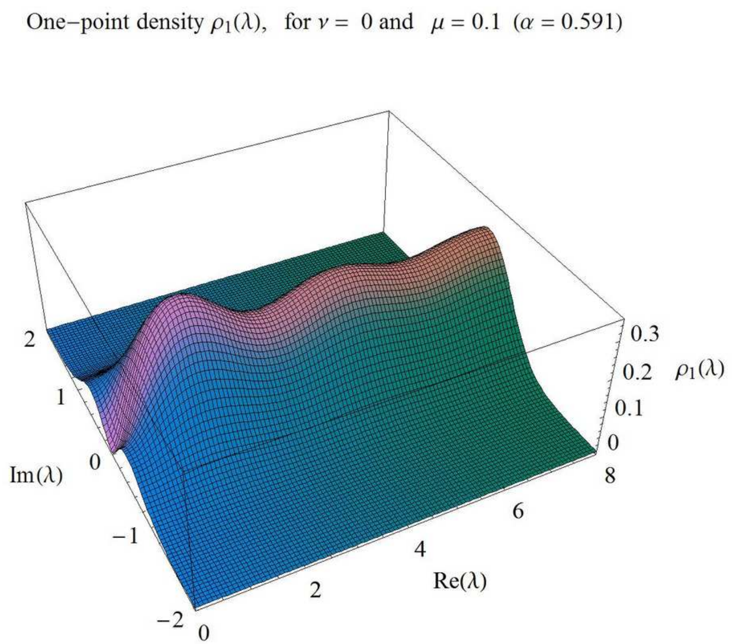
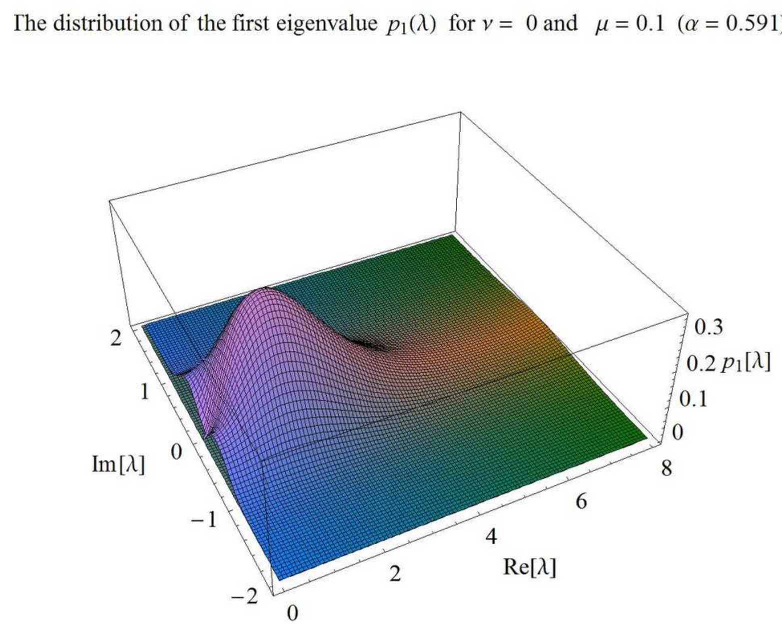
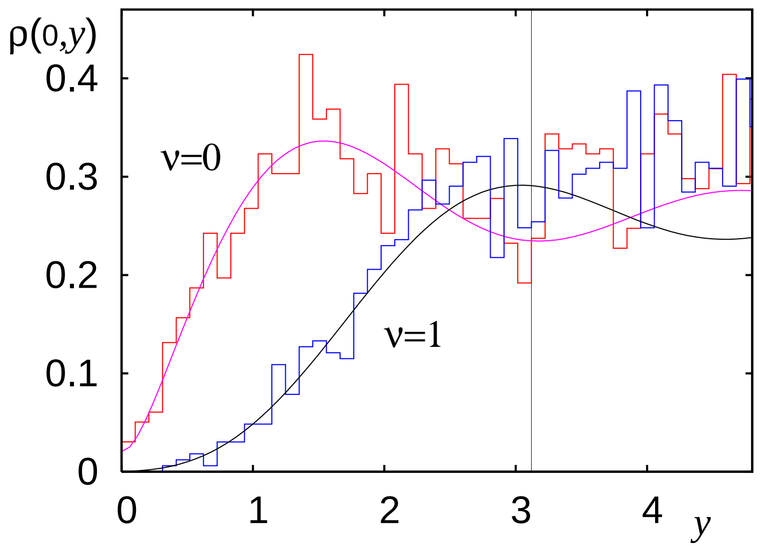
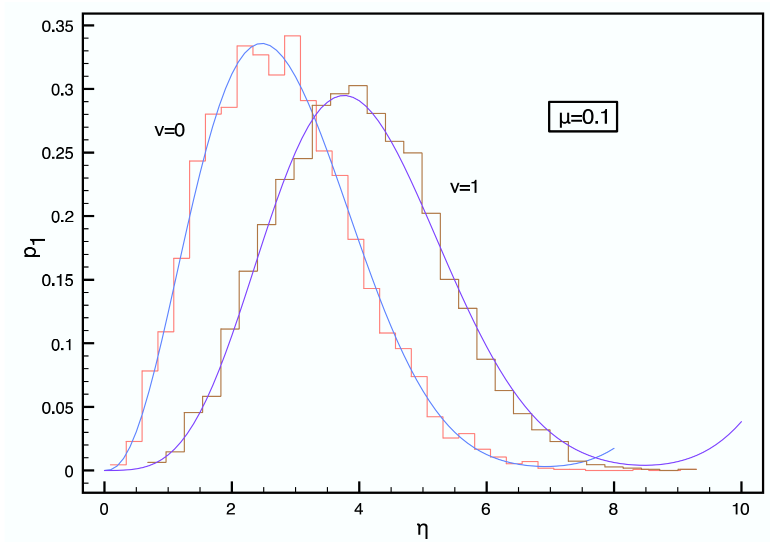


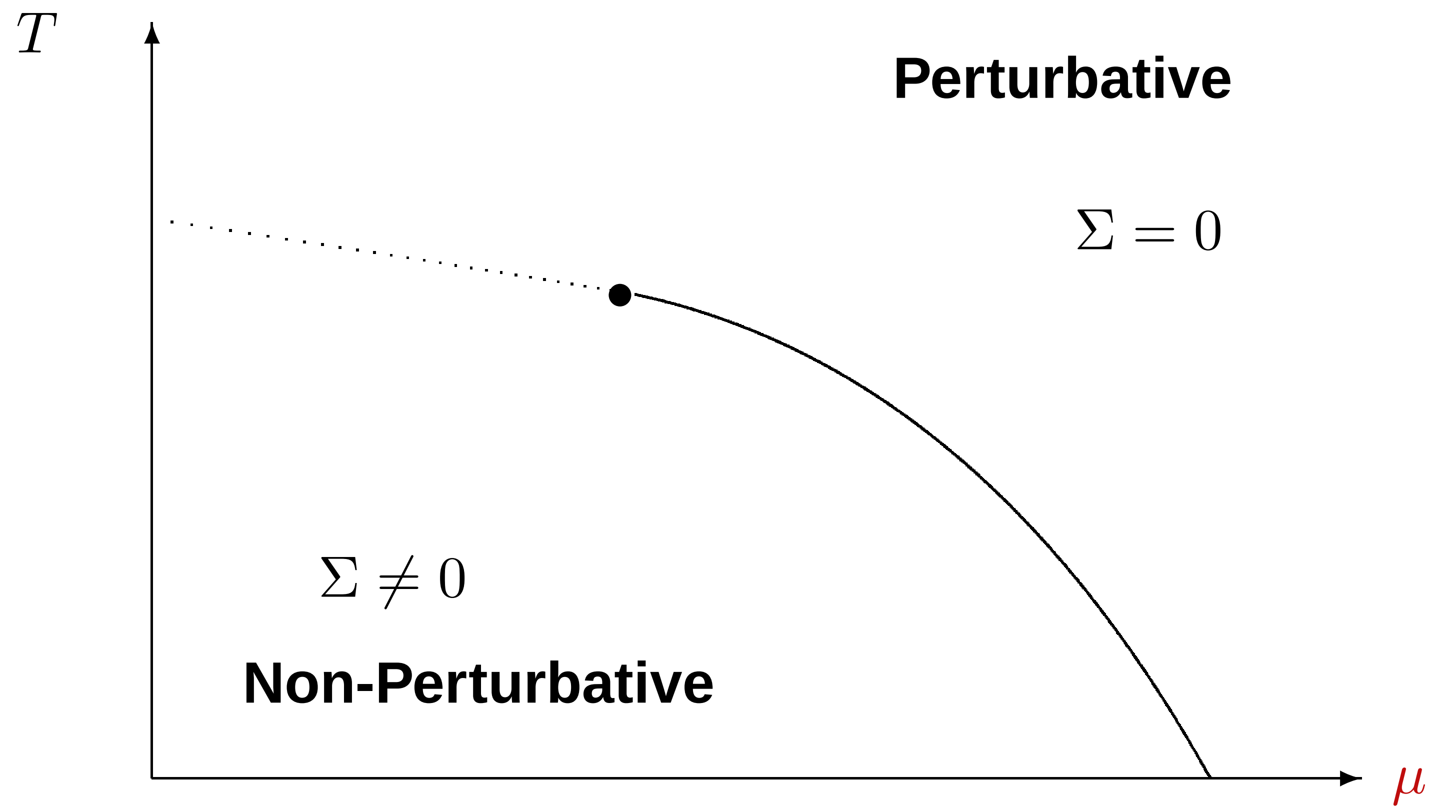






- density (left) [Bloch, Wettig 06], smallest eigenvalue p_1 (right) [+ G.A., Shifrin 07]

Example III: Cartoon of the QCD Phasediagram



- schematic phase diagram for QCD+ $N_f = 2$ light flavors [Halasz et al. 98]

  - derived from a very simple phenomenological Random Matrix Model

- **Chiral Perturbation Theory**
- **The Epsilon Regime**
- **Main Result: Partition Function**
- **What Can We Learn From That?**

Chiral Perturbation Theory

Chiral Symmetry in QCD

- symmetry of the QCD Lagrangian under global rotation

  quarks $\boxed{\mathcal{L} = \bar{\Psi} D \Psi + \bar{\Psi} M \Psi}$, D Dirac op., M=diag$(m_u, m_d, \ldots)$

  $$\Psi = \begin{pmatrix} u \\ d \\ \vdots \end{pmatrix} \rightarrow U\Psi \;, \;\; \bar{\Psi} = \Psi^\dagger \gamma_0 \rightarrow \bar{\Psi} U^\dagger \;\; U \in U(N_f)$$

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  chiral symmetry: rotate Ψ $\ R(L)$ independently:

$$\boxed{\Psi_L \to U_L \Psi_L \ , \ \ \bar{\Psi}_L \to \bar{\Psi}_L U_L^\dagger \ ; \ \Psi_R \to U_R \Psi_R \ , \bar{\Psi}_R \to \bar{\Psi}_R U_R^\dagger}$$

  with $U_R, U_L \in U(N_f)$

  symmetry group

$$\boxed{U_L(N_f) \times U_R(N_f)} = SU_L(N_f) \times SU_R(N_f) \times U_V(1) \times U_A(1)$$

Chiral Symmetry Breaking in QCD

- **explicit breaking**:

  mass term $\bar{\Psi} M \Psi = \bar{\Psi}_R M \Psi_L \; + \; \bar{\Psi}_L M \Psi_R$

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- **analogy Ferromagnet**: ground state breaks rotation symmetry of $\mathcal{H}$

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Low Energy Effective Theory of QCD

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- **example** $N_f = 2$ up and down quark only:

  3 Goldstone Bosons $\pi_0,\ \pi_{\pm}$ very light

  non-Goldstone scale $\Lambda \approx 1$ Gev

- for $N_f > 2$ more mesons are considered as Goldstone Bosons: $\rho, \eta, K, \ldots$

  we will loosely speaking call all these ”pions”

- **determined by symmetries:**

  1) standard kinetic term, 2) no interaction at zero energy, 3) inv. under chiral rotations

- Goldstone Bosons live in Goldstone manifold:

  $\boxed{U(x) = \exp[i\sqrt{2}\xi(x)/F] \in SU(N_f)}$ with F Pion decay constant,

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- **effective Lagrangian**

  $\mathcal{L} = \frac{1}{4}F^2 \text{Tr}[\partial_\alpha U \partial_\alpha U^\dagger] + L_1(\text{Tr}[\partial_\alpha U \partial_\alpha U^\dagger])^2 + \ldots$

  with invariance $\boxed{U(x) \to U_R U(x) U_L^\dagger}$

  - kinetic term ok: $\mathcal{L} = \frac{1}{2}\text{Tr}[\partial_\alpha \xi(x) \partial_\alpha \xi(x)] + \ldots$ expanded in ξ's

  - coupling constants $F,\ L_1,\ \ldots$: difficult to get from QCD($\to$ Lattice)

  - in the limit we consider later (epsilon regime) all but the first term will be subleading

Mass terms and LO XPT

- we want to keep mass terms as probes of XSB (and $m_\pi \neq 0$)

  - mass term in **QCD**: $\bar{\Psi}_R M \Psi_L \; + \; \bar{\Psi}_L M^\dagger \Psi_R$ can be made invariant if $M \to U_L M U_R^\dagger$ transform like a field, ”spurion technique”

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- **invariant mass term in XPT**:

  $\mathcal{L}_m = \frac{1}{2}\Sigma \mathrm{Tr}[MU + M^\dagger U^\dagger]$

- **leading order Lagrangian** in XPT

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- $\exists$ **higher order terms in** $U(x)$: e.g.

  $\mathcal{L}^{(2)} = a W_8 \mathrm{Tr}[U^2 + U^{\dagger\,2}] + \ldots$

  effect of finite lattice spacing a from D-Wilson in XPT

Chemical Potential and XPT

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  (e.g. $N_1 = N_2$ isospin, $N_2 = 0$ baryon chemical potential)

- incorporate in XPT as **covariant derivative**

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- Summary **starting point Euclidean path integral of XPT**

  $$\boxed{\mathcal{Z}_{\chi PT} \equiv \int_{SU(N_f)} [dU(x)] \ \exp[-\int d^4x \ \mathcal{L}(U, \mathcal{D}U)]}$$

  parameters: N_f, m_q, μ and low energy constants (LEC) F, Σ ($W_8, \ldots$ etc.) to be determined

The Epsilon-Regime

Finite-Volume QFT (XPT)

- consider **box** $V = L^4$ ($= TL^3$ for finite Temperature):

  - momenta $p \sim 1/L \equiv \epsilon$

  - XPT valid for $1/L \ll \Lambda$ non-Goldstone scale

  - dimensionless kinetic term: $\partial_\alpha \sim 1/L$ and $\xi(x) \sim 1/L$

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  propagator $\Delta(p) = \frac{1}{V(p^2+m_\pi^2)} \sim 1/L^2$

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- ϵ**-regime counting:** $\boxed{m_\pi \sim 1/L^2 \ll 1/L}$ [Gasser, Leutwyler 87]
  - **new regime**, need to treat zero-modes non-perturbatively

The ε-Regime of Gasser and Leutwyler

- $\boxed{U(x) = U_0 \exp[i\sqrt{2}\xi(x)/F]}$ where $U_0 \in SU(N_f)$ const

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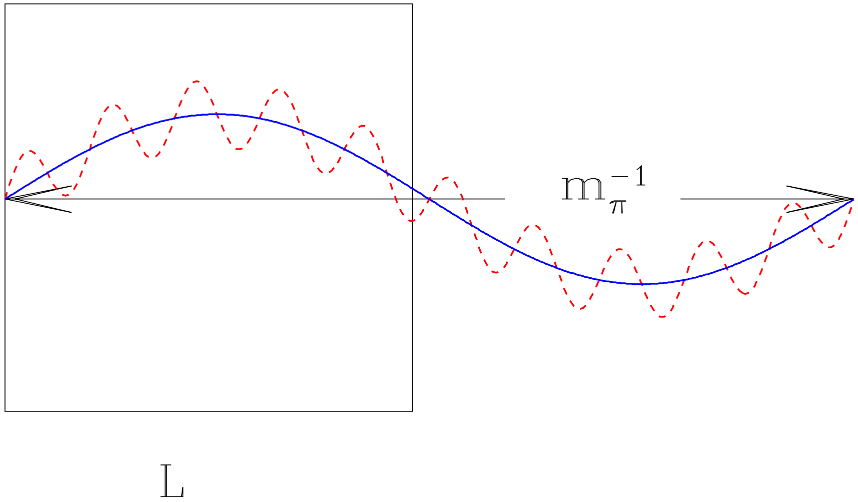
  U_0 **group integral** $\times$ **fluctuating Gaussian fields**
- the ε-regime is an **unphysical regime:**(fig. from [physics/0309072])



Compton wave length

- Lattice-QCD: can **choose parameters to be in the ε-regime**

  - determine LECs F, Σ etc. $\to$ use in physical p-regime

  - check algorithms with exact lattice chiral symmetry

1. Main Result: Partition Function of Epsilon-Regime

Partition function of $\varepsilon\chi$PT

- in the ε counting $\boxed{M \sim 1/L^4}$ (from m_π) and $\boxed{\mu \sim 1/L^2}$ we obtain

$$
\begin{aligned}
\mathcal{Z}_{\varepsilon\chi\text{PT}} &= \int_{SU(N_f)} dU_0 \exp\left[-\frac{1}{2}\Sigma V \text{Tr}\left(M(U_0 + U_0^\dagger)\right)\right] \\
&\quad \times \exp\left[+\frac{1}{4} V F^2 \mu^2 \text{Tr}[\Gamma, U_0][\Gamma, U_0^\dagger]\right] \\
&\quad \times \int [d\xi(x)] \exp\left[-\int d^4x \frac{1}{2}(\partial\xi)^2\right]
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- after Fourier trafo (fixing topology) the group integrals can be done exactly

  $\int d\Theta e^{i\nu\Theta} \int_{SU(N_f)} dU = \int_{U(N_f)} dU \det[U]^\nu$

Examples for exact group integrals

- $N_f = 2$ flavours with equal mass $\hat{m} = V\Sigma m$, at $\mu = 0$ (or $\mu_B \neq 0$)

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- **tools for evaluation**:

- expansion in group characters for $\mu = 0$ (e.g. Balantekin)

- direct parametrisation

- asymptotic of orthogonal polynomials in map to random matrices

Corrections and Universality

- partition function of eXPT **equivalent to Matrix Model** = universal

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- **1-loop corrections (NLO)** from non-zeromodes:
  $\rightarrow$ effective LECs with corrections in $1/\sqrt{V} \sim \bar{\Delta}(0)$

  $$\boxed{\Sigma_{eff} = \Sigma\left(1 - \frac{N_f^2-1}{N_f F}\bar{\Delta}(0)\right)}$$ [Gasser, Leutwyler 87]

  $$\boxed{F_{eff} = F\left(1 - \frac{1}{N_f F}(\bar{\Delta}(0) - \int d^4x[\partial_0\bar{\Delta}(x)]^2)\right)}$$ [G.A., Basile, Lellouch; Damgaard, DeGrand, Fukaya 08]

- $\Rightarrow$ factorised eXPT partition functions etc. holds to NLO!
  - at NNLO factorisation and universality destroyed [Lehner, Wettig, et al.]

What can we learn from the partition function of $\varepsilon\chi$PT?

Duality and Sum Rules

- **”flavour-topology duality”** [Verbarschot]:

$$\boxed{\lim_{m\to 0} \mathcal{Z}_{\nu}^{(N_f)}/m^{\nu} \sim \mathcal{Z}_{\nu+1}^{(N_f-1)}}$$ [G.A., Damgaard, Dalmazi, Verbaarschot 01]

  example

  ”two massless flavours $N_f = 2$ at $\nu = 0$

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- **Leutwyler-Smilga sum rules:**

  expectation values of Dirac operator eigenvalues $\longrightarrow$ **QCD**

Setup QCD Partition Function

- full theory (Euclidean) $+\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ topological term

$$\boxed{\mathcal{Z}_{\mathbf{QCD}} \;\equiv\; \int [dA][d\Psi]\;\exp[-\mathrm{Tr}\int \bar{\Psi}(\not{D}+M)\Psi \;+\; FF \;+\; i\Theta F\tilde{F}]}$$

with N_f quarks Ψ of masses M, gauge fields A and field strength F

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- diagonalise: $\boxed{D\psi_k = i\lambda_k\psi_k}$ in finite V, Euclidean $D^\dagger = -D$

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  $\rightarrow$ that's why we Fourrier trafo

- **compare quark mass dependence**

  - Taylor expand in m_q formal expression for $\mathcal{Z}_{\mathbf{QCD}}$ in terms of λ_k

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- the same should hold for $\mu_B \neq 0$: density of **complex Dirac eigenvalues**

Exercises

- Compute the partition function of $\varepsilon\chi$PTfor $N_f = 1$ starting from the group integral, both with and without Fourier trafo.

  Does the one without Fourier trafo have zero's as a function of m, and what does that imply for chiral symmetry breaking? (hint: Yang-Lee zero's)

- Compute the partition function of $\varepsilon\chi$PTfor $N_f = 2$ at equal mass after Fourier trafo (you may use that diagonalising $U \rightarrow$diag($e^{i\theta_1}, e^{i\theta_2}$) has a Jacobian $\sim |e^{i\theta_1} - e^{i\theta_2}|^2$). [The inversion of the Fourier trafo can be found in hep-th/0108166]

Some More Literature

- H. Leutwyler, A. Smilga, Phys. Rev. **D46** (1992) 5607-5632
- Maarten Golterman, Applications of chiral perturbation theory to lattice QCD, Les Houches Lecture Notes, arXiv:0912.4042v4 [hep-lat]
- P.H. Damgaard, Chiral Random Matrix Theory and Chiral Perturbation Theory, Lecture notes, arXiv:1102.1295v1 [hep-ph]