Introducing chemical potential in the overlap formalism

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Outline

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1. Introduction

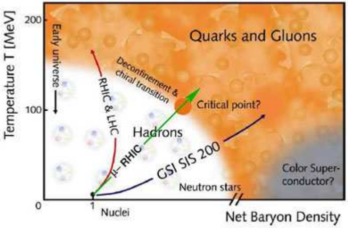
2. Overlap fermions at finite density

Introduction

- Understanding the QCD phase diagram is an important research area for both experimental and theoretical physics.

There are dedicated experimental facilities at RHIC in BNL(USA), b) ALICE in CERN and c) FAIR, GSI in Germany(upcoming) aimed at:

- Predicting location of the critical point(CP).
- The nature and interactions of different phases.



[http://www.bnl.gov/rhic/news]

- The order of the chiral phase transition and existence of CP dependent on the number of quark flavours and correct chiral anomaly. [Pisarski & Wilczek, 83].
- Lattice QCD is a non-perturbative tool: can give first principle estimate of the critical point and understanding of the QCD phase diagram in general.

Introduction to lattice fermions

- Naive discretization of the Dirac operator leads to the presence of unphysical fermions $\rightarrow$ Fermion doubling problem.
- A lattice No-go theorem [Nielsen Ninomiya(82)] states that following properties are not simultaneously allowed for lattice fermion operator $D(p)$
  1) Is a periodic analytic function of $p_\mu \rightarrow$ locality.
  2) Is invertible everywhere except at $p_\mu = 0$.
  3) $\{\gamma_5, D\} = 0$.
  4) Doubler-free spectrum.
- To remove doublers
  1) break chirality completely $\rightarrow$ Wilson fermions
  2) reduce to a remnant chiral symmetry group $\rightarrow$ Staggered fermions
  3) sacrifice ultra-locality but have exact chiral symmetry
  $\rightarrow$ Overlap fermions. Have analogs of continuum chiral symmetry and exact index theorem on the lattice.

Motivation

- Current results for critical point on lattice use staggered fermions (Gavai& Gupta 2008; Schmidt 2009): Partial chiral symmetry but flavour symmetry explicitly broken$\rightarrow$ N_f is ill defined on the lattice.
- Desirable to have fermions with chiral symmetry at finite density on the lattice for an exact order parameter $\rightarrow$ important for studying the phase diagram and the critical point.
- Known overlap operator at finite μ (Bloch & Wettig, 2006) couples μ to both physical and unphysical fermion modes.
  Topological charge depends on μ contrary to what is known in continuum QCD

Hamiltonian formulation of the Overlap fermions

- In a five dimensional Euclidean spacetime, the fifth dimension taken as timelike, the corresponding Hamiltonian is $= \gamma_5 D$, D is 4D Dirac operator.
- If there is a domain wall like profile $\phi(s) = M, s > 0$ and $\phi(s) = \Lambda, s < 0$, then the overlap between the ground states $|\pm\rangle$ of the fermion Hamiltonians,

$$\mathcal{H}_+ = \gamma_5(D + M) \ , \ \mathcal{H}_- = \gamma_5(D - \Lambda)$$

  gives the standard overlap operator as $s \to \infty$(Narayanan & Neuberger, 1994, 1998)
- On the lattice, the continuum Dirac operator replaced by D_W and $\Lambda \to \infty$.
- In Weyl notation, the D_W is a $2n \times 2n$ matrix, $n = N^3 \times N_T \times N_c$.

$$\mathcal{H}_+ = \begin{pmatrix} B - M & C \\ C^\dagger & -B + M \end{pmatrix}$$

$$B_{xy} = \frac{1}{2}\sum_{\mu=1}^{4}\left(2\delta_{y,x} - U_\mu(x)\delta_{y,x+\mu} - U_\mu^\dagger(y)\delta_{y,x-\mu}\right)$$
$$C_{xy} = \frac{1}{2}\sum_{\mu}\sigma_\mu\left(U_\mu(x)\delta_{y,x+\mu} - U_\mu^\dagger(y)\delta_{y,x-\mu}\right) \ , \ 0 < M < 2$$

Overlap fermions: continued

- D_W can be diagonalized by the unitary operator in a fixed background field,

$$U = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

- The ground states of the Hamiltonians $\mathcal{H}_\pm$ are,
$|+\rangle_{R,L} = u'^{\dagger}_{R,L_n} u'^{\dagger}_{R,L_{n-1}} \cdots u'^{\dagger}_{R,L_2} u'^{\dagger}_{R,L_1} |0\rangle$
$|-\rangle_{R,L} = u^{\dagger}_{R,L_n} u^{\dagger}_{R,L_{n-1}} \cdots u^{\dagger}_{R,L_2} u^{\dagger}_{R,L_1} |0\rangle$.

$d^\dagger$ and $u^\dagger$ creates fermion, anti fermion of $\mathcal{H}_-$.
$d'^\dagger$ and $u'^\dagger$ for $\mathcal{H}_+$.

- The superposition between the ground states $\rightarrow$ det of overlap operator.

$${}_R\langle -|+\rangle_R \quad {}_L\langle +|-\rangle_L = \det D_{ov}.$$

- The overlap operator has the form,

$$D_{ov} = 1 + \gamma_5 sgn(\gamma_5 D_W) \ , \ sgn(A) = A/\sqrt{A.A}$$

Outline

Overlap fermions at finite μ

- We introduce massless fermion sources ξ_L and ξ_R in the partition function.
- Chemical potential couples with the term $u^\dagger d$: the conserved number.

$$Z = \int \mathcal{D}U \, \mathrm{e}^{-\mathrm{S_G}} \, {}_{\mathrm{R}}\langle - | \mathrm{e}^{\bar{\xi}_\mathrm{R} \mathrm{d_R} + \xi_\mathrm{R} \mathrm{u}_\mathrm{R}^\dagger + \mathrm{u}_\mathrm{R}^\dagger \hat{\mu}_\mathrm{R} \mathrm{d_R}} | + \rangle_{\mathrm{RL}} \langle + | \mathrm{e}^{\xi_\mathrm{L} \mathrm{d}_\mathrm{L}^\dagger + \bar{\xi}_\mathrm{L} \mathrm{u_L} - \mathrm{d}_\mathrm{L}^\dagger \hat{\mu}_\mathrm{L} \mathrm{u_L}} | - \rangle_\mathrm{L},$$

- For QCD: $\hat{\mu}_R(0) = 0$, $\hat{\mu}_L(\mu) = -\hat{\mu}_R^\dagger(-\mu)$.

- $\hat{\mu}_R$ can be an operator and a general function of μ. In the naive method $\hat{\mu}_R = i\mu a/2M$.
  In Hasenfratz-Karsch(83) prescription:

$$\hat{\mu}_R = \frac{i}{2M} \frac{(\mathrm{e}^{\mu \mathrm{a}} - 1)\mathrm{T}_4 - (\mathrm{e}^{-\mu \mathrm{a}} - 1)\mathrm{T}_4^\dagger}{2} \ , \ T_4 \psi(x) = U_4(x) \psi(x + 4) \ .$$

Overlap fermions at finite μ: Our results

- The partition function can be derived in terms of the operators,
  $Z = \int \mathcal{D}U \, \mathrm{e}^{-\mathrm{S_G}} \times$

$$\det\alpha \det\left(1+\hat{\mu}_R\beta\alpha^{-1}\right)\det\alpha^\dagger \det\left(1-\hat{\mu}_L\left[\beta\alpha^{-1}\right]^\dagger\right) \times e^{\bar{\xi}_R \frac{1}{\alpha\beta^{-1}+\hat{\mu}_R}\xi_R} e^{\bar{\xi}_L \frac{1}{[\alpha\beta^{-1}]^\dagger-\hat{\mu}_L}\xi_L}$$

$\downarrow$ in Dirac notation, naive method

$$D_{ov}(\hat{\mu}) = D_{ov}(0) + \frac{\mu a}{2M}\gamma_4(2-D_{ov}(0)).$$

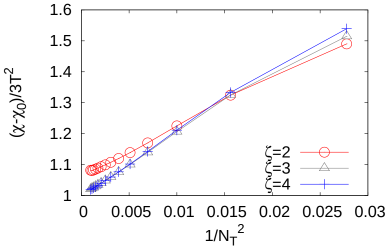
$\downarrow$ in Dirac notation

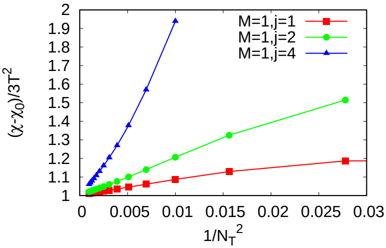
$$\left(\frac{D_{ov}(0)}{2-D_{ov}(0)} + \frac{\mu a\gamma_4}{2M}\right)^{-1}$$

- Under chiral transformation of the sources, $\xi \to \mathrm{e}^{\mathrm{i}\phi}\xi$ and $\bar{\xi} \to \bar{\xi}\mathrm{e}^{-\mathrm{i}\phi}$ the action remains invariant.
- In 4D notation, $\delta\eta \to i\phi\gamma_5\eta, \quad \eta \equiv (\xi_R \quad \xi_L)$, the action remains invariant as $\{\gamma_5, \left(\frac{D_{ov}}{2-D_{ov}} + \frac{\mu a\gamma_4}{2M}\right)^{-1}\} = 0 \Rightarrow$ chiral symmetry is respected.
- The operators appearing in the determinant and the propagator are different for chiral fermions.
- The operator in the determinant can be obtained from the domain wall formalism by coupling μ to the physical fermion localized on the wall and integrating out the bulk modes.

Susceptibility of overlap fermions

- The χ has $1/a^2$ terms on the lattice$\rightarrow$ zero temperature subtraction necessary for $\hat{\mu} \sim \mu a$.
- Usual $exp(\pm\mu a)$ prescription does not remove such divergences $\rightarrow$ combine two different methods of introducing μ.
- We make a general choice $\hat{\mu}_R = \frac{i}{M} \sinh \frac{\mu a}{2} T_4^j$.
- Susceptibility of free fermions can be exactly computed. The $1/a^2$ term is computed on a symmetric lattice with infinite temporal extent and subtracted to obtain the physical value on the lattice.





Conclusions

- The topological charge is μ dependent in the existing overlap formalism unlike in the continuum.
- We propose a first principle method of introducing μ in the overlap formalism
- The chiral symmetry is exact if massless source terms are introduced in the action.
- Has topological charge which is μ-independent.

Conclusions...

- Potentially divergent $1/a^2$ term in the expression for χ for different prescriptions. Its removal is important$\rightarrow$ correct continuum limit.
- Two proposals for full QCD:
  (a) Estimate it on a symmetric lattice with same volume and coupling const.
  (b) Using the linear combination $\hat{\mu} = \sum_j c_j(\mu a)^j$ and adjusting c_j's so that the divergence is canceled for free fermions.
- This operator could be useful in estimating the CP using the radius of convergence estimate of second order baryon number susceptibility.