Universal polylogarithms in Feynman integral computations

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Outlook:

- Motivation: Homotopy invariance and Chen's theorem
- Construction of the class B_m of universal polylogarithms of several variables
- Integration over Feynman parameters by use of B_m

Iterated integrals

$$I\left(a_w,\, a_{w-1},\, \ldots,\, a_2,\, a_1;\, x\right) \quad = \quad \int_0^x f_{a_w}(x^{(w)})dx^{(w)} \ldots \int_0^{x'''} f_{a_2}(x'')dx'' \int_0^{x''} f_{a_1}(x')dx'$$

where the differential 1-forms belong to a chosen set Ω.

Example:
$\Omega_{\mathrm{Polylogs}} = \left\{ \frac{dx}{x},\, \frac{dx}{1-x} \right\}$ determines **classical polylogarithms**:
$\mathrm{Li}_w(x) = \left[\frac{dx}{x} | \ldots | \frac{dx}{x} | \frac{dx}{1-x} \right]$ or **multiple polylogarithms in one variable**:
$\mathrm{Li}_{n_1 n_2 \ldots}(x) = [\ldots | \frac{dx}{x} | \ldots | \frac{dx}{x} | \frac{dx}{1-x} | \frac{dx}{x} | \ldots | \frac{dx}{x} | \frac{dx}{1-x}]$

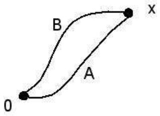
Alternatively: View I as an **integral along a path** γ

$$I\left(a_w,\, a_{w-1},\, \ldots,\, a_2,\, a_1;\, x\right) = \int_\gamma \omega_{a_w} \ldots \omega_{a_1}$$

where γ and the smooth 1-forms $\omega_{a_1},\, \ldots,\, \omega_{a_w}$ are defined on a smooth manifold.

Homotopy invariance

Definition: Two paths γ_A and γ_B are called **homotopic** if they begin and end in the same points and can be continuously transformed into each other.



Definition: An integral $I[\gamma] = \int_\gamma \omega_w \ldots$ is called **homotopy invariant** or a **homotopy functional**, if for homotopic paths it defines the same function:

γ_A, γ_B homotopic $\Rightarrow I[\gamma_A] = I[\gamma_B]$.

- A homotopy functional **depends on the endpoint**, but not on γ itself.
- All prominent iterated integrals used in physics (HPLs, 2dHPLs, ...) are homotopy functionals.
- However, if we consider **more than one variable**, homotopy invariance is **not automatic**.

- Simple case:

$$I[\gamma](z) = \int_\gamma \omega$$

  - one-fold integral with only **one smooth 1-form** ω,
  - γ with fixed initial point 0 and variable endpoint, given by **one variable** z.

Condition: I is homotopy invariant if and only if ω is **closed**.

(Proof by Stokes' theorem.)

- Generic case:

$$I[\gamma](z_1, \ldots, z_n) = \sum \int_\gamma \omega_w \ldots \omega_1$$

  - (linear combination of) iterated integrals over **several smooth 1-forms**,
  - γ with fixed initial point 0 and variable endpoint in n dimensions, given by **several variables** $z_1, \ldots, z_n$.

What is the condition for I to be homotopy invariant?

Chen 1977:

Consider tensor products of 1-forms in a given set Ω :

$$\omega_1 \otimes \ldots \otimes \omega_m \equiv [\omega_1|\ldots|\omega_m]\,.$$

Define a map D by

$$D\left([\omega_1|\ldots|\omega_m]\right) = \sum_{i=1}^{n} [\omega_1|\ldots|\omega_{i-1}|d\omega_i|\omega_{i+1}|\ldots|\omega_m] + \sum_{i=1}^{m-1} [\omega_1|\ldots|\omega_{i-1}|\omega_i \wedge \omega_{i+1}|\ldots|\omega_m]\,.$$

We say that $[\omega_1|\ldots|\omega_m]$ satisfies the **integrability condition** if

$$D\left([\omega_1|\ldots|\omega_n]\right) = 0.$$

Define $B_m(\Omega)$ to be the set of all **integrable words**, i.e. the (linear combinations of) tensor products, satisfying this condition:

$$B_m(\Omega) = \left\{ \xi = \sum_{l=0}^{m} \sum_{i_1,\,\ldots,\,i_l} \left[\omega_{i_1}|\ldots|\omega_{i_l}\right] \text{ such that } D\xi = 0 \right\}.$$

Consider the simple **integration map** from tensor products to iterated integrals

$$\sum_{l=0}^{m} \sum_{i_1, \ldots, i_l} \left[\omega_{i_1} | \ldots | \omega_{i_l}\right] \mapsto \sum_{l=0}^{m} \sum_{i_1, \ldots, i_l} \int_{\gamma} \omega_{i_1} \ldots \omega_{i_l} .$$

Chen's theorem (1977):

The integration map is an **isomorphism** from $B_m(\Omega)$ to the set of **all homotopy invariant iterated** integrals of length m with 1-forms in Ω.

- Every integrable word $\xi \in B_m(\Omega)$ maps to a homotopy invariant iterated integral I with forms in Ω.
- Reversely: If I with forms in Ω is homotopy invariant then it comes from an integrable word $\xi \in B_m(\Omega)$.

Remark:
The integrability condition

$$\sum_{i=1}^{m} [\omega_1|\ldots|\omega_{i-1}|d\omega_i|\omega_{i+1}|\ldots|\omega_m] + \sum_{i=1}^{m-1} [\omega_1|\ldots|\omega_{i-1}|\omega_i \wedge \omega_{i+1}|\ldots|\omega_m] = 0$$

is trivially satisfied if all ω_i are closed $f_i\, dt$ (as $f_i\, dt \wedge f_j\, dt = 0$).

Example: Harmonic Polylogarithms (Remiddi, Vermaseren 1999)

$$\Omega_{\mathrm{HPL}} = \left\{ \frac{dt}{t}, \frac{dt}{1-t}, \frac{dt}{1+t} \right\}$$

The condition is **not trivial** if we consider **several endpoint variables** z_1, ..., z_n and several dt_1, ..., dt_n in the 1-forms.

Explicit construction of $B_m(\Omega_n)$ with n variables:

Take the auxiliary set

$$\tilde{\Omega}_n = \left\{ \frac{dt_1}{t_1}, \frac{dt_1}{1-t_1}, \frac{t_2\, dt_1}{1-t_1 t_2}, \frac{t_2 t_3\, dt_1}{1-t_1 t_2 t_3}, \ldots, \frac{\left(\prod_{i=2}^n t_i\right) dt_1}{1-\left(\prod_{i=1}^n t_i\right)} \right\}$$

$B_m(\tilde{\Omega}_n)$ are all possible length-m tensor products of forms in $\tilde{\Omega}_n$.
They define functions of one variable t_1 with constant parameters $t_2, \ldots, t_n$.
(Hyperlogarithms)

$$\Omega_n = \left\{ \frac{dt_1}{t_1}, \ldots, \frac{dt_n}{t_n}, \frac{d\left(\prod_{i\in\Lambda} t_i\right)}{1-\left(\prod_{i\in\Lambda} t_i\right)} \text{ for all non-empty } \Lambda \subset \{t_1, \ldots, t_n\} \right\}.$$

Example: $\Omega_2 = \left\{ \frac{dt_1}{t_1}, \frac{dt_2}{t_2}, \frac{dt_1}{1-t_1}, \frac{dt_2}{1-t_2}, \frac{t_2\, dt_1 + t_1\, dt_2}{1-t_1 t_2} \right\}$
Here the integrability condition is non-trivial.

There is an **explicit map** $\psi : B_m\left(\tilde{\Omega}_n\right) \to B_m(\Omega_n)$ from which we obtain all elements of $B_m(\Omega_n)$. (sometimes called 'the symbol')

Example:

$$\tilde{\Omega}_2 = \left\{ \frac{dt_1}{t_1}, \frac{dt_1}{1-t_1}, \frac{t_2\, dt_1}{1-t_1\, t_2} \right\}$$

$$\Omega_2 = \left\{ \frac{dt_1}{t_1}, \frac{dt_2}{t_2}, \frac{dt_1}{1-t_1}, \frac{dt_2}{1-t_2}, \frac{t_2\, dt_1 + t_1\, dt_2}{1-t_1\, t_2} \right\}$$

$$\psi : \; B_m\left(\tilde{\Omega}_2\right) \to B_m(\Omega_2)$$

- $$\int_0^{t_1} \frac{t_2\, dt_1'}{1-t_1'\, t_2} \equiv \left[\frac{t_2\, dt_1}{1-t_1\, t_2} \right] \quad \mapsto \quad \left[\frac{t_1\, dt_2 + t_2\, dt_1}{1-t_1\, t_2} \right] \equiv \int_0^{t_2} \frac{t_1\, dt_2'}{1-t_1\, t_2'} + \int_0^{t_1} \frac{t_2\, dt_1'}{1-t_1'\, t_2}$$
  $$= \ln(1-t_1\, t_2)$$

- $$\left[\frac{dt_1}{1-t_1} \middle| \frac{t_2\, dt_1}{1-t_1\, t_2} \right] \quad \mapsto \quad \left[\frac{dt_2}{1-t_2} \middle| \frac{dt_1}{1-t_1} \right] + \left[\frac{dt_1}{1-t_1} - \frac{dt_2}{1-t_2} - \frac{dt_2}{t_2} \middle| \frac{t_2\, dt_1 + t_1\, dt_2}{1-t_1\, t_2} \right]$$
  $$= \mathrm{Li}_{1,1}(t_1,\, t_2)$$

Consider $B_1(\Omega_n)$, ..., $B_m(\Omega_n)$ and let $\mathcal{B}_m(\Omega_n)$ be the $\mathbb{Q}$-vectorspace of all corresponding **functions**.

Why is this a good class of functions?

- **Homotopy invariance $\Rightarrow$ well-defined functions of n variables**
- ψ maps complicated **functional relations** to trivial identities.
- **Closed** under taking **primitives**:

  The primitive $\int \sum dx_i f_i I_T$ (where $I_T \in \mathcal{B}_m(\Omega_n)$ and where f_i have the same denominators as $\omega \in \Omega_n$) belongs again to $\mathcal{B}_m(\Omega_n)_T$. (Brown '05)
  There is an algorithm for taking primitives.
- **Limits at zero and one** are under control:

  One obtains combinations of $I_T \in \mathcal{B}_m(\Omega_n)$ and multiple zeta values. (Brown '05)

  There is an algorithm for taking limits at zero and one.

We can integrate from zero to one and the result stays in this class of functions.

Our **Maple-program**

- contains an implementation of the map ψ and the functions $\mathcal{B}_m(\Omega_n)$.
- allows to differentiate, take primitives and limits (at 0 and 1) of functions in $\mathcal{B}_m(\Omega_n)$.
- **computes integrals** of the form

$$\tilde{I} = \int_0^1 \sum_i dx \frac{a_i}{g_0} \left[\frac{dg_1}{g_1} \,|\, \frac{dg_2}{g_2} \,|\, \ldots \right],$$

where $\left[\frac{dg_1}{g_1} \,|\, \frac{dg_2}{g_2} \,|\, \ldots \right], \in \mathcal{B}_m(\Omega_n)$

Denominator condition: All g_0, g_1, ... are denominators of $\omega_i \in \Omega_n$.

- The result lands in the same class of functions (with MZV prefactors)

Remark: This integration step was trivial for one-variable iterated integrals,

$$\text{e.g.} \int_0^1 \underbrace{dx\, f_i(x)}_{\in \Omega_{\mathrm{HPL}}} \mathrm{HPL}(\ldots; x) = \mathrm{HPL}(a_i, \ldots; 1) \text{ by definition.}$$

For n-variable iterated integrals, the step is not trivial (but algorithmically solved).

Systematic Integration over Feynman Parameters (Brown '08)

- Iteratively integrate out all n Feynman parameters by use of $\mathcal{B}_m(\Omega_n)$.
- *Start with* finite integrals, e.g. of a primitively divergent vacuum graph:

$$\int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^N dx_i\right) \delta\left(1-\sum_{i=1}^N x_i\right) \frac{1}{\mathcal{U}^2}$$

- **Condition:** After each integration, the new integrand can be written as as a sum of terms

$$\frac{b_i}{f_0}\left[\frac{df_1}{f_1} \mid \frac{df_2}{f_2} \mid \ldots\right]$$

  such that there is a x_r in which the f_i are **linear polynomials**.

  Then they can be mapped to the above type of integrals $\tilde{I}$.

  $\Rightarrow$ equivalent to the above Denominator Condition

- Map the denominators in the integrand the denominators of Ω_n and integrate with our program.

Which Feynman graph satisfies the condition? In which order can we integrate?

$\Rightarrow$ **Polynomial reduction algorithm**

Polynomial reduction (Brown '08)

- **Input:** A first Symanzik polynomial $\mathcal{U}$ and an ordering $(x_{r_1}, x_{r_2}, ..., x_{r_n})$ of the n Feynman parameters.

- **Output:** A sequence $S_1, ..., S_n$ of sets of polynomials in the Feynman parameters.
  These polynomials are the denominators f_i which we map to the denominators of Ω_n.

- Call the graph **polynomial reducible** if for all $1 \leq k \leq n$ every polynomial in S_k is linear in $x_{r_{k+1}}$.
  $\Rightarrow$Then the denominator condition is satisfied.
  $\Rightarrow$ The integral can be computed by use of $B_m(\Omega_n)$.

A very large class of graphs is already known to be polynomial reducible. (Brown '09)

Using hyperlogarithms, the strategy was already applied to relevant integrals of QCD.

(Ablinger, Blümlein, Hasselhuhn, Schneider, Wissbrock '12)

At first we consider primitively (UV-) divergent vacuum graphs ($\Rightarrow$ corresponding propagator integrals) which are polynomial-reducible.

How severe are these restrictions?

- Beyond primitive divergence: By a method of Brown and Kreimer (2012) we can consider graphs with sub-divergences,
- Beyond vacuum graphs: Try to include the second Symanzik polynomial $\mathcal{F}$ in the above algorithms.

  First question: Which graphs have reducible second Symanzik polynomials?
- Beyond polynomial-reducibility?: Maybe by extending the class of functions. However, we are optimistic that many interesting graphs are polynomial reducible.

Conclusions:

The functions $\mathcal{B}_m(\Omega_n)$:

- From an appropriate set of 1-forms Ω_n we construct **integrable words**.
- Chen's theorem $\Rightarrow$ These give homotopy invariant iterated integrals, i.e. **well-defined functions of** n **variables**.
- We can compute definite integrals **without leaving this class of functions** (together with **multiple zeta values**).

Integrating out Feynman parameters with $\mathcal{B}_m(\Omega_n)$:

- Polynomial reduction tells us, **if** and in **which order** we can integrate out the Feynman parameters.
- We iteratively integrate without leaving the class of functions $\mathcal{B}_m(\Omega_n)$.

A well known **functional equation** is the five-term-relation:

$$-\mathrm{Li}_2\left(\frac{1-y}{1-\frac{1}{x}}\right)-\mathrm{Li}_2\left(\frac{1-x}{1-\frac{1}{y}}\right)+\mathrm{Li}_2(xy)-\mathrm{Li}_2(x)-\mathrm{Li}_2(y)=\frac{1}{2}\ln^2(1-x)+\frac{1}{2}\ln^2(1-y)$$

Writing each function as iterated integral on the total space (using ψ), the relation becomes obvious:

$$\mathrm{Li}_2\left(\frac{1-y}{1-\frac{1}{x}}\right)=\left[\frac{dx}{x}+\frac{dx}{1-x}-\frac{dy}{1-y}\middle|\frac{xdy+ydx}{1-xy}\right]-\left[\frac{dx}{1-x}\middle|\frac{dy}{1-y}\right]-\left[\frac{dx}{x}+\frac{dx}{1-x}\middle|\frac{dx}{1-x}\right]$$

$$\mathrm{Li}_2\left(\frac{1-x}{1-\frac{1}{y}}\right)=\left[\frac{dy}{y}+\frac{dy}{1-y}-\frac{dx}{1-x}\middle|\frac{xdy+ydx}{1-xy}\right]+\left[\frac{dx}{1-x}\middle|\frac{dy}{1-y}\right]-\left[\frac{dy}{y}+\frac{dy}{1-y}\middle|\frac{dy}{1-y}\right]$$

$$\mathrm{Li}_2(xy)=\left[\frac{dx}{x}+\frac{dy}{y}\middle|\frac{xdy+ydx}{1-xy}\right],\ \mathrm{Li}_2(x)=\left[\frac{dx}{x}\middle|\frac{dx}{1-x}\right],\ \mathrm{Li}_2(y)=\left[\frac{dy}{y}\middle|\frac{dy}{1-y}\right]$$

Differential Equations Method (Kotikov '91, Remiddi '97, Gehrmann, Remiddi 2000, Mastrolia, Argeri '07)

Let I_1, ..., I_m be a set of IBP-master-integrals with a **common physical variable** z (e.g. squared momentum or mass).
For each I_j derive a first order differential equation, using IBP:

$$\underbrace{\frac{\partial}{\partial z} I_j + a_j I_j}_{\text{the MI we want to solve}} \quad = \quad \underbrace{\sum_{i \neq j} a_i I_i}_{\text{the other MIs}}$$

(**Recent alternative:** consider the **Picard-Fuchs equation** of a Feynman period integral (Müller-Stach, Weinzierl, Zayadeh '11))

Assume that the inhomogeneous part of an equation is known in terms of **iterated integrals in one variable**.

Solving the equation involves integrating over these terms.

E.g. for harmonic polylogarithms (HPL), try to write the integrand as

$$\sum_i f_i(x) \mathrm{HPL}(...; x) \text{ with } f_i \in \{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}\}$$

such that integration is trivial

$$\int_0^y \underbrace{dx\, f_i(x)}_{\in \Omega_{\mathrm{HPL}}} \mathrm{HPL}(...; x) = \mathrm{HPL}(a_i, ...; y) \text{ by definition.}$$

Assume some I_i are known as iterated integrals with **several parameters** z_1, ..., z_n. It may be advantageous to consider differential equations in several variables:

$$
\begin{aligned}
\frac{\partial}{\partial z_1} I_j + a_j I_j &= \sum_{i \neq j} a_i I_i, \\
\frac{\partial}{\partial z_2} I_k + b_k I_k &= \sum_{i \neq k} b_i I_i, \\
&\dots
\end{aligned}
$$

Then the $I_i(z_1, z_2, \dots)$ have to be **functions in n variables** and the (non-trivial) integration step can be done with our program.

A recent application of the 'symbol'

Del Duca, Duhr, Smirnov '10: **Computation** of the two-loop hexagon Wilson loop in $\mathcal{N}=4$ SYM

- using the Mellin-Barnes representation, the authors reduce to a **finite parametric integral**:

$$\int_0^1 dv_1 \int_0^1 dv_2 \int_0^1 dv_3 \int_0^{u_3} du\,(1-(1-u_1)v_1)^{-1}\left(1-v_2\left(1-\frac{u_2(1-v_2)}{1-(1-u_1)v_1}\right)\right)^{-1}$$
$$\times\,(1-v_3(1-uv_1v_2))^{-1}$$

- they obtain aresult in terms of iterated integrals, **filling 17 pages**

Goncharov et al '10: **Simplification** to only **classical polylogs**, **filling 5 lines**

- consider the **'symbol'** of the result as tensor product of differential forms,
- construct a much simpler function with the same 'symbol',
- adjust constant terms in the new expression