Discoveries in lattice QCD from perturbation theory

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Lattice perturbation theory

If one naively discretizes the continuum QCD action, gauge invariance is lost on the lattice – and only in the continuum limit can be recovered again

Instead, one must start from a continuum theory of non-interacting quarks, discretize it, and then build a gauge invariant theory directly on the lattice

In this way gauge invariance can be then maintained at any finite value of the lattice spacing a

Main consequence: the fields

$$U_\mu(x) = \mathrm{e}^{\mathrm{i} g_0 a\, T^a\, A^a_\mu(x)} \qquad (a = 1, \ldots, N_c^2 - 1)$$

appear in a lattice QCD action, instead of the usual A's

The fields $U_\mu(x)$ live on the links which connect two neighboring lattice sites – these variables are naturally defined in the middle point of a link

These lattice gluon variables belong to the **group** $SU(N_c)$ rather than to the corresponding **Lie algebra**, as is the case in the continuum

Lattice perturbation theory

Wilson action:

$$
\begin{aligned}
S_W &= S_W^f + S_W^g \\
S_W^g &= \frac{1}{g_0^2}\, a^4 \sum_{x,\mu\nu} \left[N_c - \mathrm{Re}\,\mathrm{Tr}\left[U_\mu(x) U_\nu(x+a\hat{\mu}) U_\mu^\dagger(x+a\hat{\nu}) U_\nu^\dagger(x) \right] \right] \\
S_W^f &= a^4 \sum_x \left[-\frac{1}{2a} \sum_\mu \left[\overline{\psi}(x)(r-\gamma_\mu) U_\mu(x) \psi(x+a\hat{\mu}) \right.\right. \\
&\qquad \left.\left. + \overline{\psi}(x+a\hat{\mu})(r+\gamma_\mu) U_\mu^\dagger(x) \psi(x) \right] + \overline{\psi}(x) \left(m_0 + \frac{4r}{a} \right) \psi(x) \right] \\
&= a^4 \sum_x \overline{\psi}(x) \left[\frac{1}{2} \left(\gamma_\mu (\widetilde{\nabla}_\mu^\star + \widetilde{\nabla}_\mu) - ar \widetilde{\nabla}_\mu^\star \widetilde{\nabla}_\mu \right) + m_0 \right] \psi(x)
\end{aligned}
$$

where the (forward) lattice covariant derivative is defined as

$$
\widetilde{\nabla}_\mu \psi(x) = \frac{U_\mu(x)\psi(x+a\hat{\mu}) - \psi(x)}{a}
$$

Lattice perturbation theory

Lattice perturbation theory is a saddle-point expansion around $U_\mu(x) = 1$

Its degrees of freedom are given by the components of the potential, $A^a_\mu(x)$

So, while the fundamental gauge variables for Monte Carlo simulations are the U_μ's, and the action is relatively simple when expressed in terms of these variables, in perturbation theory the true dynamical variables are the A_μ's

This **mismatch** is responsible for many complications of lattice PT

One cannot escape the complications of LPT: a gauge-invariant lattice regularization requires the U's, but the degrees of freedom of LPT are the A's

Now, the Wilson action becomes very complicated when written in terms of the variables A_μ:

$$U_\mu = 1 + \mathrm{i} g_0 a A_\mu - \frac{1}{2} g_0^2 a^2 A_\mu^2 + \cdots$$

Moreover, it consists of an infinite number of terms, which give rise to an infinite number of interaction vertices – *with an arbitrary number of fields*

$\rightarrow$ example: $\overline{\psi}\, A\, A \cdots A\, \psi$ *(no continuum counterparts. . .)*

Lattice perturbation theory

Is lattice gauge theory then non-renormalizable?

Lattice perturbation theory

Is lattice gauge theory then non-renormalizable?

Fortunately, only a finite number of vertices is needed to any given order in g_0

All but a few vertices are "irrelevant" – they are proportional to some positive power of the lattice spacing a and so they vanish in the naive continuum limit

However, this does not mean that they can be thrown away in the computation of Feynman diagrams!

Quite on the contrary: they usually contribute to correlation functions in the continuum limit, through divergent loop corrections ($\sim 1/a^n$)

These irrelevant vertices are indeed important in many cases, they contribute to the renormalization of masses, coupling constants and wave-functions

All these vertices are in fact necessary to ensure the gauge invariance of physical amplitudes

Only when they are included can gauge-invariant Ward Identities be constructed, and the renormalizability of the lattice theory proven

Lattice perturbation theory

Quark-quark-gluon vertex in momentum space, for Wilson fermions:
(euclidean space!)

$$S_{qqg} = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p'}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p+k-p')$$

$$\times \mathrm{i}g_0 \sum_\mu \overline{\psi}(p') \left(\gamma_\mu \cos \frac{a(p+p')_\mu}{2} - \mathrm{i}r \sin \frac{a(p+p')_\mu}{2} \right) A_\mu(k)\psi(p)$$

(color factor not included)

The 3-gluon vertex is *(with $p+q+r=0$, and gluons are all incoming and assigned clockwise)*:

$$W^{abc}_{\mu\nu\lambda}(p,q,r) = -\mathrm{i}g_0 f^{abc} \frac{2}{a} \left\{ \delta_{\mu\nu} \sin \frac{a(p-q)_\lambda}{2} \cos \frac{ar_\mu}{2} \right.$$

$$\left. + \delta_{\nu\lambda} \sin \frac{a(q-r)_\mu}{2} \cos \frac{ap_\nu}{2} + \delta_{\lambda\mu} \sin \frac{a(r-p)_\nu}{2} \cos \frac{aq_\lambda}{2} \right\}$$

In the formal $a \to 0$ limit one recovers the well-known continuum expressions

Peculiar issues on the lattice

Computer codes are needed to compute but the simplest matrix elements
(for example: the algebraic manipulation program FORM)

The increasing complexities can be easily seen in the calculation of the renormalization of the moments of unpolarized structure functions

Reason: the covariant derivative is proportional to the inverse of the lattice spacing, $D \sim 1/a$, and so

$$\langle x^n \rangle \quad \sim \quad \langle \overline{\psi}\, \gamma_\mu\, D_{\mu_1} \cdots D_{\mu_n}\, \psi \rangle \quad \sim \quad \frac{1}{a^n}$$

Thus, to compute the n-th moment, one needs to Taylor expand to order n in a every single quantity (propagators, vertices, operator insertions, counterterms)

It is not difficult to see how many terms can come out of that

It is sufficient to look at the Wilson quark-quark-gluon vertex to order a^2

$$(V^a)^{bc}_\mu(k, ap) \quad = \quad -g_0\,(T^a)^{bc} \cdot \left\{ \mathrm{i}\gamma_\mu \left[\cos\frac{k_\mu}{2} - \frac{1}{2} a p_\mu \, \sin\frac{k_\mu}{2} - \frac{1}{8} a^2 p_\mu^2 \, \cos\frac{k_\mu}{2} \right] \right.$$

$$\left. + r \left[\sin\frac{k_\mu}{2} + \frac{1}{2} a p_\mu \, \cos\frac{k_\mu}{2} - \frac{1}{8} a^2 p_\mu^2 \, \sin\frac{k_\mu}{2} \right] + O\left(a^3 p_\mu^3\right) \right\}$$

Peculiar issues on the lattice

or to the expansion of the Wilson quark propagator to just order a:

$$S^{ab}(k+aq,am_0) = \delta^{ab}\cdot\left\{\frac{-\mathrm{i}\sum_\mu \gamma_\mu \sin k_\mu + 2r\sum_\mu \sin^2\frac{k_\mu}{2}}{\sum_\mu \sin^2 k_\mu + \left[2r\sum_\mu \sin^2\frac{k_\mu}{2}\right]^2}\right.$$

$$+a\cdot\left[\frac{-\mathrm{i}\sum_\mu \gamma_\mu q_\mu \cos k_\mu + r\sum_\mu q_\mu \sin k_\mu + m_0}{\sum_\mu \sin^2 k_\mu + \left[2r\sum_\mu \sin^2\frac{k_\mu}{2}\right]^2}\right.$$

$$\left.\left.-\left(-\mathrm{i}\sum_\rho \gamma_\rho \sin k_\rho + 2r\sum_\rho \sin^2\frac{k_\rho}{2}\right)\frac{\sum_\mu q_\mu \sin 2k_\mu + 4r\sum_\mu \sin^2\frac{k_\mu}{2}\left(r\sum_\nu q_\nu \sin k_\nu + m_0\right)}{\left\{\sum_\mu \sin^2 k_\mu + \left[2r\sum_\mu \sin^2\frac{k_\mu}{2}\right]^2\right\}^2}\right]\right\}$$

$$+O\left(a^2 q_\mu^2\right)$$

The algebraic manipulations become thus quite complex

Overlap or domain-wall fermions, improved gauge actions, . . . , produce much more complicated expressions

Main consequence of all this: generation of a huge number of terms, at least in the initial stages of the manipulations

Peculiar issues on the lattice

Lorentz symmetry is broken on the lattice

. . . one cannot make a rotation of an arbitrary angle

The Lorentz group $O(4)$ is broken to the hypercubic group $H(4)$

A whole new kind of problems stem from this, one of which is that the Einstein summation convention ($k_\mu k^\mu = k^2$) is not valid anymore

One of the biggest challenges of computer codes for lattice perturbation theory is to deal with the fact that the summation convention on repeated indices is suspended

FORM, and other similar symbolic manipulations programs, have been developed having in mind the usual continuum calculations

There are therefore many useful built-in features of FORM that are in principle somewhat of a hindrance when one does lattice perturbative calculations

These built-in functions cannot then be used straightforwardly in the computations on the lattice

Peculiar issues on the lattice

This is for example what FORM would do by default, because it assumes that two equal indices have to be contracted:

$$\sum_\lambda \gamma_\lambda p_\lambda \quad \longrightarrow \quad \not{p}$$

$$\sum_\lambda \gamma_\lambda p_\lambda \sin k_\lambda \quad \longrightarrow \quad \not{p} \sin k_\lambda$$

$$\sum_\lambda \gamma_\lambda \sin k_\lambda \cos^2 k_\lambda \quad \longrightarrow \quad (\gamma \cdot \sin k) \cos^2 k_\lambda$$

$$\sum_{\lambda,\rho} \gamma_\rho \gamma_\lambda \gamma_\rho \sin k_\lambda \cos^2 k_\rho \quad \longrightarrow \quad -2 \sum_\lambda \gamma_\lambda \sin k_\lambda \cos^2 k_\rho$$

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On the lattice however monomials typically contain more than twice the same index

Only the first case is then correctly handled by FORM

For example, in the last case the right answer is instead

$$-\sum_{\lambda,\rho} \gamma_\lambda \sin k_\lambda \cos^2 k_\rho + 2 \sum_\rho \gamma_\rho \sin k_\rho \cos^2 k_\rho$$

Some discoveries from lattice PT

Lattice perturbation theory is widely applied for the renormalization of couplings, masses, operators *(weak matrix elements, structure functions, . . .)*

. . . but not only this . . .

Helpful in the investigation of unknown properties of new lattice formulations

or also: to elucidate the mixing structures of operators

Sometimes nonperturbative calculations are difficult or expensive, or their implications are not unambiguous

In the rest of this talk: we discuss in some detail three cases where some important features of lattice QCD simulations were first discovered using perturbation theory

Three different situations:

- mixings for lattice matrix elements of moments of structure functions
- minimally doubled fermions
- domain-wall fermions for a finite extension of the fifth dimension

An operator mixing

The n-th moment $\langle x^n \rangle$ of unpolarized structure functions is measured by matrix elements of the *(symmetric and traceless)* operator

$$O_{\{\mu\mu_1\cdots\mu_n\}}(x) = \overline{\psi}(x)\,\gamma_{\{\mu}\,D_{\mu_1}\cdots D_{\mu_n\}}\,\psi(x)$$

Second moment: the operator is $O_{\{\mu\nu\sigma\}} = \overline{\psi}\,\gamma_{\{\mu}\,D_\nu D_{\sigma\}}\,\psi$

We have three choices here for the symmetrized components

One is given by the operator $O_{\{123\}}$, which belongs to the $\mathbf{4_2}$ representation of the hypercubic group, and is multiplicatively renormalizable

However, this choice is quite unsatisfactory when one considers simulations, because two components of the hadron momentum must be different from zero and from each other, leading to rather large systematic errors

One should minimize these systematic errors by including as few nonzero components of the hadron momentum as possible

From this point of view, the optimal choice is the operator $O_{\{111\}}$, which belongs to the $\mathbf{4_1}$ representation

An operator mixing

Unfortunately this operator mixes with $\overline{\psi}\,\gamma_1\,\psi$, which is a $\mathbf{4_1}$ as well

Moreover, the coefficient of this mixing can be seen from dimensional arguments to be power divergent , $\sim 1/a^2$

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There is an intermediate choice between having the indices all different or all equal, and is given by the operator

$$O_S = O_{\{011\}} - \frac{1}{2}(O_{\{022\}} + O_{\{033\}})$$

which does not have any power divergences due to this particular combination

This operator belongs to an irreducible representation of the hypercubic group, the $\mathbf{8_1}$

Nonetheless, is not multiplicatively renormalizable, and undergoes a mixing with another operator

The way in which this happens is not trivial

An operator mixing

It turns out that two $\mathbf{8_1}$ operators mix with each other

This mixing can be best seen in the following way

The nonsymmetrized operators

$$O_A = O_{011} - \frac{1}{2}(O_{022} + O_{033})$$

$$O_B = O_{101} + O_{110} - \frac{1}{2}(O_{202} + O_{220} + O_{303} + O_{330})$$

turn out to have different 1-loop corrections on the lattice, and they renormalize with different numerical factors which form a nontrivial mixing matrix:

$$\widehat{O}_A = Z_{AA}\, O_A + Z_{AB}\, O_B$$
$$\widehat{O}_B = Z_{BA}\, O_A + Z_{BB}\, O_B$$

Notice that the two covariant derivatives have the same index in O_A but two different indices in O_B, and the two operators have different tree levels:

$$\gamma_0 p_1^2 - \frac{1}{2}(\gamma_0 p_2^2 + \gamma_0 p_3^2)$$

$$2\gamma_1 p_0 p_1 - (\gamma_2 p_0 p_2 + \gamma_3 p_0 p_3)$$

An operator mixing

The operator that we want to measure *(the "continuum" one)*,

$$O_S = O_{\{011\}} - \frac{1}{2}(O_{\{022\}} + O_{\{033\}}) = \frac{1}{3}(O_A + O_B),$$

does not transform into itself under 1-loop renormalization,

$$\widehat{O}_S = \frac{1}{3}(Z_{AA} + Z_{BA})\, O_A + \frac{1}{3}(Z_{AB} + Z_{BB})\, O_B,$$

because on the lattice $Z_{AA} + Z_{BA} \neq Z_{AB} + Z_{BB}$ *(as explicit calculations have shown)*

In other words, the symmetric combination is lost after renormalization, and O_S mixes with an operator of mixed symmetry *(with a small coefficient)*

The choice of indices for the operator for $\langle x^2 \rangle$ is thus very important , for the Monte Carlo simulations as well as for the calculation of renormalization factors

In the continuum all $O_{\{\mu\nu\sigma\}}$ cases, including $O_{\{111\}}$, belong to the $(\mathbf{\frac{3}{2}}, \mathbf{\frac{3}{2}})$

Thus, they have the same renormalization constant, and no mixing problem

All these (unexpected) features of the renormalization of $\langle x^2 \rangle$ were discovered while doing 1-loop perturbative calculations!

An operator mixing

Discovered in

- **G. Beccarini, M. Bianchi, S. C. and G.C. Rossi**
  "Deep Inelastic Scattering in Improved Lattice QCD. II. The second moment of structure functions", Nuclear Physics B456 (1995) 271

At first one could not believe that the 1-loop corrections for O_A and O_B were not the same – after all, only the ordering of the indices is different . . .

It was the first time . . . for $\langle x \rangle$ there was no such problem *(it was not possible)*

. . . and before our calculation: the renormalization for O_A and O_B had been estimated using tadpole dominance $\longrightarrow$ **same numbers!**

Our discovery then also spurred the search for mixings in many other operators *(third moment, . . .)* $\rightarrow$ careful analyses with the hypercubic group

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The $\langle x^2 \rangle$ mixing, although small in perturbation theory, cannot be overlooked

It could be small also nonperturbatively *(but this is not yet known . . .)*

However, just taking the result for the matrix element of the simmetrized operator O_S would bias the physical numbers , even if the mixing is small

Minimally doubled fermions

Minimally doubled fermions (2 flavors):

realize the minimal doubling allowed by the Nielsen-Ninomiya theorem

Preserve an exact chiral symmetry for a degenerate doublet of quarks

chiral symmetry protects mass renormalization

$\rightarrow$ *no additive renormalization* $\rightarrow$ *no tuning of masses . . .*

At the same time, also remain strictly local

$\rightarrow$ *fast for simulations*

`A cheap realization of chiral symmetry at nonzero lattice spacing`

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`A cheap realization of chiral symmetry at nonzero lattice spacing`

We can construct a conserved axial current, which has a simple expression

Compared with staggered fermions:

- same kind of $U(1) \otimes U(1)$ chiral symmetry
- 2 flavors instead of 4

  $\Rightarrow$ no uncontrolled extrapolations to 2 physical light flavors
- no complicated intertwining of spin and flavor

Minimally doubled fermions

Ideal for $N_f = 2$ simulations: no rooting needed!

Much cheaper and simpler than Ginsparg-Wilson fermions
(overlap, domain-wall, fixed-point)

We considered two realizations of minimally doubled fermions: Boriçi-Creutz and Karsten-Wilczek fermions

For them, we also constructed the **conserved** vector and **axial** currents

They have simple expressions which involve only nearest-neighbors sites

One of the very few lattice discretizations in which one can give a simple expression (and ultralocal) for a conserved axial current

This conserved axial current is even ultralocal

These features could turn out to be very useful also in numerical simulations

Boriçi-Creutz fermions

The work of Boriçi and Creutz leads to a fermionic action whose free Dirac operator in momentum space reads

$$D(p) = i \sum_{\mu} (\gamma_\mu \sin p_\mu + \gamma'_\mu \cos p_\mu) - 2i\Gamma + m_0$$

where

$$\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \qquad (\Gamma^2 = 1)$$

and

$$\gamma'_\mu = \Gamma \gamma_\mu \Gamma = \Gamma - \gamma_\mu$$

Useful relations:

$$\sum_{\mu} \gamma_\mu = \sum_{\mu} \gamma'_\mu = 2\Gamma, \quad \{\Gamma, \gamma_\mu\} = 1, \quad \{\Gamma, \gamma'_\mu\} = 1$$

The action vanishes at $p_1 = (0,0,0,0)$ and $p_2 = (\pi/2, \pi/2, \pi/2, \pi/2)$

This ingenious construction represents a special linear combination of two (physically equivalent) naive fermions , corresponding to the first two terms in the action

Karsten-Wilczek fermions

Already in the Eighties: Karsten (1981) and then Wilczek (1987) proposed some particular kind of minimally doubled fermions

Unitary equivalent to each other, after phase redefinitions

Wilczek [PRL 59, 2397 (1987)] proposed a special choice of the function $P_\mu(p)$ which minimizes the numbers of doublers

The free Karsten-Wilczek Dirac operator

$$D(p) = i \sum_{\mu=1}^{4} \gamma_\mu \sin p_\mu + i\gamma_4 \sum_{k=1}^{\mathbf{3}} (1 - \cos p_k)$$

has zeros at $p_1 = (0,0,0,0)$ and $p_2 = (0,0,0,\pi)$

Drawback: it destroys the equivalence of the four directions under discrete permutations

Mixings with new operators then arise

This is also true for the Boriçi-Creutz action

Hypercubic breaking

The actions of minimally doubled fermions have two zeros

$\Rightarrow$ there is always a special direction in euclidean space
(given by the line that connects these two zeros)

Thus, these actions cannot maintain a full hypercubic symmetry

They are symmetric only under the **subgroup** of the hypercubic group which preserves (up to a sign) a **fixed direction**

For the Boriçi-Creutz action this is a major hypercube diagonal, while for other minimally doubled actions it may not be a diagonal – for example for the Karsten-Wilczek action is the x_4 axis

Although the distance between 2 two Fermi points is the same ($p_2^2 - p_1^2 = \pi^2$), these two realization of minimally doubled fermions are not equivalent

The breaking of the hypercubic symmetry implies the appearance of mixings with operators of different dimensionality, like $\overline{\psi}\Gamma\psi$, $\overline{\psi}\gamma_4\psi$ or $\overline{\psi}\gamma_4 D_4\psi$

Self-energy

At 1 loop, for Boriçi-Creutz fermions:

$$\Sigma(p,m_0) = i\not{p}\,\Sigma_1(p) + m_0\,\Sigma_2(p) + c_1(g_0)\cdot i\,\Gamma\sum_\mu p_\mu + c_2(g_0)\cdot i\,\frac{\Gamma}{a}$$

with

$$\Sigma_1(p) = 1+\frac{g_0^2}{16\pi^2}\,C_F\left[\log a^2p^2+6.80663+(1-\alpha)\left(-\log a^2p^2+4.792010\right)\right]+O(g_0^4)$$

$$\Sigma_2(p) = 1+\frac{g_0^2}{16\pi^2}\,C_F\left[4\,\log a^2p^2-29.48729+(1-\alpha)\left(-\log a^2p^2+5.792010\right)\right]+O(g_0^4)$$

$$c_1(g_0) \;=\; 1.52766\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$$

$$c_2(g_0) \;=\; 29.54170\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$$

The full inverse propagator at one loop can be written as

$$\Sigma^{-1}(p,m_0) = \left(1-\Sigma_1\right)\cdot\left\{i\not{p}+m_0\left(1-\Sigma_2+\Sigma_1\right)-\frac{ic_1}{2}\sum_\mu\gamma_\mu\sum_\nu p_\nu-\frac{ic_2}{a}\,\Gamma\right\}$$

Self-energy

At 1 loop, for Karsten-Wilczek fermions:

$$\Sigma(p, m_0) = i\not{p}\,\Sigma_1(p) + m_0\,\Sigma_2(p) + d_1(g_0)\cdot i\,\gamma_4 p_4 + d_2(g_0)\cdot i\,\frac{\gamma_4}{a}$$

where

$$\Sigma_1(p) = \frac{g_0^2}{16\pi^2}\,C_F\left[\log a^2p^2 + 9.24089 + (1-\alpha)\Big(-\log a^2p^2 + 4.792010\Big)\right]$$

$$\Sigma_2(p) = \frac{g_0^2}{16\pi^2}\,C_F\left[4\,\log a^2p^2 - 24.36875 + (1-\alpha)\Big(-\log a^2p^2 + 5.792010\Big)\right]$$

$$d_1(g_0) = -0.12554\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$$

$$d_2(g_0) = -29.53230\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$$

The full inverse propagator at one loop can be written as

$$\Sigma^{-1}(p, m_0) = \Big(1-\Sigma_1\Big)\cdot\Big(i\not{p} + m_0\Big(1-\Sigma_2+\Sigma_1\Big) - id_1\,\gamma_4 p_4 - \frac{id_2}{a}\,\gamma_4\Big)$$

Vacuum polarization

Our focus here: the radiative corrections to the gluon propagator due to fermion loops

Contributions to the vacuum polarization due to loops of gluons and ghosts: independent of the lattice fermionic action chosen (*at one loop*)

$\Rightarrow$ do not provide informations relevant for hypercubic breaking

Only the fermionic loops are able to generate hypercubic-breaking terms (as it in the end happens for both Karsten-Wilczek and Boriçi-Creutz fermions)

The fermionic contribution to the vacuum polarization for one flavor of Wilson fermions *(where neither breaking of hypercubic symmetry nor fermion doubling occur)* is

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu} p^2 \right) \left[\frac{g_0^2}{16\pi^2} C_t \left(-\frac{4}{3} \log p^2 a^2 + 4.337002 \right) \right]$$

where $\mathrm{Tr}\,(\mathrm{t^a t^b}) = \mathrm{C_2}\,\delta^{\mathrm{ab}}$

We can see that this (gauge invariant) result satisfies the Ward identity $p^\mu \Pi^{(f)}_{\mu\nu}(p) = 0$, which expresses the conservation of the fermionic current

Vacuum polarization

For Boriçi-Creutz fermions:

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu} p^2\right)\left[\frac{g_0^2}{16\pi^2} C_2 \left(-\frac{8}{3}\log p^2 a^2 + 23.6793\right)\right]$$
$$-\left((p_\mu + p_\nu)\sum_\lambda p_\lambda - p^2 - \delta_{\mu\nu}\left(\sum_\lambda p_\lambda\right)^2\right)\frac{g_0^2}{16\pi^2} C_2 \cdot 0.9094$$

For Karsten-Wilczek fermions:

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu} p^2\right)\left[\frac{g_0^2}{16\pi^2} C_2 \left(-\frac{8}{3}\log p^2 a^2 + 19.99468\right)\right]$$
$$-\left(p_\mu p_\nu\left(\delta_{\mu 4} + \delta_{\nu 4}\right) - \delta_{\mu\nu}\left(p^2\,\delta_{\mu 4}\delta_{\nu 4} + p_4^2\right)\right)\frac{g_0^2}{16\pi^2} C_2 \cdot 12.69766$$

There are new terms, compared with a standard situation like Wilson fermions

Although each of these actions breaks hypercubic symmetry in its appropriate and peculiar way, these new terms still satisfy the Ward identity $p^\mu \Pi^{(f)}_{\mu\nu}(p) = 0$

Very important: there are no power-divergences ($1/a^2$ or $1/a$) in our results for the vacuum polarization!

Counterterms

Each of these two bare actions does not contain all possible operators allowed by the respective symmetries (broken hypercubic group)

Radiative corrections generate new contributions whose form is not matched by any term in the original bare actions

Counterterms are then necessary for a consistent renormalized theory

This consistency requirement will uniquely determine their coefficients

Our task: add to the bare actions all possible counterterms allowed by the remaining symmetries (after hypercubic symmetry has been broken)

They are lattice artefacts peculiar to minimally doubled fermions

In the following we will consider the massless case $m_0 = 0$

Chiral symmetry strongly restricts the number of possible counterterms

For Boriçi-Creutz fermions, operators are allowed where summations over just single indices are present (in addition to the standard Einstein summation over two indices)

Then objects like $\sum_\mu \gamma_\mu = 2\Gamma$ appear

Counterterms

We find that there can be only one dimension-4 counterterm: $\boxed{\overline{\psi}\,\Gamma\sum_\mu D_\mu\psi}$

Possible discretization: form similar to the hopping term in the action

$$c_4(g_0)\;\frac{1}{2a}\sum_\mu\Big(\overline{\psi}(x)\,\Gamma\,U_\mu(x)\,\psi(x+a\widehat{\mu})-\overline{\psi}(x+a\widehat{\mu})\,\Gamma\,U_\mu^\dagger(x)\,\psi(x)\Big)$$

There is also one counterterm of dimension three: $\boxed{\dfrac{ic_3(g_0)}{a}\;\overline{\psi}(x)\,\Gamma\,\psi(x)}$

This is already present in the bare action, but with a fixed coefficient , $-2/a$

The appearance of this counterterm means that in the general renormalized action the coefficient of this operator must be kept general

For Karsten-Wilczek fermions we find an analogous situation

Here objects are allowed in which we constrain any index to be equal to 4

Only gauge-invariant counterterm of dimension four: $\boxed{\overline{\psi}\,\gamma_4 D_4\,\psi}$

A suitable discretization:

$$d_4(g_0)\;\frac{1}{2a}\Big(\overline{\psi}(x)\,\gamma_4\,U_4(x)\,\psi(x+a\widehat{4})-\overline{\psi}(x+a\widehat{4})\,\gamma_4\,U_4^\dagger(x)\,\psi(x)\Big)$$

Counterterms

There is also one counterterm of dimension three, $\boxed{\frac{id_3(g_0)}{a}\ \overline{\psi}(x)\,\gamma_4\,\psi(x)}$

(already present in the bare Karsten-Wilczek action, with a fixed coefficient)

In perturbation theory the coefficients of all these counterterms are functions of the coupling which start at order g_0^2

They give rise at one loop to additional contributions to fermion lines

The rules for the corrections to fermion propagators, needed for our one-loop calculations, can be easily derived

For external lines, they are given in momentum space respectively by

$$-ic_4(g_0)\ \Gamma \sum_\nu p_\nu, \quad -\frac{ic_3(g_0)}{a}\,\Gamma$$

for Boriçi-Creutz fermions, and by

$$-id_4(g_0)\ \gamma_4\,p_4, \quad -\frac{id_3(g_0)}{a}\,\gamma_4$$

for Karsten-Wilczek fermions

Counterterms

We can determine all these coefficients (at one loop) by requiring that the renormalized self-energy assumes its standard form

For Boriçi-Creutz fermions we obtain then from our 1-loop calculations:

$$c_3(g_0) = 29.54170 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4)$$

$$c_4(g_0) = 1.52766 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4)$$

For Karsten-Wilczek fermions the coefficients are:

$$d_3(g_0) = -29.53230 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4)$$

$$d_4(g_0) = -0.12554 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4)$$

Note: for each action, one of two coefficients seems to be rather small

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Note: for each action, one of two coefficients seems to be rather small

Counterterm interaction vertices are generated as well – but they are at least of order g_0^3, and thus they cannot contribute at one loop

Counterterms

We need counterterms also for the pure gauge part of the actions of minimally doubled fermions

Although at the bare level the breaking of hypercubic symmetry is a feature of the fermionic actions only, in the renormalized theory it propagates *(via the interactions between quarks and gluons)* also to the pure gauge sector

These counterterms must be of the form $\mathrm{tr}\, FF$, but with nonconventional choices of the indices, reflecting the breaking of the hypercubic symmetry

Only purely gluonic counterterm possible for the Boriçi-Creutz action:

$$c_P(g_0) \sum_{\lambda\rho\tau} \mathrm{tr}\, F_{\lambda\rho}(x)\, F_{\rho\tau}(x)$$

At one loop this counterterm is relevant only for gluon propagators

Denoting the fixed external indices at both ends with μ and ν, all possible lattice discretizations of this counterterm give in momentum space the same Feynman rule:

$$-c_P(g_0) \left[(p_\mu + p_\nu) \sum_\lambda p_\lambda - p^2 - \delta_{\mu\nu} \Big(\sum_\lambda p_\lambda \Big)^2 \right]$$

The presence of this counterterm is essential for the correct renormalization of the vacuum polarization

Counterterms

It is not hard to imagine that in the case of Karsten-Wilczek fermions the temporal plaquettes will be renormalized differently from the other plaquettes

Indeed, the counterterm to be introduced contains an asymmetry between these two kinds of plaquettes, and can be written in continuum form as

$$d_P(g_0) \sum_{\rho\lambda} \mathrm{tr}\, F_{\rho\lambda}(x)\, F_{\rho\lambda}(x)\, \delta_{\rho 4}$$

This is the only purely gluonic counterterm needed for this action, since introducing also a $\delta_{\lambda 4}$ in the above expression will produce a vanishing object

It is immediate to write a lattice discretization for it, using the plaquette:

$$d_P(g_0)\, \frac{\beta}{2} \sum_{\rho\lambda} \left(1 - \frac{1}{N_C}\, \mathrm{tr}\, P_{4\lambda}(x)\right)$$

The Feynman rule for this counterterm reads

$$-d_P(g_0)\, \left[p_\mu p_\nu\, (\delta_{\mu 4} + \delta_{\nu 4}) - \delta_{\mu\nu}\, \left(p^2\, \delta_{\mu 4}\delta_{\nu 4} + p_4^2\right)\right]$$

and again is exactly what is needed in the vacuum polarization

Counterterms

The hypercubic breaking terms of the vacuum polarization disappear when the coefficient of the gluonic counterterm has the value

$$c_P(g_0) = -0.9094 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4)$$

for Boriçi-Creutz fermions and

$$d_P(g_0) = -12.69766 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4)$$

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All counterterms remain of the same form at all orders of perturbation theory – only the values of their coefficients depend on the number of loops

The same counterterms appear at the nonperturbative level, and will be required for consistent Monte Carlo simulations of these fermions

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Counterterms not only provide additional Feynman rules for the calculation of loop amplitudes

They can **modify Ward identities** and hence, in particular, contribute additional terms to the conserved currents

Conserved currents

Z_V and Z_A (of the local currents) are not equal to one

The local vector and axial currents are not conserved

We need to consider the chiral Ward identities in order to work with currents which are protected from renormalization

We have constructed the conserved vector and axial currents, and verified that at one loop their renormalization constants are equal to one

We act on the Boriçi-Creutz action in position space

$$S = a^4 \sum_x \left[\frac{1}{2a} \sum_\mu \left[\overline{\psi}(x) \left(\gamma_\mu + i\gamma'_\mu\right) U_\mu(x) \, \psi(x + a\widehat{\mu}) \right.\right.$$

$$\left.\left. - \overline{\psi}(x + a\widehat{\mu}) \left(\gamma_\mu - i\gamma'_\mu\right) U^\dagger_\mu(x) \, \psi(x) \right] + \overline{\psi}(x) \left(m_0 - \frac{2i\Gamma}{a} \right) \psi(x) \right]$$

with the vector transformation

$$\delta_V \psi = i\alpha \, \psi, \quad \delta_V \overline{\psi} = -i\alpha \, \overline{\psi}$$

or the axial transformation

$$\delta_A \psi = i\alpha \, \gamma_5 \psi, \quad \delta_A \overline{\psi} = i\alpha \, \overline{\psi} \gamma_5$$

Conserved currents

We then obtain the conserved vector current for Boriçi-Creutz fermions as

$$V_\mu^{cons}(x) = \frac{1}{2}\left[\overline{\psi}(x)\,(\gamma_\mu + i\,\gamma'_\mu)\,U_\mu(x)\,\psi(x+a\widehat{\mu}) + \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\,\gamma'_\mu)\,U_\mu^\dagger(x)\,\psi(x)\right]$$

while the axial current (conserved in the case $m_0 = 0$) is

$$A_\mu^{cons}(x) = \frac{1}{2}\left[\overline{\psi}(x)\,(\gamma_\mu + i\,\gamma'_\mu)\,\gamma_5\,U_\mu(x)\,\psi(x+a\widehat{\mu}) + \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\,\gamma'_\mu)\,\gamma_5\,U_\mu^\dagger(x)\,\psi(x)\right]$$

We have computed the renormalization of these point-split currents

The sum of vertex, sails and operator tadpole gives *(in the vector case)*

$$\frac{g_0^2}{16\pi^2}\,C_F\,\gamma_\mu\left[-\log a^2p^2 - 6.80664 + (1-\alpha)\left(\log a^2p^2 - 4.79202\right)\right] \qquad + c_1^{cv}(g_0)\,\Gamma$$

where the coefficient of the mixing is $c_1^{cv}(g_0) = -1.52766 \cdot \frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$

The term proportional to γ_μ exactly compensates the contribution of $\Sigma_1(p)$ from the quark self-energy (wave-function renormalization)

Conserved currents

We then obtain the conserved vector current for Boriçi-Creutz fermions as

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The term proportional to γ_μ exactly compensates the contribution of $\Sigma_1(p)$ from the quark self-energy (wave-function renormalization)

But what about the mixing term, proportional to Γ ?

We should take into account the counterterms . . .

Conserved currents

The counterterm $\overline{\psi}(x)\,\dfrac{i\Gamma}{a}\,\psi(x)$ does **not** modify the Ward identities

On the contrary, the counterterm

$$\frac{c_4(g_0)}{4}\sum_\mu\sum_\nu\left(\overline{\psi}(x)\,\gamma_\nu\,U_\mu(x)\,\psi(x+a\widehat{\mu})+\overline{\psi}(x+a\widehat{\mu})\,\gamma_\nu\,U_\mu^\dagger(x)\,\psi(x)\right)$$

generates new terms in the Ward identities and then in the conserved currents

The additional term in the conserved vector current so generated reads

$$\frac{c_4(g_0)}{4}\left[\overline{\psi}(x)\left(\sum_\nu\gamma_\nu\right)U_\mu(x)\,\psi(x+a\widehat{\mu})+\overline{\psi}(x+a\widehat{\mu})\left(\sum_\nu\gamma_\nu\right)U_\mu^\dagger(x)\,\psi(x)\right]$$

Its 1-loop contribution is easy to compute (c_4 *is already of order* g_0^2 !): $c_4(g_0)\,\Gamma$

The value of c_4 is known from the self-energy $\quad\Rightarrow\quad c_4(g_0)\,\Gamma\;=\;-c_1^{cv}(g_0)\,\Gamma$

Only this value of c_4 exactly cancels the Γ mixing term present in the 1-loop conserved current without counterterms

Thus, we obtain that the renormalization constant of these point-split currents is one – which confirms that they are conserved currents

Everything is consistent. . .

Conserved currents

Let us now consider the Karsten-Wilczek action in position space:

$$S = a^4 \sum_x \left[\frac{1}{2a} \sum_{\mu=1}^{4} \left[\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1 - \delta_{\mu 4}\right)\right) U_\mu(x) \, \psi(x + a\widehat{\mu}) \right.\right.$$

$$\left.\left. - \overline{\psi}(x + a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1 - \delta_{\mu 4}\right)\right) U_\mu^\dagger(x) \, \psi(x) \right] + \overline{\psi}(x) \left(m_0 + \frac{3i\gamma_4}{a} \right) \psi(x) \right]$$

Conserved currents

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Adding the counterterms, application of the chiral Ward identities gives for the conserved axial current of Karsten-Wilczek fermions

$$A_\mu^{\mathrm{c}}(x) = \frac{1}{2} \left(\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1-\delta_{\mu 4}\right)\right) \gamma_5\, U_\mu(x)\, \psi(x+a\widehat{\mu}) \right.$$

$$\left. +\overline{\psi}(x+a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1-\delta_{\mu 4}\right)\right) \gamma_5\, U_\mu^\dagger(x)\, \psi(x) \right)$$

$$+ \frac{d_4(g_0)}{2} \left(\overline{\psi}(x)\, \gamma_4\gamma_5\, U_4(x)\, \psi(x+a\widehat{4}) + \overline{\psi}(x+a\widehat{4})\, \gamma_4\gamma_5\, U_4^\dagger(x)\, \psi(x) \right)$$

Once more, is a simple expression which involve only nearest-neighbour sites

We checked explicitly that its renormalization constant is one

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Importance of perturbation theory

We have learned (only using perturbation theory!) that:

- one needs to add three counterterms *(two in quenched QCD)*
- one of these counterterms comes with a very small coefficient

For more than two years, nobody had suspected that there could be counterterms – until we carried out these 1-loop calculations

Perturbation theory has thus been essential for the discovery of some key features of this class of fermions

Even with that, it took some time to understand what was happening . . .

All this is also an example of the usefulness of perturbative techniques in helping to unfold theoretical aspects of *(new)* lattice formulations

This is practically all that is known at the moment about these actions . . .

Next task: determine the coefficients of these counterterms nonperturbatively

this particular work is being done now in Mainz

Domain-wall fermions

- Domain-wall fermions *(Kaplan, 1992; Shamir, 1993)*: chiral symmetry
- The exact chiral symmetry reduces the leading discretization errors from $O(a)$ to $O(a^2)$

  $\Rightarrow$ automatic $O(a)$ improvement
- Simulations use lattices with a finite number of points N_s in the extra fifth dimension

  $\Rightarrow$ overlap between the modes on the two opposite walls

  $\Rightarrow$ violations of chiral symmetry
- Only in the theoretical limit in which the extension of the fifth dimension becomes infinite the chiral modes can fully decouple from each other, yielding an exact chiral symmetry
- Aim: investigate the magnitude of these chirality-violating effects using perturbative calculations
- The focus is at small N_s, where the simulations are currently performed
- We had to derive the required propagator functions

Domain-wall fermions

Five-dimensional domain-wall action:

$$\sum_x \sum_{s=1}^{N_s} \left[\frac{1}{2} \sum_\mu \left(\overline{\psi}_s(x)(\gamma_\mu - r) U_\mu(x) \psi_s(x+\hat{\mu}) - \overline{\psi}_s(x)(\gamma_\mu + r) U_\mu^\dagger(x-\hat{\mu}) \psi_s(x-\hat{\mu}) \right) \right.$$

$$+ \left(\overline{\psi}_s(x) P_+ \psi_{s+1}(x) + \overline{\psi}_s(x) P_- \psi_{s-1}(x) \right) + (M - 1 + 4r) \overline{\psi}_s(x) \psi_s(x)$$

$$\left. + m \sum_x \left(\overline{\psi}_{N_s}(x) P_+ \psi_1(x) + \overline{\psi}_1(x) P_- \psi_{N_s}(x) \right) \right.$$

with:

- $0 < M < 2$
  $\Rightarrow$ correct structure of chiral modes (with no doublers)
- $r = -1$

Chiral projectors: $P_\pm = (1 \pm \gamma_5)/2$

Domain-wall fermions

In (four-dimensional) momentum space the action becomes

$$\delta_{s,t} \sum_{\mu} i\gamma_\mu \sin p_\mu + (W_{st}^{+}(p) + mM_{st}^{+})P_{+} + (W_{st}^{-}(p) + mM_{st}^{-})P_{-}$$

where

$$W^{+}(p) = \begin{pmatrix} -W(p) & 1 & & \\ & -W(p) & \ddots & \\ & & \ddots & 1 \\ & & & -W(p) \end{pmatrix} \qquad M^{+} = \begin{pmatrix} & & \\ & & \\ 1 & & \end{pmatrix}$$

$$W^{-}(p) = \begin{pmatrix} -W(p) & & & \\ 1 & -W(p) & & \\ & \ddots & \ddots & \\ & & 1 & -W(p) \end{pmatrix} \qquad M^{-} = \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix}$$

and one puts

$$W(p) = 1 - M + 2\sum_{\lambda} \sin^2 \frac{p_\lambda}{2}$$

Chiral mode

This is a Wilson action endowed with an additional flavor index s, plus a special mass matrix for these flavors

It can be imagined as having several flavors of lattice Dirac fermions, mixed in a very special way so that a large mass hierarchy is generated

The mass matrix governs the mixing among the flavors and induces a sophisticated structure on the flavor space, which at the end produces one chiral mode which is nearly massless together with $N_s - 1$ heavy fermions

To determine the chiral mode one must diagonalize (in the fifth dimension) the mass matrix – however this is not hermitian $\Longrightarrow$ diagonalize $D^\dagger D$

Chiral mode:

$$\chi_0(x) = \sqrt{1 - w_0^2} \sum_s (P_+ w_0^{s-1} \psi_s(x) + P_- w_0^{N_s - s} \psi_s(x))$$

where $w_0 = W(0) = 1 - M$

We see from the damping factors w_0^{s-1} and $w_0^{N_s - s}$ that the chiral mode is exponentially localized near the boundaries of the fifth dimension

Chiral mode

However: the domain-wall height M (and thus w_0) are additively renormalized

The standard "physical" quark fields *(chiral modes)* used in Monte Carlo simulations are instead constructed only from quark fields exactly located at the boundaries:

$$\begin{aligned} q(x) &= P_+\psi_1(x) + P_-\psi_{N_s}(x) \\ \overline{q}(x) &= \overline{\psi}_{N_s}(x)P_+ + \overline{\psi}_1(x)P_- \end{aligned}$$

These physical quark fields $q(x)$ are more convenient to use than $\chi_0(x)$, because of the renormalization of w_0

Moreover, at finite N_s there is anyway an additional issue: $\chi_0(x)$ is not exactly the chiral mode

$\chi_0(x)$ is in fact an eigenvector of the mass matrix only up to terms of order

$$N_s\, e^{-N_s\alpha(0)}$$

where $\alpha(0)$ is defined by

$$\cosh(\alpha(0)) = \frac{1+w_0^2}{2|w_0|}$$

Axial current

One has to consider 4- as well as 5-dimensional objects, and the relations between them $\rightarrow$ appropriate expressions for currents and operators

Variation of the action with respect to the 5-dimensional axial transformation

$$\begin{aligned} \delta\psi_s(x) &= iq(s)\,(\alpha_A)^a_s(x)\,\frac{\lambda^a}{2}\,\psi_s(x) \\ \delta\overline{\psi}_s(x) &= -iq(s)\,\overline{\psi}_s(x)\,(\alpha_A)^a_s(x)\,\frac{\lambda^a}{2} \end{aligned}$$

For each fixed 4-dimensional slice s is like a *vector* transformation – what makes the difference *(in the fifth dimension)* is

$$q(s) = \begin{cases} 1\,, & 1 \le s \le N_s/2 \\ -1\,, & N_s/2 < s \le N_s \end{cases}$$

Consider the 5-dimensional current which for $\mu = 1, \ldots, 4$ is the conserved *vector* current for Wilson fermions (with $r = -1$):

$$j^a_\mu(x,s) = \frac{1}{2}\,\overline{\psi}_s(x)\,(\gamma_\mu + 1)U_\mu(x)\,\frac{\lambda^a}{2}\psi_s(x+\hat{\mu}) + \frac{1}{2}\,\overline{\psi}_s(x+\hat{\mu})\,(\gamma_\mu - 1)U^\dagger_\mu(x)\,\frac{\lambda^a}{2}\psi_s(x)$$

while its fifth component , which couples neighboring s-slices, is:

$$j^a_5(x,s) = \begin{cases} \overline{\psi}_s(x)\,P_+\,\frac{\lambda^a}{2}\psi_{s+1}(x) - \overline{\psi}_{s+1}(x)\,P_-\,\frac{\lambda^a}{2}\psi_s(x)\,, & 1 \le s < N_s \\ \overline{\psi}_{N_s}(x)\,P_+\,\frac{\lambda^a}{2}\psi_1(x) - \overline{\psi}_1(x)\,P_-\,\frac{\lambda^a}{2}\psi_{N_s}(x)\,, & s = N_s \end{cases}$$

Axial current

Continuity equation:

$$\sum_{\mu} \nabla_{\mu}\, j^{a}_{\mu}(x,s) = \begin{cases} -j^{a}_{5}(x,1) - m\, j^{a}_{5}(x,N_s) & , \quad s = 1 \\ -\nabla_{5}\, j^{a}_{5}(x,s) & , \quad 1 < s < N_s \\ j^{a}_{5}(x,N_s-1) + m\, j^{a}_{5}(x,N_s) & , \quad s = N_s \end{cases}$$

The 4-dimensional axial current is given by

$$A^{a}_{\mu}(x) = -\sum_{s=1}^{N_s} q(s)\, j^{a}_{\mu}(x,s)$$

and satisfies the PCAC equation (Furman e Shamir, 1995)

$$\nabla^{\mu} A^{a}_{\mu}(x) = \; 2\; j^{a}_{5}(x,\frac{N_s}{2}) + 2m\; j^{a}_{5}(x,N_s)$$

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The term $j^{a}_{5}(x,\frac{N_s}{2})$ is nonzero only for $N_s < \infty$, and measures the violations of chiral symmetry when the distance between the two walls is not infinite

It is like a γ_5 located at the middle point of the fifth dimension

In fact, since $j^{a}_{5}(x,N_s) = \overline{q}(x)\gamma_5 \frac{\lambda^a}{2} q(x)$, the term $2m\; j^{a}_{5}(x,N_s) = 2m\, P(x)$ is like a γ_5 located on the domain walls

One is in general interested in various observables which can measure the amount of chiral symmetry breaking

Propagators for finite N_s

Tree-level quark propagator $\langle \psi_s(-p)\overline{\psi}_t(p)\rangle$ for massless quarks:

$$\sum_u \left[\left(-i\gamma_\mu \sin p_\mu \, \delta_{s,u} + W^-_{su}(p) \right) G^R_{ut}(p) \, P_+ + \left(-i\gamma_\mu \sin p_\mu \, \delta_{s,u} + W^+_{su}(p) \right) G^L_{ut}(p) \, P_- \right]$$

where the expressions of the function $G^R(p)$ and $G^L(p)$ are

$$\begin{aligned} G^R_{st}(p) \quad = \quad & \frac{A(p)}{F(p)} \Big[(1 - W(p)e^{-\alpha(p)})(e^{-2N_s\alpha(p)} - 1)e^{(s+t)\alpha(p)} \\ & + 2W(p)\sinh(\alpha(p))(e^{(s-t)\alpha(p)} + e^{-(s-t)\alpha(p)}) \\ & + (1 - W(p)e^{\alpha(p)})(1 - e^{2N_s\alpha(p)})e^{-(s+t)\alpha(p)} \Big] \\ & + A(p)\left(e^{(N_s - |s-t|)\alpha(p)} + e^{-(N_s-|s-t|)\alpha(p)} \right) \end{aligned}$$

and

Propagators for finite N_s

$$
\begin{aligned}
G^L_{st}(p) \quad = \quad & \frac{A(p)}{F(p)} \Big[(e^{-2\alpha(p)} - W(p)e^{-\alpha(p)})(e^{-2N_s\alpha(p)} - 1)e^{(s+t)\alpha(p)} \\
& +2W(p)\sinh(\alpha(p))(e^{(s-t)\alpha(p)} + e^{-(s-t)\alpha(p)}) \\
& +(e^{2\alpha(p)} - W(p)e^{\alpha(p)})(1 - e^{2N_s\alpha(p)})e^{-(s+t)\alpha(p)} \Big] \\
& +A(p)\left(e^{(N_s - |s-t|)\alpha(p)} + e^{-(N_s-|s-t|)\alpha(p)} \right)
\end{aligned}
$$

where

$$
\begin{aligned}
A(p) \quad &= \quad \frac{1}{2W(p)\,\sinh(\alpha(p))} \, \frac{1}{2\sinh(N_s\alpha(p))} \\
F(p) \quad &= \quad e^{N_s\alpha(p)}(1 - W(p)e^{\alpha(p)}) \\
& \quad\quad -e^{-N_s\alpha(p)}(1 - W(p)e^{-\alpha(p)})
\end{aligned}
$$

and $\alpha(p)$ is defined by the positive solution of the equation

$$
\cosh(\alpha(p)) = \frac{1 + W^2(p) + \sum_\lambda \sin^2 p_\lambda}{2|W(p)|}
$$

Propagators for finite N_s

These are the only calculations which use the exact propagator at finite N_s:

- **S. C.**, *"Chiral violations in domain-wall QCD from one-loop perturbation theory at finite N_s"*
  Physical Review D75 (2007) 054505
- **S. C.**, *"Perturbative chiral violations for domain-wall QCD with improved gauge actions"*
  Nuclear Physics B801 (2008) 220

Matrix elements of observables contain states and operators constructed from the physical quark fields, and this requires additional propagators

The additional propagators which connect the 4-dimensional physical quark fields with the 5-dimensional quark fields which appear in the Lagrangian had not yet been computed for finite N_s

Propagators of the physical fields

We have then derived the expressions of these propagators for the case of finite N_s:

$$\begin{aligned}
\langle q(-p)\overline{\psi}_s(p)\rangle &= P_+\langle\psi_1(-p)\overline{\psi}_s(p)\rangle + P_-\langle\psi_{N_s}(-p)\overline{\psi}_s(p)\rangle \\
&= \left(\frac{i\gamma_\mu \sin p_\mu}{E(p)} + e^{-N_s\alpha(p)}\,\frac{2W(p)\sinh(\alpha(p))}{E(p)\left(1-e^{-2N_s\alpha(p)}\right)}\right) \\
&\quad\times\Big(\left(e^{-(N_s-s)\alpha(p)} - e^{-2N_s\alpha(p)}e^{(N_s-s)\alpha(p)}\right)P_+ \\
&\quad\quad+\left(e^{-(s-1)\alpha(p)} - e^{-2N_s\alpha(p)}e^{(s-1)\alpha(p)}\right)P_-\Big) \\
&\quad-\frac{1}{1-e^{-2N_s\alpha(p)}}\,e^{-\alpha(p)}\Big(\left(e^{-(s-1)\alpha(p)} - e^{-2(N_s-1)\alpha(p)}e^{(s-1)\alpha(p)}\right)P_+ \\
&\quad\quad+\left(e^{-(N_s-s)\alpha(p)} - e^{-2(N_s-1)\alpha(p)}e^{(N_s-s)\alpha(p)}\right)P_-\Big)
\end{aligned}$$

where

$$E(p) = 1 - W(p)e^{\alpha(p)} - e^{-2N_s\alpha(p)}\left(1 - W(p)e^{-\alpha(p)}\right)$$

$\langle\psi_s(-p)\overline{q}(p)\rangle$ gives a similar expression

Propagators of the physical fields

For the calculation of perturbative amplitudes one also needs the knowledge of the expressions of these new propagators for small momentum :

$$\langle q(-p)\overline{\psi}_s(p)\rangle_c = -\frac{1-w_0^2}{1-w_0^{2N_s}}\,\frac{ip\!\!\!/ + w_0^{N_s}(1-w_0^2)}{p^2 + w_0^{2N_s}(1-w_0^2)^2}\Big(\big(w_0^{N_s-s} - w_0^{2N_s}w_0^{-(N_s-s)}\big)P_+ + \big(w_0^{s-1} - w_0^{2N_s}w_0^{-(s-1)}\big)P_-\Big)$$

$$-\frac{1}{1-w_0^{2N_s}}\,w_0\Big(\big(w_0^{s-1} - w_0^{2(N_s-1)}w_0^{-(s-1)}\big)P_+ + \big(w_0^{N_s-s} - w_0^{2(N_s-1)}w_0^{-(N_s-s)}\big)P_-\Big)$$

and a similar expression for $\langle \psi_s(-p)\overline{q}(p)\rangle_c$

The factors $1-w_0^2$ are related to the sums of the tree-level exponential damping factors over the fifth dimension:

$$\lim_{N_s\to\infty}\sum_{s=1}^{N_s}\Big(w_0^{N_s-s}P_+ + w_0^{s-1}P_-\Big)^2 = \frac{1}{1-w_0^2}$$

Since $w_0 = e^{-\alpha(0)}$, it is easy to see that the terms which are proportional to $w_0^{N_s} = e^{-N_s\alpha(0)}$ rapidly approach zero when N_s becomes large

Residual mass at tree level

Finally, we also need the tree-level propagator of the physical fields

The function that describes the propagation of the physical fields alone is

$$\langle q(-p)\overline{q}(p)\rangle = \frac{1}{E(p)}\left(i\gamma_\mu \sin p_\mu \,(1-e^{-2N_s\alpha(p)}) + e^{-N_s\alpha(p)}\cdot 2W(p)\sinh(\alpha(p))\right)$$

In the limit of small momentum this expression becomes

$$\langle q(-p)\overline{q}(p)\rangle_c = -(1-w_0^2)\,\frac{i\not{p} + w_0^{N_s}(1-w_0^2)}{p^2 + w_0^{2N_s}(1-w_0^2)^2}$$

Thus, domain-wall fermions present at finite N_s some new peculiar features

Although in the Lagrangian all quark fields are massless , the truncation at finite N_s generates already at tree level a nonvanishing residual mass of the physical fields (Vranas, . . .)

$$a\,m_{res}^{(0)} = -w_0^{N_s}(1-w_0^2) = -(1-M)^{N_s}\cdot M(2-M)$$

Residual mass at tree level

As expected, this tree-level residual mass vanishes when $N_s = \infty$

The sign can be inferred from the general expression of a fermion propagator of mass μ for small momentum in Euclidean space:

$$\frac{-i\not{p} + \mu}{p^2 + \mu^2} = \frac{1}{i\not{p} + \mu}$$

Since we work with even N_s (where the fermion determinant can be proven to be positive) , $m_{res}^{(0)}$ is always a negative quantity

With our calculations we have thus reproduced, up to a sign, the result for $m_{res}^{(0)}$ found by Shamir, Vranas, Kikukawa, Neuberger, Blum, Wingate, Soni, ...

That result was derived by considering the quadratic operator $D^\dagger D$, which could perhaps explain the sign discrepancy

Radiative corrections also give additional contributions to m_{res}

We will see that when the one-loop corrections are taken into account, the residual mass changes sign and becomes positive

Residual mass at tree level

Residual mass at tree level in lattice units.

M	$N_s = 8$	$N_s = 12$	$N_s = 16$	$N_s = 20$	$N_s = 24$	$N_s = 28$	$N_s = 32$	$N_s = 48$	$N_s = \infty$
0.1	**-12.91556**	**-8.47390**	**-5.55973**	**-3.64774**	**-2.39328**	**-1.57023**	**-1.03023**	**-0.19090**	**0**
0.2	**-9.53767**	**-3.90663**	**-1.60015**	**-0.65542**	**-0.26846**	**-0.10996**	**-0.04504**	**-0.00127**	**0**
0.3	**-4.64274**	**-1.11472**	**-0.26764**	**-0.06426**	**-0.01543**	**-0.00370**	**-0.00089**	**0.00000**	**0**
0.4	**-1.69750**	**-0.22000**	**-0.02851**	**-0.00370**	**-0.00048**	**-0.00006**	**-0.00001**	**0.00000**	**0**
0.5	**-0.46264**	**-0.02891**	**-0.00181**	**-0.00011**	**-0.00001**	**0.00000**	**0.00000**	**0.00000**	**0**
0.6	**-0.08693**	**-0.00223**	**-0.00006**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
0.7	**-0.00943**	**-0.00008**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
0.8	**-0.00039**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
0.9	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.0	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.1	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.2	**-0.00039**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.3	**-0.00943**	**-0.00008**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.4	**-0.08693**	**-0.00223**	**-0.00006**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0**
1.5	**-0.46264**	**-0.02891**	**-0.00181**	**-0.00011**	**-0.00001**	**0.00000**	**0.00000**	**0.00000**	**0**
1.6	**-1.69750**	**-0.22000**	**-0.02851**	**-0.00370**	**-0.00048**	**-0.00006**	**-0.00001**	**0.00000**	**0**
1.7	**-4.64274**	**-1.11472**	**-0.26764**	**-0.06426**	**-0.01543**	**-0.00370**	**-0.00089**	**0.00000**	**0**
1.8	**-9.53767**	**-3.90663**	**-1.60015**	**-0.65542**	**-0.26846**	**-0.10996**	**-0.04504**	**-0.00127**	**0**
1.9	**-12.91556**	**-8.47390**	**-5.55973**	**-3.64774**	**-2.39328**	**-1.57023**	**-1.03023**	**-0.19090**	**0**

1-loop results for Σ_0

Quark self-energy *(where for brevity $\bar{g}^2 = (g_0^2/16\pi^2)\, C_F$)* :

$$\Sigma_q(p) = \frac{\bar{g}^2}{1-w_0^2}\left[\frac{\Sigma_0}{a} + i\not{p}\left(c_{\Sigma_1}^{(N_s,M)}\log a^2p^2 + \Sigma_1\right) - \left(i\not{p} - w_0^{N_s}(1-w_0^2)\right)\frac{2w_0}{1-w_0^2}\,\Sigma_3\right]$$

Coefficient of $\bar{g}^2$ for the complete result of Σ_0, in Feynman gauge.

M	$N_s=8$	$N_s=12$	$N_s=16$	$N_s=20$	$N_s=24$	$N_s=28$	$N_s=32$	$N_s=48$	$N_s=\infty$
0.1	17.85919	19.67880	19.25277	17.32109	14.66546	11.87550	9.30218	2.86309	0
0.2	21.82264	15.20660	8.97397	4.82019	2.44504	1.19446	0.56831	0.02488	0
0.3	15.28940	5.98758	1.99997	0.61538	0.18022	0.05106	0.01412	0.00007	0
0.4	7.55847	1.55410	0.27613	0.04548	0.00713	0.00108	0.00016	0.00000	0
0.5	2.79190	0.27543	0.02358	0.00186	0.00014	0.00001	0.00000	0.00000	0
0.6	0.76252	0.03191	0.00115	0.00004	0.00000	0.00000	0.00000	0.00000	0
0.7	0.15015	0.00250	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.8	0.02561	0.00031	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0
0.9	0.00774	0.00012	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.0	0.00418	0.00008	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.1	0.00283	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.2	-0.00172	0.00006	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.3	-0.06252	-0.00073	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0
1.4	-0.44327	-0.01733	-0.00060	-0.00002	0.00000	0.00000	0.00000	0.00000	0
1.5	-1.85863	-0.18051	-0.01531	-0.00122	-0.00009	-0.00001	0.00000	0.00000	0
1.6	-5.42629	-1.12306	-0.20001	-0.03298	-0.00521	-0.00080	-0.00012	0.00000	0
1.7	-11.49207	-4.61264	-1.55867	-0.48269	-0.14194	-0.04034	-0.01120	-0.00006	0
1.8	-16.77186	-12.20090	-7.36730	-4.00994	-2.05105	-1.00765	-0.48138	-0.02129	0
1.9	-13.69620	-15.97157	-16.18935	-14.90821	-12.82561	-10.50396	-8.29629	-2.59956	0

Automation of the calculations

At one loop two diagrams contribute to Σ_0 (at order zero in p) and enter in the calculation of the residual mass: the half-circle (or sunset) and the tadpole diagrams

We have automated the calculations of the half-circle (and vertex) diagrams by developing suitable **FORM codes**, integrating afterwards the corresponding expressions by means of Fortran codes

With these codes one is able to compute matrix elements for general values of N_s and M

The numbers that we have obtained are valid both in the quenched and unquenched cases, because at one loop internal quark loops can never appear in the diagrams

In addition to running the standard numerical integration in 6 dimensions, we have also redone the computation of the half-circle diagram by hand , including the calculation of the gamma algebra and the explicit exact evaluation of the sums over the fifth-dimensional indices

This provides a rather strong check of our calculations, and also saves 2 dimensions in the numerical integration

Conclusions

- Lattice perturbation theory more involved than in the continuum:
  - more complicated propagators and vertices
  - more diagrams
  - lenghty integrals of trigonometric functions
- Heavy use of algebraic manipulation programs *(FORM)*
- LPT frequently applied for renormalizing operators
- But there are also cases where significant discoveries were first made using lattice perturbation theory:
  - nontrivial mixing structures of operators for structure functions
  - unexpected properties of hypercubic-breaking chiral actions
  - properties of domain-wall fermions for a finite extension of the fifth dimension

    $\rightarrow$ in particular, the residual mass at tree level:

    $$a\, m_{res}^{(0)} = -(1-M)^{N_s} \cdot M(2-M)$$

    more 1-loop observables measuring the breaking of chirality are planned to be calculated . . . (with G. Rossi)
- Lattice perturbation theory is still very useful . . .