

Bielefeld

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Five-loop Konishi in $\mathcal{N} = 4$ super Yang-Mills theory

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0 Introductory remarks

- The **integrable system** describing the **spectrum** of anomalous dimensions in planar $\mathcal{N} = 4$ SYM quickly runs into difficulties with **finite size effects**.
- Way out: **Lüscher corrections** or **thermodynamic Bethe ansatz** (TBA)
- **Testing ground:** Anomalous dimension of the **Konishi operator**

$$\gamma_5 = \frac{237}{16} + \frac{27}{4} \zeta_3 - \frac{135}{16} \zeta_5 - \frac{81}{16} \zeta_3^2 + \frac{945}{32} \zeta_7.$$

[Bajnok, Hegedus, Janik, Lukowski (2009)]

- Prediction based on Lüscher corrections. We **confirm** this by a **field theory calculation**.
[Eden, Heslop, Korchemsky, Sokatchev, Smirnov ([hep-th/1202.5733](#))]
- Technique: **OPE limit** on the **four-point function** of stress tensor multiplets \mathcal{T} .
Bubble integrals reduced to four loops by **IRR**, **IBP** reduction, evaluation of two **new masters**.
- **Conformal integrals** contributing to the planar four-point function **classified up to six loops**.
Coefficients fixed by criteria about **suppression of singularities** in the integrand.
[Eden, Heslop, Korchemsky, Sokatchev (2011,2012)]

1 OPE limit of $\langle \mathcal{T}\mathcal{T}\mathcal{T}\mathcal{T} \rangle$

Harmonics y, \mathcal{Y} of $SU(4)$, $SO(6)$.

$$\mathcal{T} = \mathcal{Y}_{A_1 A_2} \mathcal{Y}_{B_1 B_2} \mathcal{O}_{\mathbf{20}'}^{[A_1, A_2][B_1, B_2]}, \quad \mathcal{Y}_{A_1 A_2} = -\frac{1}{2} \epsilon_{a_1 a_2} y_{A_1}^{a_1} y_{A_2}^{a_2} \quad \mathcal{O}_{\mathbf{20}'} = \text{tr}(\phi\phi).$$

Leading terms of the **OPE**

$$\begin{aligned} \lim_{x_1 \rightarrow x_2} \mathcal{T}_1 \mathcal{T}_2 &= c_{\mathcal{I}}(N) \frac{y_{12}^4}{x_{12}^4} \mathcal{I} + c_{\mathcal{O}}(N) \frac{y_{12}^2}{x_{12}^2} \mathcal{Y}_{1, A_1 A_2} \mathcal{Y}_{2, B_1 B_2} \mathcal{O}_{\mathbf{20}'}^{[A_1, A_2][B_1, B_2]}(x_2) + \\ &+ c_{\mathcal{K}}(a, N) \frac{y_{12}^4}{(x_{12}^2)^{1-\frac{\gamma_{\mathcal{K}}}{2}}} \mathcal{K}(x_2) + \dots \end{aligned}$$

Double limit $x_1 \rightarrow x_2, x_3 \rightarrow x_4$ of $\langle \mathcal{T}\mathcal{T}\mathcal{T}\mathcal{T} \rangle$ contains

$$\langle \mathcal{I} \mathcal{I} \rangle = 1, \quad \langle \mathcal{K}(x_2) \mathcal{K}(x_4) \rangle = d_{\mathcal{K}}(a, N) \frac{1}{(x_{24}^2)^{2+\gamma_{\mathcal{K}}}}$$

and

$$\begin{aligned} \langle \mathcal{O}_{\mathbf{20}'}^{[A_1, A_2][B_1, B_2]}(x_2) \mathcal{O}_{\mathbf{20}'}^{[C_1, C_2][D_1, D_2]}(x_4) \rangle &= \\ d_{\mathcal{O}}(N) \frac{\epsilon^{A_1 A_2 C_1 C_2} \epsilon^{B_1 B_2 D_1 D_2} + \epsilon^{A_1 A_2 D_1 D_2} \epsilon^{B_1 B_2 C_1 C_2} - \frac{1}{3} \epsilon^{A_1 A_2 B_1 B_2} \epsilon^{C_1 C_2 D_1 D_2}}{x_{24}^4}. \end{aligned}$$

We should thus find:

$$\begin{aligned} x_{13}^8 \lim_{x_2 \rightarrow x_1, x_4 \rightarrow x_3} \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle &= \frac{y_{12}^4 y_{34}^4}{X^2} c_{\mathcal{I}}(N)^2 + \frac{y_{12}^2 y_{34}^2 (y_{13}^2 y_{24}^2 + y_{14}^2 y_{23}^2)}{X} c_{\mathcal{O}}(N)^2 d_{\mathcal{O}}(N) \\ &+ \frac{y_{12}^4 y_{34}^4}{X} \left(c_{\mathcal{K}}(a, N)^2 d_{\mathcal{K}}(a, N) X^{\frac{\gamma_{\mathcal{K}}}{2}} - \frac{1}{3} c_{\mathcal{O}}(N)^2 d_{\mathcal{O}}(N) \right) + \dots \end{aligned}$$

where $X = (x_{12}^2 x_{34}^2)/x_{13}^4$.

By **Lagrangian insertions** and **symmetry**:

[Eden, Petkou, Schubert, Sokatchev (2000)], [Eden, Heslop, Korchemsky, Sokatchev (2011)]

$$\begin{aligned} \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle_{a^0} &= \frac{(N^2 - 1)^2}{4(4\pi^2)^4} \left(\frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} + \frac{y_{13}^4 y_{24}^4}{x_{13}^4 x_{24}^4} + \frac{y_{14}^4 y_{23}^4}{x_{14}^4 x_{23}^4} \right) \\ &+ \frac{(N^2 - 1)}{(4\pi^2)^4} \left(\frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{14}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2} + \frac{y_{12}^2 y_{24}^2 y_{34}^2 y_{13}^2}{x_{12}^2 x_{24}^2 x_{34}^2 x_{13}^2} + \frac{y_{13}^2 y_{23}^2 y_{24}^2 y_{14}^2}{x_{13}^2 x_{23}^2 x_{24}^2 x_{14}^2} \right) \\ \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle_{a^l} &= \frac{R}{x_{12}^2 x_{34}^2} \frac{2(N^2 - 1)}{(4\pi^2)^4} \sum_{l=1}^{\infty} a^l F^{(l)}, \quad a = \frac{g^2 N}{4\pi^2}, \end{aligned}$$

with the polynomial

$$\begin{aligned} R &= y_{12}^4 y_{34}^4 + u y_{13}^4 y_{24}^4 + \frac{u}{v} y_{14}^4 y_{23}^4 \\ &- \frac{u+v-1}{v} y_{12}^2 y_{23}^2 y_{34}^2 y_{14}^2 - (u+1-v) y_{12}^2 y_{24}^2 y_{34}^2 y_{13}^2 - \frac{u(1+v-u)}{v} y_{13}^2 y_{23}^2 y_{24}^2 y_{14}^2. \end{aligned}$$

Cross ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

We do find

$$\begin{aligned} x_{24}^8 \lim_{x_1 \rightarrow x_2, x_3 \rightarrow x_4} \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle &= \frac{y_{12}^4 y_{34}^4}{X^2} \frac{1}{4} (N^2 - 1)^2 + \frac{y_{12}^2 y_{34}^2 (y_{13}^2 y_{24}^2 + y_{14}^2 y_{23}^2)}{X} (N^2 - 1) + \\ &+ \frac{y_{12}^4 y_{34}^4}{X} \left(a G^{(1)} + a^2 G^{(2)} + a^3 G^{(3)} + \dots \right) (N^2 - 1) \\ &\dots \end{aligned}$$

where $G^{(l)}(x)$ represent the loop-corrections in the coincidence limit.

Put

$$c_{\mathcal{K}}(a, N)^2 d_{\mathcal{K}}(a, N) = \alpha(a, N) (N^2 - 1).$$

OPE constraint on the anomalous dimension of the Konishi operator:

$$a G^{(1)} + a^2 G^{(2)} + a^3 G^{(3)} + \dots = \left(\frac{1}{3} + \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3 + \dots \right) X^{\frac{1}{2}(\gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + \dots)} - \frac{1}{3}.$$

Up to three loops:

$$\begin{aligned}
F^{(1)} &= g(1, 2, 3, 4), \\
F^{(2)} &= \frac{1}{2} g(1, 2, 3, 4)^2 \left(x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2 \right) \\
&\quad + 2 (h(1, 2; 3, 4) + h(1, 3; 2, 4) + h(1, 4; 2, 3)), \\
F^{(3)} &= 2 g(1, 2, 3, 4) \left(x_{12}^2 x_{34}^2 h(1, 2; 3, 4) + x_{13}^2 x_{24}^2 h(1, 3; 2, 4) + x_{14}^2 x_{23}^2 h(1, 4; 2, 3) \right) \\
&\quad + 6 (l(1, 2; 3, 4) + l(1, 3; 2, 4) + l(1, 4; 2, 3)) \\
&\quad + 4 (E(1; 3, 4; 2) + E(1; 2, 4; 3) + E(1; 2, 3; 4)) \\
&\quad + (s+t) H(1, 2; 3, 4) + \frac{1+t}{s} H(1, 3; 2, 4) + \frac{1+s}{t} H(1, 4; 2, 3).
\end{aligned}$$

Definition of the integrals:

$$\begin{aligned}
g(1, 2, 3, 4) &= -\frac{1}{4\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}, \\
h(1, 2; 3, 4) &= +x_{34}^2 \frac{1}{(4\pi^2)^2} \int \frac{d^4 x_5 d^4 x_6}{(x_{15}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2)}, \\
l(1, 2; 3, 4) &= -x_{34}^4 \frac{1}{(4\pi^2)^3} \int \frac{d^4 x_5 d^4 x_6 d^4 x_7}{(x_{15}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{36}^2 x_{46}^2) x_{67}^2 (x_{27}^2 x_{37}^2 x_{47}^2)}, \\
E(1; 3, 4; 2) &= -x_{23}^2 x_{24}^2 \frac{1}{(4\pi^2)^3} \int \frac{d^4 x_5 d^4 x_6 d^4 x_7 x_{16}^2}{(x_{15}^2 x_{25}^2 x_{35}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2) x_{67}^2 (x_{17}^2 x_{27}^2 x_{47}^2)}, \\
H(1, 2; 3, 4) &= -x_{12}^2 x_{34}^4 \frac{1}{(4\pi^2)^3} \int \frac{d^4 x_5 d^4 x_6 d^4 x_7 x_{57}^2}{(x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{36}^2 x_{46}^2) x_{67}^2 (x_{17}^2 x_{27}^2 x_{37}^2 x_{47}^2)}.
\end{aligned}$$

MB, numerics and fitting; asymptotic expansion of Feynman integrals

$$\begin{aligned}
x_{24}^4 \lim g(1, 2, 3, 4) &= \frac{1}{4} \log(X) - \frac{1}{2} + O(X) \\
x_{24}^4 \lim h(1, 2; 3, 4) &= \frac{3}{8} \zeta_3, \\
x_{24}^4 \lim h(1, 3; 2, 4), h(1, 4; 2, 3) &= \frac{1}{32} \log(X)^2 - \frac{3}{16} \log(X) + \frac{3}{8}, \\
x_{24}^4 \lim l(1, 2; 3, 4) &= -\frac{5}{16} \zeta_5, \\
x_{24}^4 \lim l(1, 3; 2, 4), l(1, 4; 2, 3) &= \frac{1}{384} \log(X)^3 - \frac{1}{32} \log(X)^2 + \frac{5}{32} \log(X) - \frac{5}{16}, \\
x_{24}^4 \lim E(1, 2; 3, 4) &= \frac{1}{192} \log(X)^3 - \frac{3}{64} \log(X)^2 + \frac{5}{32} \log(X), \\
&\quad - \left(\frac{5}{16} \zeta_5 - \frac{3}{32} \zeta_3 + \frac{5}{32} \right), \\
x_{24}^4 \lim E(1, 3; 2, 4), E(1, 4; 2, 3) &= \frac{3}{32} \zeta_3 \log(X) - \frac{3}{16} \zeta_3, \\
x_{24}^4 \lim H(1, 2; 3, 4) &= \frac{3}{16} \zeta_3 X \log(X) - \frac{3}{16} \zeta_3 X, \\
x_{24}^4 \lim H(1, 3; 2, 4), H(1, 4; 2, 3) &= \frac{1}{192} \log(X)^3 - \frac{1}{16} \log(X)^2 + \frac{9}{32} \log(X) - 0.1359(6).
\end{aligned}$$

It follows

$$\begin{aligned}
G^{(1)} &= 2x_{24}^4 \lim_{1 \rightarrow 2,3 \rightarrow 4} F^{(1)} = \frac{1}{2} \log(X) - 1 + O(X), \\
G^{(2)} &= 2x_{24}^4 \lim_{1 \rightarrow 2,3 \rightarrow 4} F^{(2)} = \frac{3}{8} \log(X)^2 - 2 \log(X) + \frac{3}{2} \zeta_3 + \frac{7}{2} + O(X), \\
G^{(3)} &= 2x_{24}^4 \lim_{1 \rightarrow 2,3 \rightarrow 4} F^{(3)} = \frac{3}{16} \log(X)^3 - \frac{15}{8} \log(X)^2 + \left[\frac{9}{4} \zeta_3 + \frac{61}{8} \right] \log(X) \\
&\quad - \left[\frac{25}{4} \zeta_5 + 3 \zeta_3 + \frac{41}{4} + 0.544(3) \right] + O(X).
\end{aligned}$$

Identify

$$c_{\mathcal{I}} = \frac{1}{2}(N^2 - 1), \quad c_{\mathcal{O}}(N)^2 d_{\mathcal{O}}(N) = N^2 - 1, \quad c_{\mathcal{K}}(a, N)^2 d_{\mathcal{K}}(a, N)|_{a=0} = \frac{1}{3}(N^2 - 1),$$

$$\alpha_1 = -1, \quad \alpha_2 = \frac{3}{2} \zeta_3 + \frac{7}{2}, \quad \alpha_3 = - \left(\frac{25}{4} \zeta_5 + 3 \zeta_3 + \frac{41}{4} + 0.544(3) \right)$$

and in full agreement with the literature

$$\gamma_1 = 3, \quad \gamma_2 = -3, \quad \gamma_3 = \frac{21}{4}.$$

2 More power for γ_5

Logarithm of the OPE condition on $\gamma_{\mathcal{K}}$:

$$\ln \left(1 + 3 \sum_{l \geq 1} a^l G^{(l)}(x_i) \right) \longrightarrow \frac{1}{2} \gamma_{\mathcal{K}}(a) \ln X + O(X^0)$$

- Above: Take **limit** $x_2 \rightarrow x_1$, $x_4 \rightarrow x_3$ of **finite integrals**, study large logarithms.
- Now: **Identify** $x_2 = x_1$, $x_4 = x_3$ and introduce **dimreg**, study pole/logs.
- **single log** on r.h.s. \Leftrightarrow **single pole**.

Example: $G^{(1)} \propto g(1, 2, 3, 4)$

Let $x_{13} = p$, $x_{15} = k$. Find:

$$G_{\epsilon}^{(1)} = -2 \frac{\bar{\mu}^{2\epsilon}}{4\pi^2} \int \frac{d^{4-2\epsilon}k p^4}{k^4(p-k)^4} = (\bar{\mu}^2/x_{13}^2)^{\epsilon} \left(\frac{1}{\epsilon} + 1 + O(\epsilon^2) \right) = \frac{1}{\epsilon} + \ln \left(\frac{\bar{\mu}^2}{x_{13}^2} \right) + \dots$$

- Reproduces $\gamma_1 = 3a$ if $x_{12}^2 = \bar{\mu}^2 = x_{34}^2$.

- l -loop contribution goes like $\frac{\gamma_l}{3l\epsilon} (\bar{\mu}^2/x_{13}^2)^{l\epsilon}$.

Two loops:

$$G_\epsilon^{(2)} - \frac{3}{2} (G_\epsilon^{(1)})^2 = 2 \left(\frac{\bar{\mu}^{2\epsilon}}{4\pi^2} \right)^2 \int d^{4-2\epsilon} x_5 d^{4-2\epsilon} x_6 \frac{2x_{13}^6(x_{15}^2 x_{36}^2 + x_{16}^2 x_{35}^2 - x_{13}^2 x_{56}^2)}{(x_{15}^4 x_{16}^4) x_{56}^2 (x_{35}^4 x_{36}^4)}$$

First two numerator terms:

$$M^{(2)} = \left(\frac{\bar{\mu}^{2\epsilon}}{4\pi^2} \right)^2 \int \frac{d^{4-2\epsilon} k_1 d^{4-2\epsilon} k_2}{k_1^4 k_2^2 (k_1 - k_2)^2 (p - k_1)^2 (p - k_2)^4} = (\bar{\mu}^2/x_{13}^2)^{2\epsilon} \left(\frac{1}{8\epsilon^2} + \frac{3}{16\epsilon} - \frac{1}{16} + O(\epsilon) \right).$$

- Last term $-\frac{1}{4} (G^{(1)})^2$ so **double pole cancels!** Single pole $-1/(2\epsilon) = \gamma_2/(6\epsilon)$.
- *Mincer* gets the three-loop anomalous dimension right.
- At four-loops Laporta algorithm needed, but indices relatively high.

- Most singular regions $x_5 \rightarrow x_1$, $x_6 \rightarrow x_3$ and $x_5 \rightarrow x_3$, $x_6 \rightarrow x_1$ suppressed.
- Numerator vanishes for $\mathbf{x}_5 \rightarrow \mathbf{x}_1$ and $x_5 \rightarrow x_3$ with x_6 in generic.
- Weaker singularity $\mathbf{x}_5, \mathbf{x}_6 \rightarrow \mathbf{x}_1$ (or both to x_3)
- Introduce $\delta^2 \ll x_{13}^2$ and look ball domain Ω_δ around \mathbf{x}_1 , say.
- Replace $\mathbf{x}_{35}^2, \mathbf{x}_{36}^2 \rightarrow \mathbf{x}_{13}^2$.

$$\begin{aligned}
G_\epsilon^{(2)} - \frac{3}{2} (G_\epsilon^{(1)})^2 &\sim 8 \left(\frac{\bar{\mu}^{2\epsilon}}{4\pi^2} \right)^2 \int_{\Omega_\delta} d^{4-2\epsilon} x_5 d^{4-2\epsilon} x_6 \frac{x_{15}^2 + x_{16}^2 - x_{56}^2}{(x_{15}^4 x_{16}^4) x_{56}^2} \\
&= \left(\frac{\bar{\mu}^{2\epsilon}}{\pi^2} \right)^2 \int_{\Omega_\delta} d^{4-2\epsilon} x_5 d^{4-2\epsilon} x_6 \frac{(x_{15} \cdot x_{16})}{(x_{15}^4 x_{16}^4) x_{56}^2} \\
&= \bar{\mu}^{4\epsilon} \int_0^\delta \frac{dr_5 dr_6}{(r_5 r_6)^{2\epsilon}} \frac{r_<}{r_>} = (\bar{\mu}^2 / \delta^2)^{2\epsilon} \left(-\frac{1}{2\epsilon} + O(\epsilon^0) \right),
\end{aligned}$$

Here $r_5 = |x_{15}|$, $r_6 = |x_{16}|$, expand $1/x_{56}^2$, integrate out angle.

Notation: $r_< = \min(r_5, r_6)$ and $r_> = \max(r_5, r_6)$

- Half as many propagators, so **half as many propagators with high indices!**

3 Loop reduction

Example: **Non-planar** sector at **four loops**

$$I(x_{13}) = (\bar{\mu}^2)^{4\epsilon} \int \frac{(x_{13}^2)^4 d^D x_5 \dots d^D x_8}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{35}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2}.$$

- **Indices 1** although $D = 4 - 2\epsilon$. **Vital for IBP** reduction.

Has a **simple pole** in ϵ

$$I(x_{13}) = (\bar{\mu}^2/x_{13}^2)^{4\epsilon} \left[\frac{C}{\epsilon} + O(\epsilon^0) \right]$$

from the region where $x_5, \dots, x_8 \rightarrow x_1$ or x_3 . First case:

$$F(x_1, x_5, \dots, x_8) = \frac{1}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2}$$

Ultralocal counterterm:

$$\Delta(x_1, x_5, \dots, x_8) = \frac{C}{2\epsilon} \delta(x_1 - x_5) \dots \delta(x_1 - x_8)$$

- **p-space** log divergent integrals: **Counterterm constant \Leftrightarrow rearrange momenta** (IRR)
- $p = 0 \Leftrightarrow \int x$
- **Can drop an x-integral!**

Treat x_5 as an outer point:

$$F(x_1, x_5) = \frac{1}{x_{15}^2} \int \frac{d^D x_6 d^D x_7 d^D x_8}{x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{57}^2 x_{68}^2 x_{78}^2} = \frac{C}{2\epsilon} \delta(x_1 - x_5) + O(\epsilon^0) = \frac{f(\epsilon)}{(x_{15}^2)^{2+3\epsilon}}$$

Dual graph 3-loop **FA** propagator bubble. **Finite!** $f(\epsilon) = 20\zeta_5 + O(\epsilon)$

Simple **pole** in ϵ : **Hidden in $(x_{15}^2)^{-2-3\epsilon}$**

$$\frac{1}{\pi^{D/2}} \int d^D x \epsilon^{ipx} \frac{f(\epsilon)}{(x^2)^{2+3\epsilon}} = \frac{4^{-4\epsilon} \Gamma(-4\epsilon)}{\Gamma(2+3\epsilon)} \frac{f(\epsilon)}{(p^2)^{-4\epsilon}} = -\frac{f(0)}{4\epsilon} + O(\epsilon^0),$$

so

$$C = -\frac{1}{2} f(0) = -10\zeta_5.$$

Logarithm of the **correlator** at \mathbf{a}^3 in the limit $x_{12}, x_{34} \rightarrow 0$:

$$I^{(3)} = 3 [G^{(3)} - 3 G^{(1)} G^{(2)} + 3 (G^{(1)})^3]$$

- **Symmetrise** over integration points and outer points ($S_{l+4} = S_7$ symmetry), see Paul Heslop's talk.
- Restrict to $\mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7 \rightarrow \mathbf{x}_1$ region.

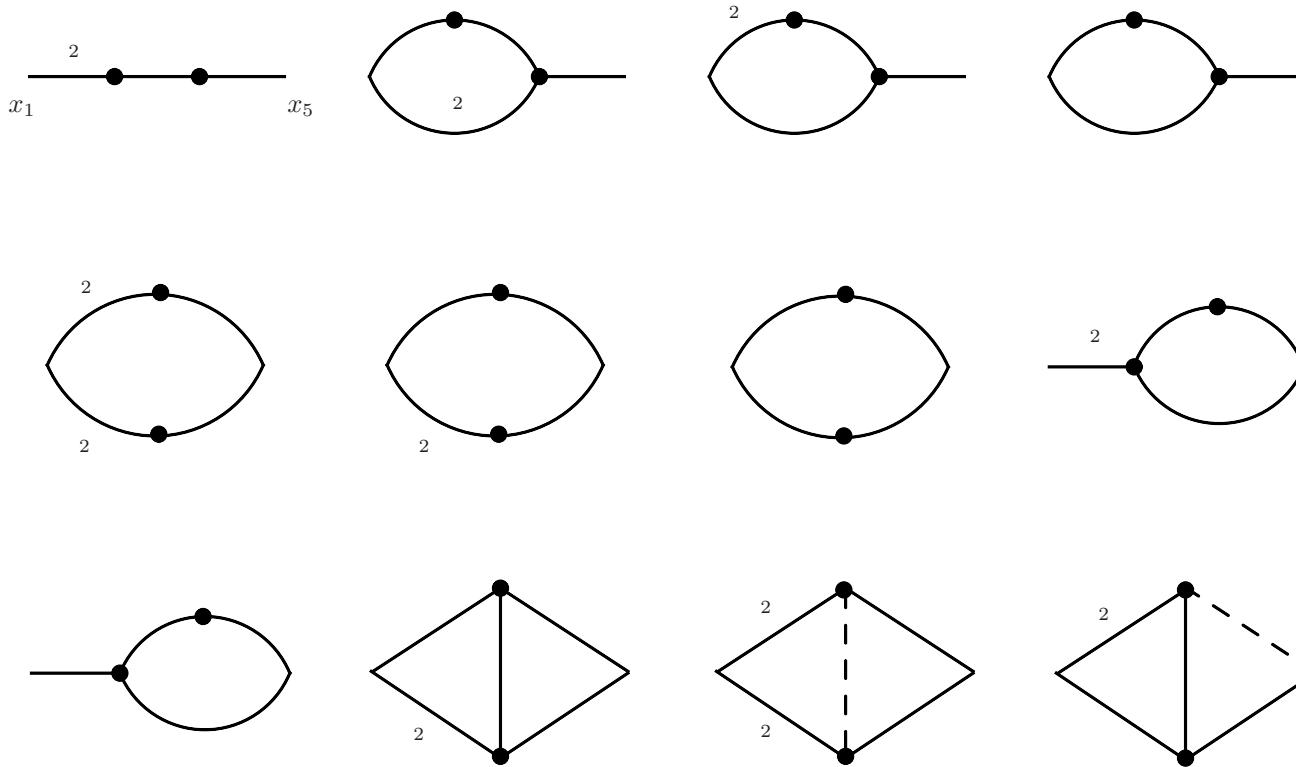
$$I^{(3)} \rightarrow -\frac{2}{x_{15}^4 x_{16}^4 x_{17}^4 x_{56}^2 x_{57}^2 x_{67}^2} \left[4 x_{15}^4 x_{16}^2 + x_{15}^4 x_{67}^2 + x_{15}^2 x_{67}^4 + 2 x_{15}^2 x_{16}^2 x_{67}^2 + x_{15}^2 x_{16}^2 x_{56}^2 - 10 x_{15}^2 x_{56}^2 x_{67}^2 + 3 x_{56}^2 x_{57}^2 x_{67}^2 + S_3 \text{ permutations} \right]$$

Integrate over x_6, x_7 with x_5 fixed

$$\int d^D x_6 d^D x_7 I^{(3)}(x_1, x_5; x_6, x_7) = \frac{f(\epsilon)}{(x_{15}^2)^{2+2\epsilon}}$$

Two-loop integrals mostly trivial, triangle rule for number 10.

- **f(ϵ) finite**, find $\gamma_3 = 21/4$.



- Four loops → three-loop bubbles. Can use *Mincer*. Did use *FIRE* (Laporta algorithm). Reproduce [Fiamberti, Santambrogio, Sieg, Zanon (2007,2008)], [Bajnok, Janik (2008)]

$$\gamma_4 = -\frac{39}{4} + \frac{9}{4}\zeta_3 - \frac{45}{8}\zeta_5 .$$

4 Five loops

- Reduction to **four loops by IRR**
- **IBP Reduction:** *FIRE* for $O(17000)$ integrals
- Known **masters** + two new cases (FT of non-planar three-loop graph with nonstandard indices)

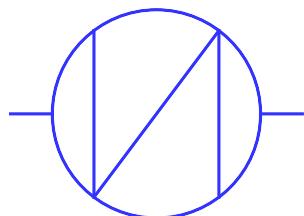
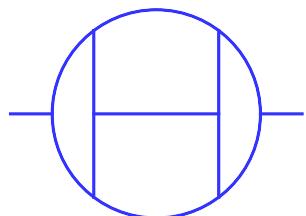
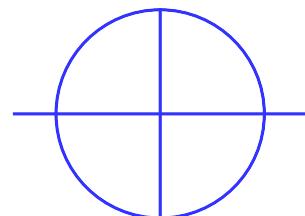
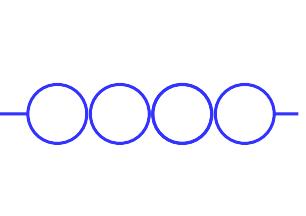
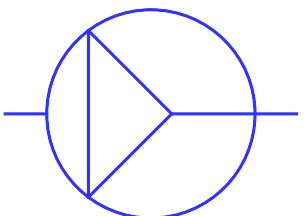
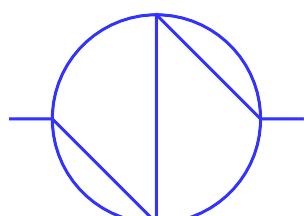
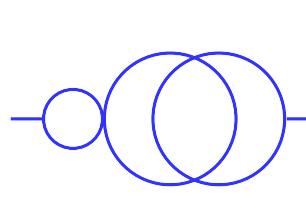
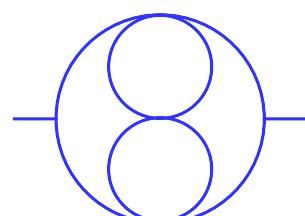
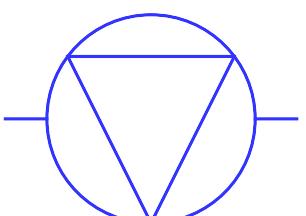
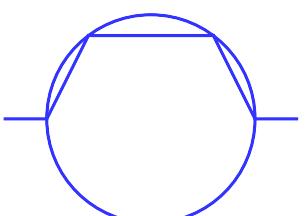
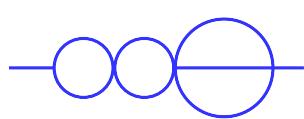
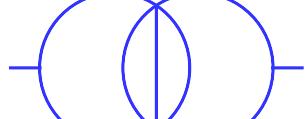
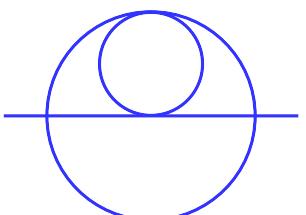
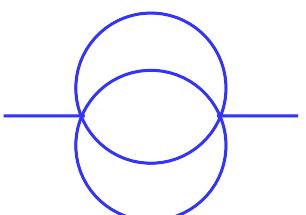
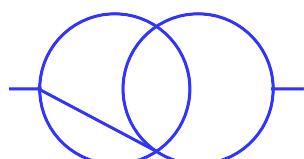
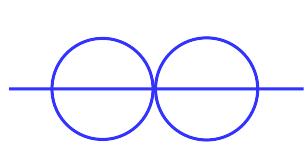
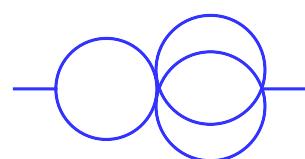
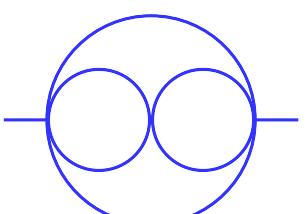
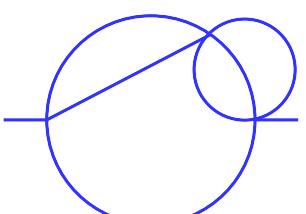
$$\begin{aligned} C_4 = & w_{44}M_{44} + w_{61}M_{61} + w_{36}M_{36} + w_{31}M_{31} + w_{35}M_{35} + w_{22}M_{22} + w_{32}M_{32} \\ & + w_{33}M_{33} + w_{34}M_{34} + w_{25}M_{25} + w_{23}M_{23} + w_{27}M_{27} + w_{24}M_{24} + w_{26}M_{26} \\ & + w_{01}M_{01} + w_{21}M_{21} + w_{12}M_{12} + w_{11}M_{11} + w_{14}M_{14} + w_{13}M_{13} + w_1I_1 + w_2I_2, \end{aligned}$$

Leading orders of the two new non-planar master integrals I_1 and I_2 :

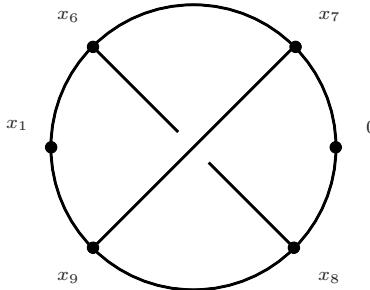
$$\begin{aligned} I_1 = & \frac{5\zeta_5}{\epsilon} + \frac{5}{378}\pi^6 - 13\zeta_3^2 + 35\zeta_5 + \left(-\frac{13}{30}\pi^4\zeta_3 - 91\zeta_3^2 + 195\zeta_5 - \frac{5}{3}\pi^2\zeta_5 + \frac{345}{4}\zeta_7 + \frac{5}{54}\pi^6 \right)\epsilon + \dots \\ I_2 = & -\frac{20\zeta_5}{\epsilon} - \frac{10}{189}\pi^6 - 8\zeta_3^2 - 40\zeta_5 + \left(-\frac{4}{15}\pi^4\zeta_3 - 16\zeta_3^2 - 80\zeta_5 + \frac{20}{3}\pi^2\zeta_5 - 520\zeta_7 - \frac{20}{189}\pi^6 \right)\epsilon + \dots \end{aligned}$$

Their coefficient functions are

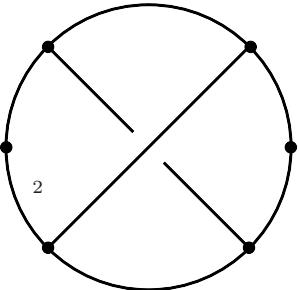
$$w_1 = \frac{3}{80}\epsilon^{-1} - \frac{21}{80} + \frac{741}{640}\epsilon + O(\epsilon^2), \quad w_2 = \frac{9}{160}\epsilon^{-1} - \frac{9}{80} + \frac{807}{320}\epsilon + O(\epsilon^2).$$

 M_{44}  M_{61}  M_{36}  M_{31}  M_{35}  M_{22}  M_{32}  M_{33}  M_{34}  M_{25}  M_{23}  M_{27}  M_{24}  M_{26}  M_{01}  M_{21}  M_{12}  M_{11}  M_{14}  M_{13}

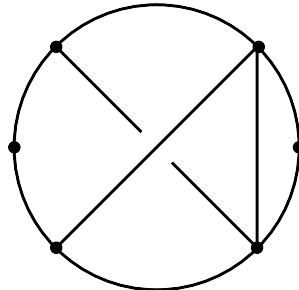
$$\begin{aligned}
w_{44} &= -\frac{3}{80} + \dots, \\
w_{61} &= \frac{3}{128} + \frac{3}{32}\epsilon + \dots, \\
w_{36} &= \frac{1}{10}\epsilon^{-1} - \frac{189}{320} + \frac{11}{16}\epsilon + \dots, \\
w_{31} &= -\frac{3}{64}\epsilon^{-2} + \frac{81}{320}\epsilon^{-1} - \frac{27}{40} + \frac{57}{80}\epsilon - \frac{9}{40}\epsilon^2 + \dots, \\
w_{35} &= -\frac{3}{80}\epsilon^{-2} + \frac{233}{480}\epsilon^{-1} - \frac{189}{80} + \frac{2131}{480}\epsilon - \frac{509}{80}\epsilon^2 + \dots, \\
w_{22} &= -\frac{11}{20}\epsilon^{-3} + \frac{295}{64}\epsilon^{-2} - \frac{85}{32}\epsilon^{-1} - \frac{39711}{640} + \frac{349167}{1280}\epsilon - \frac{358191}{512}\epsilon^2 + \frac{1748923}{1024}\epsilon^3 + \dots, \\
w_{32} &= -\frac{3}{80}\epsilon^{-3} + \frac{9}{32}\epsilon^{-2} - \frac{261}{160}\epsilon^{-1} + \frac{1281}{160} - \frac{1953}{64}\epsilon + \frac{3879}{40}\epsilon^2 - \frac{2313}{8}\epsilon^3 + \frac{1701}{2}\epsilon^4 + \dots, \\
w_{33} &= -\frac{7}{80}\epsilon^{-3} + \frac{57}{64}\epsilon^{-2} - \frac{179}{40}\epsilon^{-1} + \frac{4911}{320} - \frac{6269}{160}\epsilon + \frac{1723}{20}\epsilon^2 - \frac{2631}{16}\epsilon^3 + \frac{17481}{80}\epsilon^4 + \dots, \\
w_{34} &= -\frac{27}{320}\epsilon^{-3} + \frac{21}{20}\epsilon^{-2} - \frac{3629}{640}\epsilon^{-1} + \frac{1141}{64} - \frac{5643}{128}\epsilon + \frac{475}{4}\epsilon^2 - \frac{25953}{80}\epsilon^3 + \frac{8817}{10}\epsilon^4 + \dots, \\
w_{25} &= \frac{9}{80}\epsilon^{-4} - \frac{313}{160}\epsilon^{-3} + \frac{4221}{320}\epsilon^{-2} - \frac{25583}{640}\epsilon^{-1} + \frac{42073}{640} - \frac{3341}{32}\epsilon + \frac{12279}{64}\epsilon^2 - \frac{4131}{32}\epsilon^3 + \dots, \\
w_{23} &= -\frac{3}{80}\epsilon^{-4} - \frac{3}{10}\epsilon^{-3} + \frac{81}{80}\epsilon^{-2} + \frac{9939}{640}\epsilon^{-1} - \frac{83889}{640} + \frac{183273}{320}\epsilon - \frac{622629}{320}\epsilon^2 + \frac{978141}{160}\epsilon^3 + \dots, \\
w_{27} &= \frac{21}{160}\epsilon^{-4} - \frac{753}{320}\epsilon^{-3} + \frac{5319}{320}\epsilon^{-2} - \frac{9963}{160}\epsilon^{-1} + \frac{27913}{160} - \frac{42339}{80}\epsilon + \frac{139719}{80}\epsilon^2 - \frac{448299}{80}\epsilon^3 + \dots, \\
w_{24} &= -\frac{3}{20}\epsilon^{-4} + \frac{439}{160}\epsilon^{-3} - \frac{5141}{320}\epsilon^{-2} + \frac{20549}{640}\epsilon^{-1} + \frac{16623}{1280} - \frac{630099}{2560}\epsilon + \frac{5544039}{5120}\epsilon^2 - \frac{8490959}{2048}\epsilon^3 + \dots, \\
w_{26} &= \frac{19}{80}\epsilon^{-4} - \frac{1329}{320}\epsilon^{-3} + \frac{127}{4}\epsilon^{-2} - \frac{43443}{320}\epsilon^{-1} + \frac{504477}{1280} - \frac{2516491}{2560}\epsilon + \frac{10666651}{5120}\epsilon^2 - \frac{22997991}{10240}\epsilon^3 + \dots, \\
w_{01} &= -\frac{213}{4}\epsilon^{-5} + \frac{25311}{40}\epsilon^{-4} - \frac{1038647}{960}\epsilon^{-3} - \frac{8046763}{480}\epsilon^{-2} + \frac{38217863}{320}\epsilon^{-1} - \frac{398050007}{960} + \frac{2071600273}{1920}\epsilon + \dots, \\
w_{21} &= -\frac{133}{160}\epsilon^{-4} + \frac{61}{6}\epsilon^{-3} - \frac{30023}{480}\epsilon^{-2} + \frac{84321}{320}\epsilon^{-1} - \frac{329747}{384} + \frac{9067319}{3840}\epsilon - \frac{14362073}{2560}\epsilon^2 + \frac{59375477}{5120}\epsilon^3 + \dots, \\
w_{12} &= -\frac{61}{64}\epsilon^{-5} + \frac{2149}{160}\epsilon^{-4} - \frac{122841}{1280}\epsilon^{-3} + \frac{576611}{1280}\epsilon^{-2} - \frac{2066537}{1280}\epsilon^{-1} + \frac{6439581}{1280} - \frac{1803913}{128}\epsilon + \frac{11613069}{320}\epsilon^2 + \dots, \\
w_{11} &= -\frac{471}{160}\epsilon^{-5} + \frac{15583}{320}\epsilon^{-4} - \frac{346883}{960}\epsilon^{-3} + \frac{509833}{320}\epsilon^{-2} - \frac{5155033}{960}\epsilon^{-1} + \frac{1322649}{80} - \frac{1855317}{40}\epsilon + \frac{9142761}{80}\epsilon^2 + \dots, \\
w_{14} &= -\frac{51}{160}\epsilon^{-5} + \frac{361}{64}\epsilon^{-4} - \frac{4829}{80}\epsilon^{-3} + \frac{471857}{960}\epsilon^{-2} - \frac{677747}{240}\epsilon^{-1} + \frac{14881773}{1280} - \frac{289328273}{7680}\epsilon + \frac{1662945973}{15360}\epsilon^2 + \dots, \\
w_{13} &= \frac{573}{160}\epsilon^{-5} - \frac{5431}{80}\epsilon^{-4} + \frac{1084379}{1920}\epsilon^{-3} - \frac{1648533}{640}\epsilon^{-2} + \frac{16101181}{1920}\epsilon^{-1} - \frac{16307253}{640} + \frac{12145053}{160}\epsilon - \frac{65454483}{320}\epsilon^2 + \dots
\end{aligned}$$



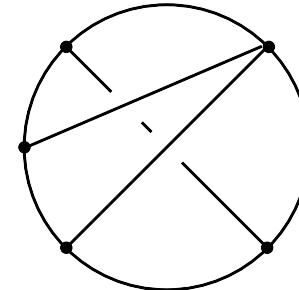
I_1



I_2



$I_3(0)$



$I_4(0)$

New master integrals (for simplicity $x_1^2 = 1$):

$$I_1 = (\bar{\mu}^2)^{4\epsilon} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_1}{\epsilon} + b_1 + c_1 \epsilon + O(\epsilon^2),$$

$$I_2 = (\bar{\mu}^2)^{4\epsilon} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 (x_{19}^2)^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_2}{\epsilon} + b_2 + c_2 \epsilon + O(\epsilon^2).$$

Fourier transform of I_3 :

$$\mathcal{F}[I_1] = 16 \left[\text{Diagram } I_3 \right] = 16 (20 \zeta_5 + O(\epsilon)) (p^2)^{-2+5\epsilon},$$

from where $a_1 = 5 \zeta_5$.

FT sends the $1/x^4$ line in I_2 to $-1/\epsilon + O(\epsilon^0)$.

We find the residue of the pole by shrinking this line to a point:

$$\mathcal{F}[I_2] = -\frac{4}{\epsilon} \left[\text{Diagram with a circle and two intersecting lines} \right] = -\frac{4}{\epsilon} \left[\text{Diagram with a circle and four radial lines} \right] = -\frac{4}{\epsilon} \times (20 \zeta_5 + O(\epsilon)) (p^2)^{-1+5\epsilon},$$

and hence $a_2 = -20 \zeta_5$.

Introduce I_3, I_4 ; at $\kappa = 0$ run Laporta to relate them to masters.

$$I_3(\kappa) = \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_7^2 x_8^2 x_{89}^2)^{1-\epsilon\kappa}},$$

$$I_4(\kappa) = \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2 x_{17}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{79}^2 x_7^2 x_8^2 x_{89}^2)^{1-\epsilon\kappa}}.$$

For finite κ the two integrals are finite as $\epsilon \rightarrow 0$:

$$I_i(\kappa) = b_i + \epsilon(c_i + \kappa d_i) + O(\epsilon^2) \quad (i = 3, 4)$$

- Compute integrals at $\kappa = 1/2, \kappa = 1$.

At $\kappa = 0$ we use IBP reduction finding:

$$\begin{aligned} b_3 &= -\frac{2}{3}b_1 - \frac{7}{3}b_2 - 70\zeta_5 + \frac{26}{3}\zeta_3^2 - \frac{65}{567}\pi^6, \\ b_4 &= -b_1 - 2b_2 - 45\zeta_5 + 7\zeta_3^2 - \frac{5}{54}\pi^6, \\ c_3 &= \frac{14}{3}b_1 + \frac{14}{3}b_2 - \frac{2}{3}c_1 - \frac{7}{3}c_2 - \frac{4667}{6}\zeta_7 + \frac{130}{9}\pi^2\zeta_5 - \frac{100}{3}\zeta_5 + \frac{13}{45}\pi^4\zeta_3, \\ c_4 &= 2b_1 - 6b_2 - c_1 - 2c_2 - \frac{4193}{4}\zeta_7 + \frac{35}{3}\pi^2\zeta_5 - 275\zeta_5 + 35\zeta_3^2 + \frac{7}{30}\pi^4\zeta_3 - \frac{25}{54}\pi^6. \end{aligned}$$

- $\kappa = 1$

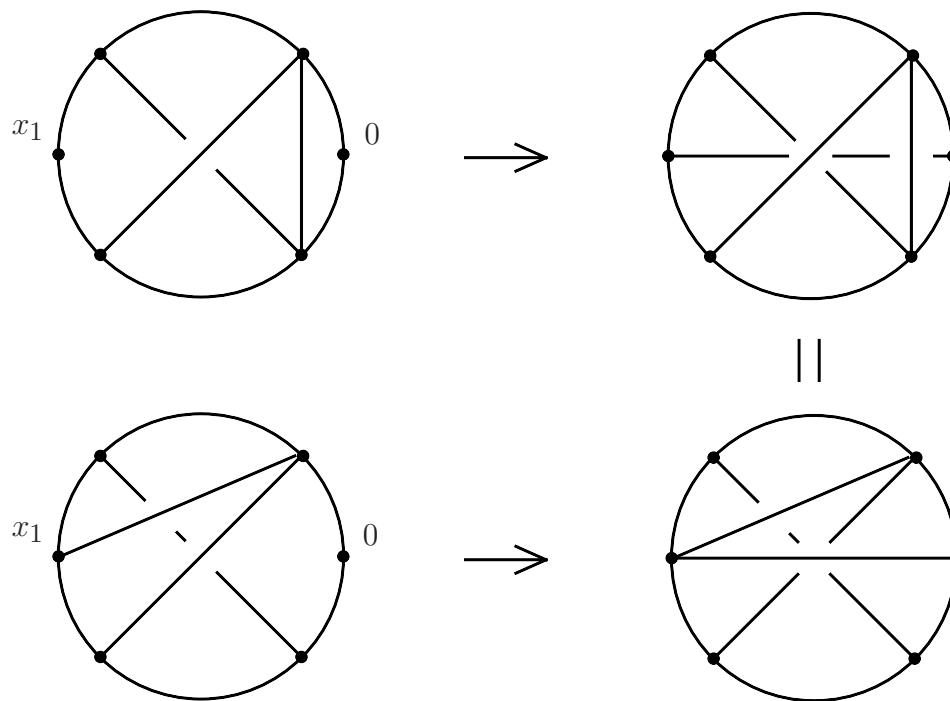
FT of I_3 is master M_{45} , FT of I_4 can be IBP reduced to the standard list.

- $\kappa = 1/2$

Use conformal inversion to get I_3 into a new form:

$$I_3(1/2) = \int \frac{d^Dx_6 d^Dx_7 d^Dx_8 d^Dx_9}{(x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_6^2 x_9^2 x_{89}^2)^{1-\epsilon/2}} = \text{Diagram}$$

$\mathbf{I}_4(1/2) = \mathbf{I}_3(1/2)$ by glueing:



$\kappa = 1/2 + \lambda/(10\epsilon)$ with λ arbitrary.

$$\mathcal{F} \left[\frac{I_i(1/2 + \lambda/(10\epsilon))}{(x_1^2)^{1-\epsilon/2-\lambda/10}} \right] = \mathcal{F} \left[\frac{C_i(\epsilon, \lambda)}{(x_1^2)^{2-\epsilon-\lambda}} \right] = C_i(\epsilon, \lambda) \frac{2^{-2\lambda} \Gamma(\lambda)}{\Gamma(2 - \epsilon - \lambda)} (p^2)^{-\lambda}$$

To show $C_3(\epsilon, 0) = C_4(\epsilon, 0)$ take residue at $\lambda = 0$.

5 Conclusions

- We agree with the integrability prediction for $\gamma_5(\mathcal{K})$.
- We used a new field theory method.
- The result confirms hypotheses about finite size effects in the AdS/CFT spectrum.
[Ambjorn, Bajnok, Hegedus, Heller, Janik, Kristiansen, Lukowski]
[Arutyunov, Frolov], [Gromov, Kazakov, Vieira], [Bombardelli, Fioravanti, Tateo]
- The result equally confirms the correctness of our results for the four-point function of stress tensor multiplets \mathcal{T} . Symmetry arguments made it possible to predict its integrand up to six loops superseeding the off-shell Feynman diagram method in yet a new way.
[Eden, Heslop, Korchemsky, Sokatchev (2011,2012)]
- It confirms the "triality" between n -point functions of \mathcal{T} , amplitudes and Wilson loops.
[Alday, Eden, Heslop, Korchemsky, Maldacena, Sokatchev]
- Last week integrability predictions for $\gamma_6(\mathcal{K}), \gamma_7(\mathcal{K})$ were published.
[Leurent, Serban, Volin], [Bajnok, Janik]
They contain only ordinary ζ values.
- Lee, Smirnov and Smirnov will attempt the calculation of $\gamma_6(\mathcal{K})$ with the methods here presented.