

# Tensor reduction for one-loop Feynman diagrams

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## Outline

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One-loop: important for LHC, testing the Standard Model

Many approaches:

- Original: Passarino-Veltman,  $e^+ e^- \rightarrow \mu^+ \mu^-$ , Nucl. Phys. '79

Applications at the time, among others :4-point functions

- Higgs Decays, JF, F. Jegerlehner Phys.Rev '81
- $e^+ e^- \rightarrow Z H$ , JF, F. Jegerlehner Nucl.Phys. '83
- $e^+ e^- \rightarrow W^+ W^-$ , JF, F. Jegerlehner, M.Zralek Z.Phys. '89

Tensor integrals are reduced to basic one-loop scalar integrals (1-point, ... 4-point: A0,B0,C0,D0). The scalar integrals were calculated originally by 't Hooft and Veltman, Nucl. Phys. '79.

Problem: appearance of inverse Gram determinants, which may become small, yielding numerical instabilities.

## One-loop: further developments

Improvement on numerical instabilities:

van Oldenborgh, Vermaseren Z.Phys. '90.

More problems: for 4-point functions NI's  $\sim$  forward scattering.

$n$ -point functions ( $n > 4$ ): NI's happen also **within** phase space.

Recent activities on  $n$ -point functions ( $n > 4$ ):

- T. Binoth, J. Guillet, G. Heinrich, E. Pilon, C. Schubert, An algebraic / numerical formalism for one-loop multi-leg amplitudes, JHEP 10 (2005) 015.
- A. Denner, S. Dittmaier, Reduction schemes for one-loop integrals, Nucl. Phys. B734 (2006)
- A. van Hameren, Computer physics Commun., 182 (2011) 2427
- JF and T.Riemann: Phys. Rev. D83 (2011), Complete reduction of 1-loop tensors.
- V.Yundin C++ package PJFry. Available at <https://github.com/Vayu/PJFry>

## Use of Algebra of signed minors and higher dimensional integrals

First talk given on the subject, in connection with Mellin-Barnes representations: **Frontiers, 15.6.2007!** Our approach based on:

- Algebra of signed Minors: D.B.Melrose, Nuovo Cim.**40** (1965),  
Reduction of Feynman diagrams, Cayley determinants. E.g.: present the scalar 5-point function in terms of 4-point functions

$$I_5 \equiv E = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \binom{0}{s}_5 I_4^s$$

- Higher dimensional integrals Davydychev:Phys.Lett. **B263** (1991),  
integrals in diff. space-time dim.
- Recursion relations Tarasov:Phys.Rev. **D54** (1996), dimensional recurrence relations.
- Demonstration JF,Jegerlehner,Tarasov: Nucl. Phys. **B566** (2000) .

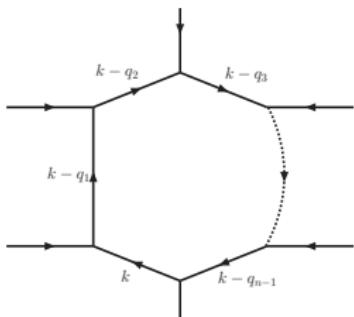
# The tensor integrals

## $n$ -point tensor integrals of rank $R$ : (n,R)-integrals

$$I_n^{\mu_1 \dots \mu_R} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^R k^{\mu_r}}{\prod_{j=1}^n c_j^{\nu_j}},$$

$d = 4 - 2\epsilon$  and denominators  $c_j$  have *indices*  $\nu_j$  and *chords*  $q_j$

$$c_j = (k - q_j)^2 - m_j^2 + i\epsilon$$



tensor integrals due to, e.g.:

- fermion propagators
- three-gauge boson couplings

## Tensors expressed in terms of integrals in higher dimension

Following [Davydychev:1991], also [J.F. et al.:2000]  
 express tensors by means of scalar integrals in higher dimensions ( $n_{ij} = \nu_{ij} = 1 + \delta_{ij}$ ,  $n_{ijk} = \nu_{ij}\nu_{jik}$ ,  $\nu_{ijk} = 1 + \delta_{ik} + \delta_{jk}$  etc.):

$$I_n^\mu = \int^d k^\mu \prod_{r=1}^n c_r^{-1} = - \sum_{i=1}^n q_i^\mu I_{n,i}^{[d+]}$$

$$I_n^{\mu\nu} = \int^d k^\mu k^\nu \prod_{r=1}^n c_r^{-1} = \sum_{i,j=1}^n q_i^\mu q_j^\nu n_{ij} I_{n,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu} I_n^{[d+]}$$

$$I_n^{\mu\nu\lambda} = \int^d k^\mu k^\nu k^\lambda \prod_{r=1}^n c_r^{-1} = - \sum_{i,j,k=1}^n q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{n,ijk}^{[d+]^3} + \frac{1}{2} \sum_{i=1}^n g^{\mu\nu} q_i^\lambda I_{n,i}^{[d+]^2}$$

$$I_n^{\mu\nu\lambda\rho} = \int^d k^\mu k^\nu k^\lambda k^\rho \prod_{r=1}^n c_r^{-1} = \sum_{i,j,k,l=1}^n q_i^\mu q_j^\nu q_k^\lambda q_l^\rho n_{ijkl} I_{n,ijkl}^{[d+]^4}$$

$$-\frac{1}{2} \sum_{i,j=1}^n g^{\mu\nu} q_i^\lambda q_j^\rho n_{ij} I_{n,ij}^{[d+]^3} + \frac{1}{4} g^{\mu\nu} g^{\lambda\rho} I_n^{[d+]^2}$$

## The integrals

$$I_{p,ijk\dots}^{[d+]^l} = \int^{[d+]^l} \prod_{r=1}^n \frac{1}{c_r^{1+\delta_{ri}+\delta_{rj}+\delta_{rk}+\dots}}, \quad \int^{\textcolor{blue}{d}} \equiv \int \frac{d^d k}{\pi^{d/2}}.$$

Integration in higher dimension, i.e.  $\textcolor{blue}{d} \equiv [d+]^l = 4 + 2l - 2\varepsilon$  ( $l = 0$  :“generic”).

“scratched” lines:

$$I_{n-2,ab}^{\{\mu_1, \dots\}, \textcolor{red}{s,t}}, \quad a, b \neq s, t$$

is obtained from

$$I_n^{\{\mu_1, \dots\}}$$

by

- scratching lines  $s, t$
- raising the powers of inverse propagators  $a, b$ .

## The metric tensor

- External momenta and  $g^{\mu\nu}$  in 4 dimensions.
- Considering an  $n$ -point tensor,  $q_n = 0$  is chosen. Scalar product

$$(q_i \cdot q_j) = \frac{1}{2} [Y_{ij} - Y_{in} - Y_{nj} + Y_{nn}].$$

Usually one assumes the metric tensor to be given by

$$\text{diag}[g^{\mu\nu}] = (1, -1, -1, -1)$$

Here we use (assume a 5-point function with 4 independent  $q_i$ ,  $i = 1 \dots 4$ ,  $q_5 = 0$ )

$$\frac{1}{2} g^{\mu\nu} = \sum_{i,j=1}^4 \frac{\binom{i}{j}}{\binom{5}{5}} q_i^\mu q_j^\nu \Rightarrow q_{a,\mu} g^{\mu\nu} q_{b,\nu} = q_a \cdot q_b.$$

- No contraction of  $g^{\mu\nu}$  with an integration momentum.
- Cancellation of ("reducibel") scalar propagators is avoided.
- $g^{\mu\nu}$  cancels for  $n = 5$ .

## Notations: modified Cayley determinant

Modified Cayley determinant  $(\cdot)_n$  of a diagram with  $n$  internal lines and chords  $q_j$ :

$$(\cdot)_n \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix},$$

with matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \dots n)$$

Gram determinant  $G_n$ :  $G_n = |2q_i q_j|$ ,  $i, j = 1, \dots, n$

For a choice  $q_n = 0$ , both determinants are related:  $(\cdot)_n = -G_{n-1}$

⇒ The determinant  $(\cdot)_n$  does not depend on the masses.

## Notations: signed minors

We also need **signed minors** of  $(\cdot)_n$ , constructed by deleting  $m$  rows and  $m$  columns from  $(\cdot)_n$ , and multiplying with a sign factor:

$$\begin{aligned} & \left( \begin{array}{cccc} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{array} \right)_n = \\ & \equiv (-1)^{\sum_i (j_i + k_i)} \operatorname{sgn}_{\{j\}} \operatorname{sgn}_{\{k\}} \left| \begin{array}{c|c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \hline \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \end{aligned}$$

where  $\operatorname{sgn}_{\{j\}}$  and  $\operatorname{sgn}_{\{k\}}$  are the signs of permutations that sort the deleted rows  $j_1 \cdots j_m$  and columns  $k_1 \cdots k_m$  into ascending order.

The signed minors are **antisymmetric** under exchange of neighboring indices  $j$  and  $k$ .

## Some algebraic relations [D.B.Melrose, Nuovo Cim.**40** (1965)]

$$\sum_{i=1}^n \binom{0}{i}_n = ()_n$$

and

$$\sum_{i=1}^n \binom{j}{i}_n = 0, \quad (j \neq 0).$$

$$()_n \binom{il}{jk}_n = \binom{i}{j}_n \binom{l}{k}_n - \binom{i}{k}_n \binom{l}{j}_n; \quad i, j, k, l = 0, \dots, n.$$

## Recursion for integrals

Following [Tarasov:1996, JF:2000]: apply recurrence relations relating scalar integrals of different indices and - or different dimensions.

$$\begin{aligned}\nu_j j^+ I_n^{(d+2)} &= \frac{1}{(0)_n} \left[ -\binom{j}{0}_5 + \sum_{k=1}^n \binom{j}{k}_n \mathbf{k}^- \right] I_n^d \\ (d - \sum_{i=1}^n \nu_i + 1) I_n^{(d+2)} &= \frac{1}{(0)_n} \left[ \binom{0}{0}_n - \sum_{k=1}^n \binom{0}{k}_n \mathbf{k}^- \right] I_n^d,\end{aligned}$$

where the operators  $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$  act by shifting the indices  $\nu_i, \nu_j, \nu_k$  by  $\pm 1$ . Example for a "scratched" integral ( $\nu_{ij} = 1 + \delta_{ij}$ ):

$$\nu_{ij} I_{4,ij}^{[d+]^2,s} = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} I_{4,i}^{[d+],s} + \frac{\binom{is}{js}_5}{\binom{s}{s}_5} I_4^{[d+],s} + \sum_{t=1}^5 \frac{\binom{ts}{js}_5}{\binom{s}{s}_5} I_{3,i}^{[d+],st}.$$

## Recursion for tensors

### 5-point tensor recursion (5-PTR):

Express any  $(5, R)$  pentagon by a  $(5, R - 1)$  pentagon plus  $(4, R - 1)$  boxes by applying recurrence relations, reducing simultaneously indices and dimension

[T.Diakonidis,JF,T.Riemann,J.B.Tausk: Phys.Lett. **B683** (2010)]

$$I_5^{\mu_1 \dots \mu_{R-1} \mu} = I_5^{\mu_1 \dots \mu_{R-1}} Q_0^\mu - \sum_{s=1}^5 I_4^{\mu_1 \dots \mu_{R-1}, s} Q_s^\mu,$$

auxiliary vectors with inverse Gram determinants

$$Q_s^\mu = \sum_{i=1}^5 q_i^\mu \frac{\binom{s}{i}_5}{\binom{0}{5}}, \quad s = 0, \dots, 5$$

For e.g.  $R = 3$ , again  $[1/\binom{0}{5}]^3$  will occur.

## Tensor of rank $R = 3$

$$\begin{aligned} I_5^{\mu\nu\lambda} &= \sum_{i,j,k=1}^5 q_i^\mu q_j^\nu q_k^\lambda E_{ijk} + \sum_{k=1}^5 g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \\ g^{[\mu\nu} q_k^{\lambda]} &= g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu. \end{aligned}$$

Some algebra allows to obtain from our (5-PTR)

$$\begin{aligned} E_{00j} &= \sum_{s=1}^5 \frac{1}{\binom{0}{0}_5} \left[ \frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \binom{s}{j}_5 I_4^{[d+]^2,s} \right], \\ E_{ijk} &= - \sum_{s=1}^5 \frac{1}{\binom{0}{0}_5} \left\{ \left[ \binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,j}^{[d+]^2,s} \right\}. \end{aligned}$$

No inverse  $(\cdot)_5$  ! Inverse 4-point Gram  $\binom{s}{s}_5$  ?

$$\begin{aligned} I_{4,i}^{[d+]^2,s} &= \frac{1}{\binom{0s}{0s} 5} \left[ - \binom{0s}{is} 5 (d-1) I_4^{[d+]^2,s} + \sum_{t=1}^5 \binom{0st}{0si} 5 I_3^{[d+],st} \right], \\ \nu_{ij} I_{4,ij}^{[d+]^2,s} &= \frac{\binom{0s}{is}}{\binom{0s}{0s}} \frac{\binom{0s}{js}}{\binom{0s}{0s}} (d-2)(d-1) I_4^{[d+]^2,s} + \frac{\binom{0si}{0sj}}{\binom{0s}{0s}} I_4^{[d+],s} \\ &\quad - \frac{\binom{0s}{js}}{\binom{0s}{0s}} \frac{d-2}{\binom{0s}{0s}} \sum_{t=1}^4 \binom{0st}{0si} I_3^{[d+],st} + \frac{1}{\binom{0s}{0s}} \sum_{t=1}^4 \binom{0st}{0sj} I_{3,i}^{[d+],st}. \end{aligned}$$

*Indices  $i, j$  are contained in signed minors !!! .*

*Integrals free of indices !*

## Small $\binom{s}{s}_5$

$$Z_4^{[d+]^l} = \frac{1}{\binom{0}{0}} \sum_{t=1}^4 \binom{t}{0} I_3^{[d+]^l, t}, \quad l = 1, \dots$$

$$I_4^{[d+]^L} = \sum_{j=0}^{\infty} a_j^L r^j \left[ Z_4^{(L+j)} - b_j^L Z_{4D}^{(L+j)} \right], \quad L = 0, \dots 4.$$

with  $r = \frac{\binom{0}{0}}{\binom{0}{0}}$  and

$$a_j^L = 2^j \frac{\Gamma(L+j+\frac{1}{2})}{\Gamma(L+\frac{1}{2})}, \quad b_j^L = \psi(L+j+\frac{1}{2}) - \psi(L+\frac{1}{2}),$$

The  $\psi(z)$  is the logarithmic derivative of the Gamma function.

On the basis of this "standard" procedure, the **C++ package PJFry** by **V.Yundin** has been written, available at <https://github.com/Vayu/PJFry>: up to tensors of rank  $R = 5$ .

# Contractions I

$$q_{i_1\mu_1} \cdots q_{i_R\mu_R} I_5^{\mu_1 \cdots \mu_R} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^R (q_{i_r} \cdot k)}{\prod_{j=1}^5 c_j},$$

$$g_{\mu_1, \mu_2} q_{i_1\mu_3} \cdots q_{i_R\mu_R} I_5^{\mu_1 \cdots \mu_R} \neq \int \frac{k^2 d^d k}{i\pi^{d/2}} \frac{\prod_{r=3}^R (q_{i_r} \cdot k)}{\prod_{j=1}^5 c_j} \quad ?$$

etc., are obtained by constructing **projection operators**

or

by calculating scalar differential cross sections  
(Born  $\times$  1-loop)

For  $q_n = 0$ ,  $a = 1, \dots, n-1$ ,  $s = 1, \dots, n$

$$(q_a \cdot Q_0) = \sum_{j=1}^{n-1} (q_a \cdot q_j) \frac{\binom{0}{j}_n}{\binom{0}{n}} = -\frac{1}{2} (Y_{an} - Y_{nn}),$$

$$(q_a \cdot Q_s) = \sum_{j=1}^{n-1} (q_a \cdot q_j) \frac{\binom{s}{j}_n}{\binom{0}{n}} = \frac{1}{2} (\delta_{as} - \delta_{ns}),$$

Considering again the **5-PTR** :

$$I_5^{\mu_1 \dots \mu_{R-1} \mu} = I_5^{\mu_1 \dots \mu_{R-1}} Q_0^\mu - \sum_{s=1}^5 I_4^{\mu_1 \dots \mu_{R-1}, s} Q_s^\mu,$$

contracting with  $q_a^\mu$  shows: immediately we get rid of the inverse  $(\ )_5$  !  
 "Scrapped" vectors for the 4-point function:

$$Q_\tau^{s,\mu} = \sum_{i=1}^5 q_i^\mu \frac{\binom{\tau s}{is}_5}{\binom{s}{s}_5}, \quad \tau = 0 \dots 5$$

## Scalar Expressions:

$$q_{a\mu} q_{b\nu} I_5^{\mu\nu} = (q_a \cdot I_5)(q_b \cdot Q_0) - \sum_{s=1}^5 (q_a \cdot I_4^s)(q_b \cdot Q_s)$$

$$(q_a \cdot I_5) = E(q_a \cdot Q_0) - \sum_{s=1}^5 I_4^s (q_a \cdot Q_s),$$

$$(q_a \cdot I_4^s) = (q_a \cdot Q_0^s) I_4^s - \sum_{t=1}^5 (q_a \cdot Q_t^s) I_3^{st} = \frac{1}{(s)_5} \left[ \Sigma_a^{2,s} I_4^s - \Sigma_a^{1,st} I_3^{st} \right]$$

and with  $Y_a = Y_{a5} - Y_{55}$  and  $D_a^s = \delta_{as} - \delta_{5s}$

$$\Sigma_a^{2,s} \equiv \sum_{i=1}^4 (q_a \cdot q_i) \binom{0s}{is}_5 = -\frac{1}{2} \left\{ \binom{s}{s}_5 Y_a + \binom{s}{0}_5 D_a^s \right\},$$

$$\Sigma_a^{1,st} \equiv \sum_{i=1}^4 (q_a \cdot q_i) \binom{ts}{is}_5 = \frac{1}{2} \left\{ \binom{s}{s}_5 D_a^t - \binom{s}{t}_5 D_a^s \right\}.$$

JF and T.Riemann, Phys.Lett. **B701** (2011) and Phys.Lett. **B707** (2012).

$$g_{\mu\nu} I_5^{\mu\nu}:$$

We take the following scalar products

$$(Q_0 \cdot Q_0) = \frac{1}{2} \left[ \frac{\binom{0}{0}_5}{\binom{0}{0}_5} + Y_{55} \right], \quad (Q_0 \cdot Q_s) = \frac{1}{2} \left[ \frac{\binom{s}{0}_5}{\binom{0}{0}_5} - \delta_{s5} \right],$$

$$(Q_0^s \cdot Q_s) = -\frac{1}{2} \delta_{s5}, \quad (Q_t^s \cdot Q_s) = 0, \quad \text{and}$$

$$(I_5 \cdot Q_0) = E(Q_0 \cdot Q_0) - \sum_{s=1}^5 I_4^s (Q_0 \cdot Q_s)$$

$$= \frac{1}{2} \left\{ \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \binom{s}{0}_5 I_4^s \cdot \left[ \frac{\binom{0}{0}_5}{\binom{0}{0}_5} + Y_{55} \right] - \sum_{s=1}^5 I_4^s \left[ \frac{\binom{s}{0}_5}{\binom{0}{0}_5} - \delta_{s5} \right] \right\}$$

- cancellation of  $\frac{1}{\binom{0}{0}_5}$  - and finally (not surprisingly)

$$g_{\mu\nu} I_5^{\mu\nu} = \frac{Y_{55}}{2} E + I_4^5.$$

Tensor of rank 3:  $I_5^{\mu\nu\lambda} = I_5^{\mu\nu} \cdot Q_0^\lambda - \sum_{s=1}^5 I_4^{\mu\nu,s} \cdot Q_s^\lambda$

The corresponding 4-point function reads ( $q_5 = 0$ ):

$$I_4^{\mu\nu,s} = \sum_{i,j=1}^4 q_i^\mu q_j^\nu \nu_{ij} I_{4,ij}^{[d+]^2,s} - \frac{1}{2} g^{\mu\nu} I_4^{[d+],s},$$

$$\nu_{ij} I_{4,ij}^{[d+]^2,s} = - \frac{\binom{0s}{js}}{\binom{s}{s} 5} I_{4,i}^{[d+],s} + \frac{\binom{is}{js}}{\binom{s}{s} 5} I_4^{[d+],s} + \sum_{t=1}^5 \frac{\binom{ts}{js}}{\binom{s}{s} 5} I_{3,i}^{[d+],st}.$$

eliminate  $g^{\mu\nu}$ :  $\frac{\binom{is}{js}}{\binom{s}{s} 5} = \frac{\binom{i}{j}}{\binom{0}{0} 5} - \frac{\binom{s}{i} \binom{s}{j}}{\binom{0}{0} \binom{s}{s} 5}$

$$\begin{aligned} q_{a\mu} q_{b\nu} I_4^{\mu\nu,s} &= (q_a \cdot I_4^s) (q_b \cdot Q_0^s) - \sum_{t=1}^5 (q_a \cdot I_3^{st}) (q_b \cdot Q_t^s) \\ &\quad - \frac{\binom{0}{5}}{\binom{s}{s} 5} (q_a \cdot Q_s) (q_b \cdot Q_s) I_4^{[d+],s} \end{aligned}$$

$(q_a \cdot I_3^{st})$ :

$$(q_a \cdot I_3^{st}) = (q_a \cdot Q_0^{st}) I_3^{st} - \sum_{u=1}^5 (q_a \cdot Q_u^{st}) I_2^{stu} = \frac{1}{\binom{st}{st} 5} \left[ \Sigma_a^{3,st} I_3^{st} - \Sigma_a^{1,stu} I_2^{stu} \right]$$

with

$$\begin{aligned} \Sigma_a^{3,st} &= -\frac{1}{2} \left\{ \binom{st}{st} 5 Y_a + \binom{st}{s0} 5 D_a^t + \binom{st}{0t} 5 D_a^s \right\}, \\ \Sigma_a^{1,stu} &= \frac{1}{2} \left\{ \binom{st}{st} 5 D_a^u - \binom{st}{su} 5 D_a^t - \binom{ts}{tu} 5 D_a^s \right\}. \end{aligned}$$

Result for  $C5_{abc} = -\sum_{s=1}^5 q_{a\mu} q_{b\nu} I_4^{\mu\nu,s} \cdot \frac{1}{2} (\delta_{cs} - \delta_{5s})$

$$C5_{abc} = \frac{1}{8} \left\{ \begin{aligned} & J_4^5 - \delta_{ab}\delta_{ac}J_4^a + Y_a Y_b (I_4^5 - I_4^c) \\ & + Y_a (I_3^{b5} + I_3^{c5} - I_3^{bc}) + Y_b (I_3^{a5} + I_3^{c5} - I_3^{ac}) \\ & + I_2^{ab5} + I_2^{ac5} + I_2^{bc5} - I_2^{abc} \\ & - Y_a (R^5 + \delta_{bc}R^b) - Y_b (R^5 + \delta_{ac}R^a) \\ & - R^{5a} - R^{5b} - R^{5c} \\ & + \delta_{ab} (R^{a5} - R^{ac}) + \delta_{ac} (R^{a5} - R^{ab}) \\ & + \delta_{bc} (R^{b5} - R^{ba}) \end{aligned} \right\}$$

$$\begin{aligned}
 R^s &\equiv \frac{1}{\binom{s}{s}_5} \left[ \binom{s}{0}_5 I_4^s - \sum_{t=1}^5 \binom{s}{t}_5 I_3^{st} \right] = \frac{1}{\binom{0s}{0s}_5} \left[ \binom{s}{0}_5 I_4^{[d+],s} - \sum_{t=1}^5 \binom{0s}{0t}_5 I_3^{st} \right] \\
 R^{st} &\equiv \frac{1}{\binom{st}{st}_5} \left[ \binom{st}{0t}_5 I_3^{st} - \sum_{u=1}^5 \binom{st}{ut}_5 I_2^{stu} \right] \\
 &= \frac{1}{\binom{0st}{0st}_5} \left[ \binom{st}{0t}_5 (d-2) I_3^{[d+],st} - \sum_{u=1}^5 \binom{0st}{0ut}_5 I_2^{stu} \right].
 \end{aligned}$$

and

$$\begin{aligned}
 J_4^s &\equiv \frac{1}{\binom{s}{s}_5} \left\{ -()_5 I_4^{[d+],s} + \binom{s}{0}_5 R^s - \sum_{t=1}^5 \binom{s}{t}_5 R^{st} \right\} \\
 &= \frac{-1}{\binom{0s}{0s}_5} \left\{ ()_5 (d-2)(d-1) I_4^{[d+]^2,s} - \binom{0}{0}_5 I_4^{[d+],s} + \sum_{t=1}^5 \binom{t}{0}_5 (d-2) I_3^{[d+],st} \right. \\
 &\quad \left. + \sum_{t=1}^5 \binom{0s}{0t}_5 R^{st} \right\}
 \end{aligned}$$

Elimination of  $\frac{1}{\binom{s}{s}_5}$  solved -  $a = b = c = s_0 = 5$  only!

## General Approach for $n$ -point tensors

Following ideas presented in

Z. Bern, L. J. Dixon, D. A. Kosower, Nucl. Phys. B412 (1994) 751–816,

an iterative approach has been systematically worked out in

T. Binoth, J. Guillet, G. Heinrich, E. Pilon, C. Schubert, JHEP 10 (2005) 015.

$$I_n^{\mu_1 \mu_2 \dots \mu_R} = - \sum_{r=1}^n C_r^{\mu_1} I_{n-1}^{\mu_2 \dots \mu_R, r} \quad \text{with the conditions}$$

$$\sum_{j=1}^n C_j^\mu q_j^\nu = \frac{1}{2} g_{[4]}^{\mu\nu}, \quad \sum_{j=1}^n C_j^\mu = 0.$$

The solution of this set is not unique. Numerical? Assume

$$\frac{1}{2} g^{\mu\nu} = \sum_{i,j=1}^{n-1} G_{ij} q_i^\mu q_j^\nu, \quad \text{then}$$

$$C_j^\mu = \sum_{i=1}^{n-1} G_{ij} q_i^\mu.$$

$n = 6$

We have to find a **proper representation of the metric tensor**. E.g. we have

$$\sum_{i,j=1}^{n-1} (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} si \\ sj \end{pmatrix}_n = \frac{1}{2} (q_a \cdot q_b) \begin{pmatrix} s \\ s \end{pmatrix}_n - \frac{1}{4} ()_n (\delta_{as} - \delta_{ns}) (\delta_{bs} - \delta_{ns}).$$

**Observation:** all our sums are valid for arbitrary  $n$  and have to be considered as identities in terms of arbitrary (symmetric)  $Y_{ij}$ .

From  $()_6 = 0$  for the "physical" ones, we obtain

$$\frac{1}{2} g^{\mu\nu} = \sum_{i,j=1}^5 \frac{\binom{si}{sj}_6}{\binom{s}{s}_6} q_i^\mu q_j^\nu, \quad \text{or}$$

$$C_r^{s,\mu} = \sum_{i=1}^5 q_i^\mu \frac{\binom{sr}{si}_6}{\binom{s}{s}_6} = Q_r^{s,\mu}, \quad s = 1 \dots 6.$$

$$\sum_{r=1}^6 \binom{sr}{si}_6 = 0, \quad s = 1, \dots 6$$

yields the second condition. Further possibility:  $Q_r^{0,\mu}$  or  $\frac{\binom{sr}{si}_6}{\binom{s}{s}_6} \Rightarrow \frac{\binom{0r}{si}_6}{\binom{0}{s}_6}$ .

$n = 7, 8 \dots$

Starting with a sum, again, to find the proper metrik tensor:

$$\begin{aligned} \sum_{i,j=1}^{n-1} (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} sti \\ stj \end{pmatrix}_n &= \frac{1}{2} (q_a \cdot q_b) \begin{pmatrix} st \\ st \end{pmatrix}_n \\ -\frac{1}{4} \left\{ \left[ \begin{pmatrix} s \\ s \end{pmatrix}_n (\delta_{at} - \delta_{nt}) - \begin{pmatrix} s \\ t \end{pmatrix}_n (\delta_{as} - \delta_{ns}) \right] (\delta_{bt} - \delta_{nt}) \right. \\ \left. + \left[ \begin{pmatrix} t \\ t \end{pmatrix}_n (\delta_{as} - \delta_{ns}) - \begin{pmatrix} s \\ t \end{pmatrix}_n (\delta_{at} - \delta_{ns}) \right] (\delta_{bs} - \delta_{ns}) \right\}, \end{aligned}$$

With dimension 4 of the chords, all  $(\ )_n$ ,  $n \geq 7$ , have rank 6. (Melrose, 1965).  
The  $(\ )_7$  is of order 8 and thus

$$\begin{pmatrix} s \\ t \end{pmatrix}_7 = 0 \text{ and } \begin{pmatrix} s \\ s \end{pmatrix}_7 = 0,$$

and therefore the whole curly bracket vanishes with the result

$$C_r^{st,\mu} = \sum_{i=1}^6 \frac{1}{\begin{pmatrix} st \\ st \end{pmatrix}_7} \begin{pmatrix} sti \\ str \end{pmatrix}_7 q_i^\mu = Q_r^{st,\mu}.$$

Similar for the 8-point function, etc..

## Complete contraction

$$I_7^{\mu_1 \mu_2 \dots \mu_R} = - \sum_{r_1=1}^7 Q_{r_1}^{s,t,\mu_1} I_6^{\mu_2 \dots \mu_R, r_1} \quad s, t : \text{redundancy indices},$$

$$I_6^{\mu_2 \dots \mu_R, r_1} = - \sum_{r_2=1}^7 Q_{r_2}^{r_1, u, \mu_2} I_5^{\mu_3 \dots \mu_R, r_1, r_2} \quad \text{with}$$

$$Q_{r_1}^{s,t,\mu_1} = \sum_{i=1}^6 \frac{1}{(st)_7} \binom{st i}{st r_1}_7 q_i^\mu$$

$$Q_{r_2}^{r_1, u, \mu_2} = \sum_{i=1}^6 \frac{1}{(r_1 u)_7} \binom{r_1 u i}{r_1 u r_2}_7 q_i^\mu$$

For  $R = 3$  : the 5-point vector remains, no redundancy index

$$I_5^{\mu, r_1 r_2} = E^{r_1 r_2} Q_0^{r_1 r_2, \mu} - \sum_{r_3=1}^7 I_4^{r_1 r_2 r_3} Q_{r_3}^{r_1 r_2, \mu} \quad \text{with}$$

$$Q_0^{r_1 r_2, \mu} = \sum_{i=1}^6 q_i^\mu \frac{\binom{r_1 r_2 0}{r_1 r_2 i}_7}{\binom{r_1 r_2}{r_1 r_2}_7}, \quad Q_{r_3}^{r_1 r_2, \mu} = \sum_{i=1}^6 q_i^\mu \frac{\binom{r_1 r_2 r_3}{r_1 r_2 i}_7}{\binom{r_1 r_2}{r_1 r_2}_7}$$

## Contractions

Also for the higher  $n$ -point functions, we can work with Contractions, e.g.

$$\begin{aligned}
 (q_a \cdot Q_r^{\textcolor{red}{st}}) &= \frac{1}{2 \binom{st}{st}_7} \left[ \binom{st}{st}_7 (\delta_{ar} - \delta_{7r}) - \binom{st}{sr}_7 (\delta_{at} - \delta_{7t}) - \binom{ts}{tr}_7 (\delta_{as} - \delta_{7s}) \right] \\
 &= \frac{1}{2} (\delta_{ar} - \delta_{7r}) \quad \text{for } \textcolor{red}{s, t \neq a, 7}
 \end{aligned}$$

Recall

$$q_{a\mu} q_{b\nu} I_5^{\mu\nu} = (q_a \cdot I_5)(q_b \cdot Q_0) - \sum_{s=1}^5 (q_a \cdot I_4^s)(q_b \cdot Q_s)$$

with

$$(q_b \cdot Q_s) = \frac{1}{2} (\delta_{bs} - \delta_{5s}).$$

## Conclusion

- Starting with the representation given by **Davydychev**,
- a systematic application of the **algebra of signed minors** according to **Melrose**
- was first presented in **JF et al.** in connection with the recursion relations of **Tarasov**.
- In **JF & T.Riemann** the complete reduction was described and with the **C++ program PJFry** by **V.Yundin** a publicly available numerical package was provided for tensors up to **rank 5**.
- After separation of the indices from the integrals
- for the **contraction** a method for the **summation over the indices** was developed, yielding analytic expressions, **JF & T.Riemann**.
- Avoidance of the inverse Gram determinants is achieved.
- Applicability for any **higher  $n$ -point functions** is demonstrated.
- **Package on contractions in preparation** in collaboration with J.Gluza.