Lattice field theories with dual variables

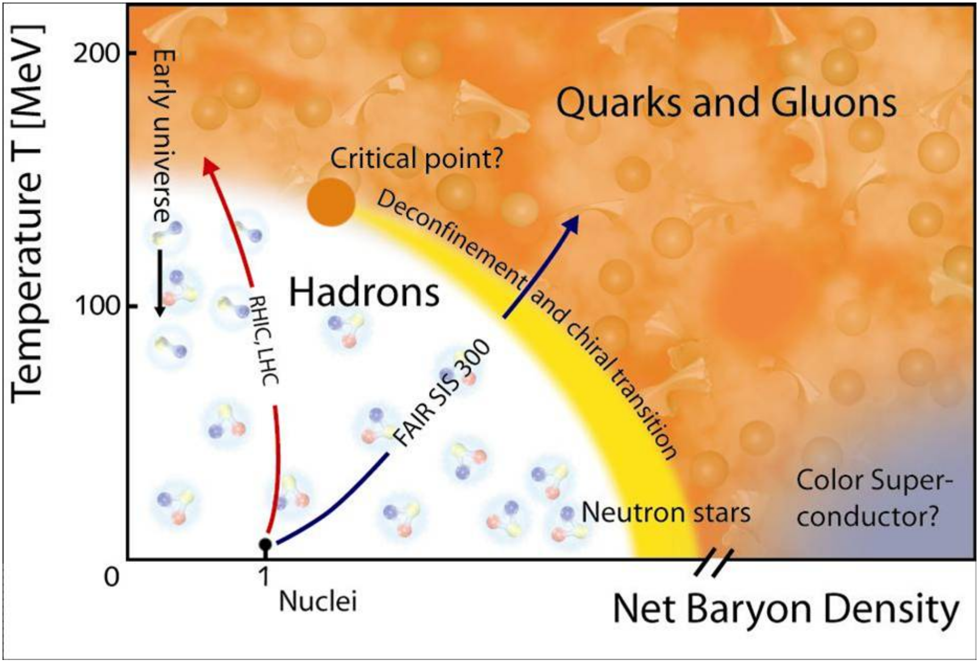
Christof Gattringer

Karl-Franzens-Universität Graz, Austria

Ydalia Delgado Mercado, Hans Gerd Evertz, Thomas Kloiber, Alexander Schmidt

Why do we explore reformulations of lattice QCD?

Something we would like to understand in detail ...



Euclidean path integral and complex action problem

- Vacuum expectation values with Feynman's path integral:

$$\langle O \rangle \;=\; \frac{1}{Z} \int D[\psi] \;\, e^{\,-\,S[\psi]} \;\, O[\psi]$$

- In a Monte Carlo simulation observables are computed as averages over field configurations ψ distributed according to

$$P[\psi] \;=\; \frac{1}{Z}\, e^{-S[\psi]}$$

- For finite chemical potential μ the action $S[\psi]$ is complex and the Boltzmann factor cannot be used as probability weight in a stochastic process.

Rewriting a system in terms of new variables where only real and positive terms appear in the partition sum could overcome the complex action problem.

General idea of a dual representation

Example: U(1) Gauge-Higgs system (here without chemical potential)

- Continuum action:

$$S = \frac{1}{4}\int d^4x F_{\rho\sigma}(x)^2 \; + \; \int d^4x \Big[\phi(x)^\star (m^2 - D_\nu^\dagger D_\nu)\phi(x) + \lambda \, |\phi(x)|^4 \Big]$$

- Compact gauge fields used on the lattice:

$$U_{\vec{n},\nu} \; = \; e^{iA_{\vec{n},\nu}} \qquad , \qquad A_{\vec{n},\nu} \; \in \; [-\pi, \pi]$$

- Action on the lattice:

$$S_G \; = \; -\beta \sum_{\vec{n}} \sum_{\rho<\sigma} Re \, U_{\vec{n},\rho} \, U_{\vec{n}+\hat{\rho},\sigma} \, U^\star_{\vec{n}+\hat{\sigma},\rho} \, U^\star_{\vec{n},\sigma}$$

$$S_H \; = \; \sum_{\vec{n}} \Big[\kappa \, |\phi_{\vec{n}}|^2 + \lambda \, |\phi_{\vec{n}}|^4 - \sum_{\nu=1}^{4} \Big(\phi^\star_{\vec{n}} \, U_{\vec{n},\nu} \, \phi_{\vec{n}+\hat{\nu}} + \phi^\star_{\vec{n}} \, U^\star_{\vec{n}-\hat{\nu},\nu} \, \phi_{\vec{n}-\hat{\nu}} \Big) \Big]$$

Loops for the matter fields:

- A single nearest neighbor term:

$$e^{\phi^\star_{\vec{n}}\, U_{\vec{n},\nu}\, \phi_{\vec{n}+\widehat{\nu}}} \;=\; \sum_{j_{\vec{n},\nu}=0}^{\infty} \frac{1}{(j_{\vec{n},\nu})!}\, (U_{\vec{n},\nu})^{j_{\vec{n},\nu}}\, (\phi_{\vec{n}})^{j_{\vec{n},\nu}}\, (\phi^\star_{\vec{n}+\widehat{\nu}})^{j_{\vec{n},\nu}}$$

- Idea: Use the expansion indices $j_{\vec{n},\nu}$ at the links of the lattice as the new degrees of freedom for the matter fields.

- Remaining $\phi \;=\; r\, e^{i\theta}$ integrals at a site $\vec{n}$:

$$\int_0^\infty dr\; r^{s_{\vec{n}}+1}\; e^{-\kappa r^2 - \lambda r^4} \int_{-\pi}^{\pi} d\theta\, e^{i\theta f_{\vec{n}}} \;=\; W(s_{\vec{n}})\; \delta(f_{\vec{n}})$$

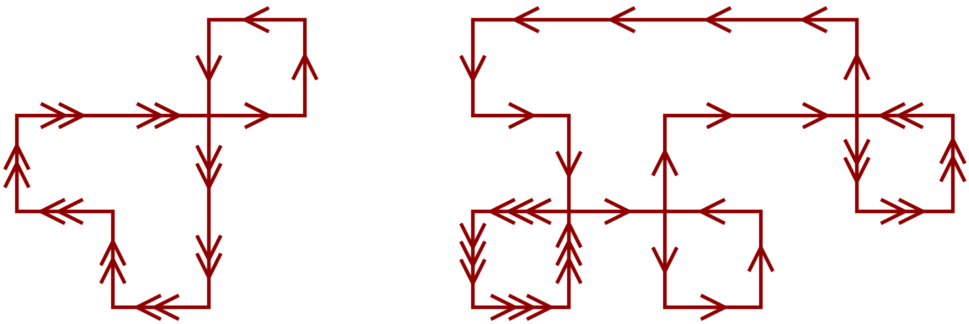
$f_{\vec{n}} = \sum_\nu \, [l_{\vec{n},\nu} - l_{\vec{n}-\widehat{\nu},\nu}]$. The $l_{\vec{n},\nu} \in \mathbb{Z}$ are linear combinations of the $j_{x,\nu}$

Constraints for the matter loops:

- Constraint from the integration over the U(1) phases:

$$\forall\, \vec{n} \;:\qquad f_{\vec{n}} \;=\; \sum_{\nu} \left[\, l_{\vec{n},\nu} - l_{\vec{n}-\widehat{\nu},\nu} \,\right] \;=\; 0$$

- Admissible configurations of dynamical variables are loops of flux dressed with gauge transporters $U_{\vec{n},\nu}$:



Surfaces for the gauge fields:

- A single plaquette term from the gauge action:

$$e^{\beta\, U_{\vec{n},\rho}\, U_{\vec{n}+\hat{\rho},\sigma}\, U^\star_{\vec{n}+\hat{\sigma},\rho}\, U^\star_{\vec{n},\sigma}} \;=\; \sum_{p_{\vec{n},\rho\sigma}} \frac{\beta^{\,p_{\vec{n},\rho\sigma}}}{(p_{\vec{n},\rho\sigma})!} \left[U_{\vec{n},\rho}\, U_{\vec{n}+\hat{\rho},\sigma}\, U^\star_{\vec{n}+\hat{\sigma},\rho}\, U^\star_{\vec{n},\sigma} \right]^{p_{\vec{n},\rho\sigma}}$$

- For gauge fields the expansion indices $p_{\vec{n},\rho\sigma}$ live on the plaquettes.

- The constraints for the gauge fields force the combined flux from the matter variables $l_{\vec{n},\nu}$ and the plaquette variable $p_{\vec{n},\rho\sigma}$ to vanish at each link of the lattice.

- Admissible configurations of the plaquette variables $p_{\vec{n},\rho\sigma}$ have the interpretation of 2-D surfaces embedded in 4-D.

- The surfaces are either closed or bounded by matter flux.

Dual form of partition function:

- The original partition sum is mapped exactly to a sum over loop and surface configurations:

$$Z = \sum_{\{p,l\}} \mathcal{W}_G(p)\, \mathcal{W}_H(l)\, \mathcal{C}_L(p,l)\, \mathcal{C}_S(l)$$

$\mathcal{W}_G(p)$: plaquette-based weight factor for gauge variables p

$\mathcal{W}_H(l)$: link-based weight factor for matter variables l

$\mathcal{C}_L(p,l)$: link-based constraint $\Rightarrow$ gauge surfaces

$\mathcal{C}_S(l)$: site-based constraint $\Rightarrow$ matter loops

- Observables are mapped to moments and correlators of p and l variables.

- Chemical potential gives different weight to forward and backward temporal propagation (affects winding loops).

Example 1: Charged scalar ϕ^4 field with chemical potential

C. Gattringer, T. Kloiber, arXiv:1206.2954

M. Endres, PRD 2007 (2+1 dimensions)

G. Aarts: Studies with complex Langevin and mean field methods

Charged scalar field with chemical potential

- Lattice action in conventional representation:

$$S = \sum_{\vec{n}} \Big[\kappa \, |\phi_{\vec{n}}|^2 + \lambda \, |\phi_{\vec{n}}|^4 - \sum_{j=1}^{3} \Big(\phi^\star_{\vec{n}} \, \phi_{\vec{n}+\widehat{j}} + \phi^\star_{\vec{n}} \, \phi_{\vec{n}-\widehat{j}} \Big) - e^{\,\mu} \, \phi^\star_{\vec{n}} \, \phi_{\vec{n}+\widehat{4}} - e^{-\mu} \, \phi^\star_{\vec{n}} \, \phi_{\vec{n}-\widehat{4}} \Big]$$

- In the conventional representation the action is complex for $\mu > 0$.

- Monte Carlo techniques are not applicable because the Boltzmann factor

$$\frac{1}{Z} \, e^{-S} \;\in\; \mathbb{C}$$

cannot be used as a probability in a stochastic process.

Charged scalar field with chemical potential

- No gauge fields $\Rightarrow$ only link variables $l_{x,\nu} \in \mathbb{Z}$

- Partition sum in terms of dual variables:

$$Z \;=\; \sum_{l} \mathcal{W}(l) \, \prod_{\vec{n}} \delta\Big(\sum_{\nu} [l_{\vec{n},\nu} - l_{\vec{n}-\widehat{\nu},\nu}] \Big) \; \exp\Big(\mu \sum_{\vec{n}} l_{\vec{n},4} \Big)$$

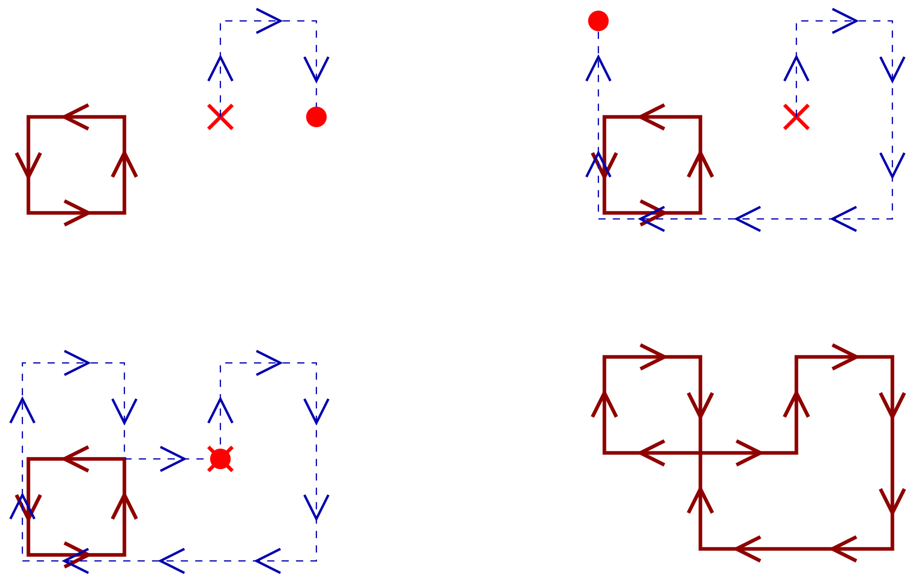
- In the dual representation all contributions to the partition sum are real and non-zero. The complex phase problem is solved.

- Challenge: Construct a Monte Carlo process that generates only admissible configurations that obey all constraints.

$\Rightarrow$ Prokofev-Svistunov worm algorithm

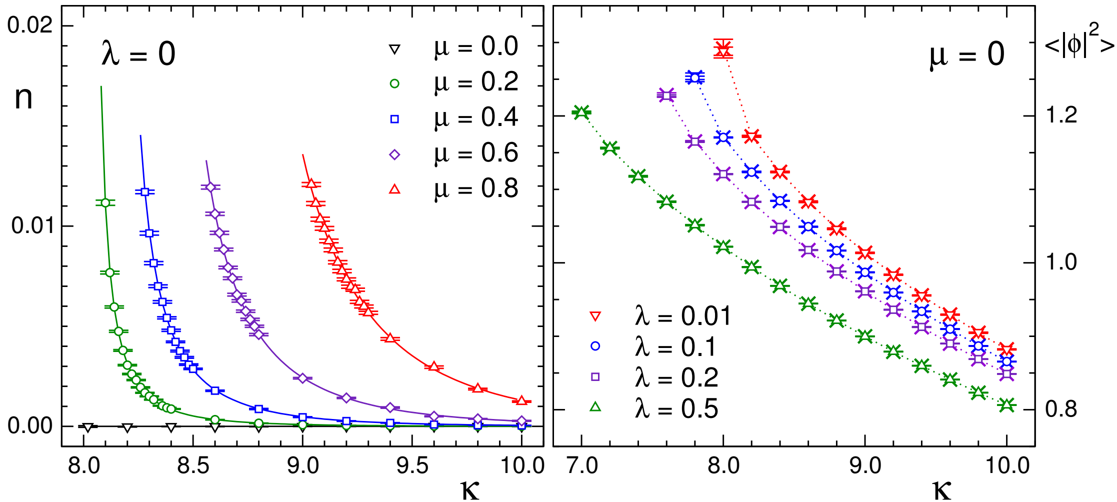
Worm algorithm

- A worm locally violates the constraint and propagates the defect until the worm closes and the constraint is healed.
- Every local change is accepted with the Metropolis probability.

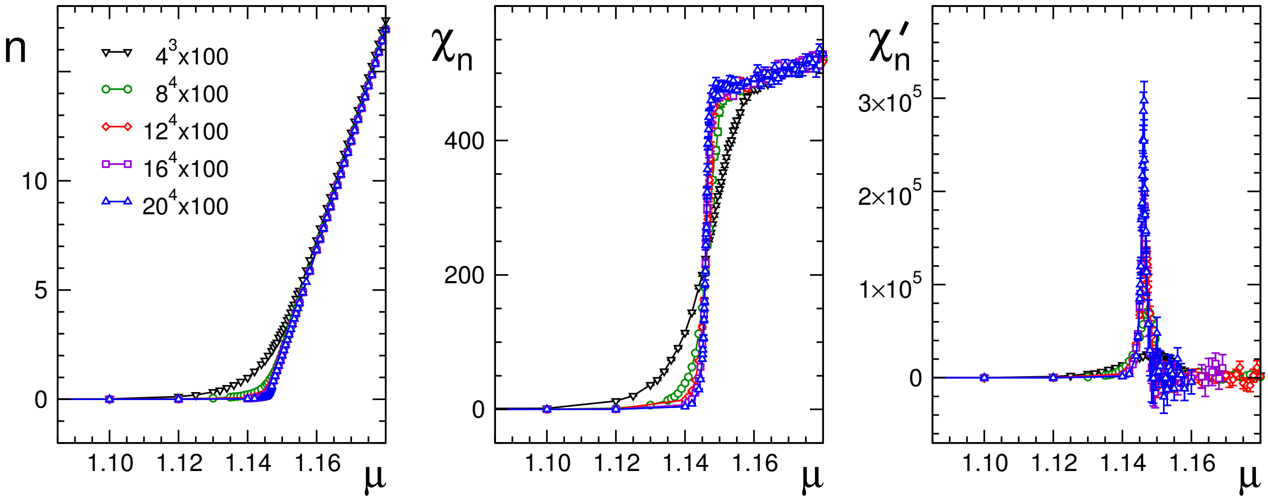


Checks

Simulation with dual variables can be checked with high precision:



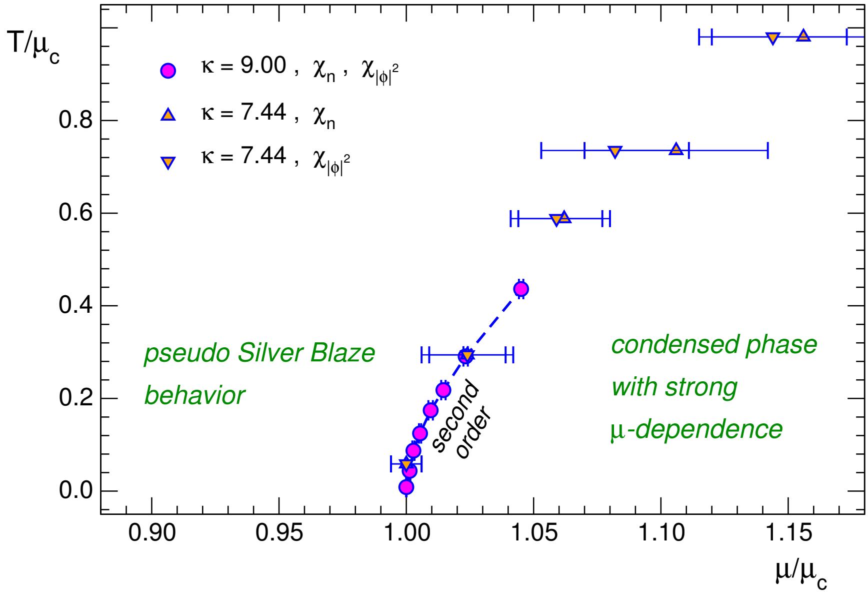
Thermodynamics at zero temperature



Second order transition at the end of the Silver Blaze region.

Phase diagram

Phase diagram in the $\mu - T$ plane:



Silver Blaze transition persists for finite temperature.

Example 2: Effective theories for QCD

Y. Delgado Mercado, H.G. Evertz, C. Gattringer, Phys. Rev. Lett., 2011

C. Gattringer, Nucl. Phys. B, 2011

Y. Delgado Mercado, C. Gattringer Nucl. Phys. B, 2012

Y. Delgado Mercado, H.G. Evertz, C. Gattringer, Comp. Phys. Comm., 2012

F. Karsch, H. Wyld, N. Bilic, H. Gausterer, S. Sanielevici, G. Aarts, F. James: Analysis with complex Langevin (at various points in time).

Frankfurt group: Study of similar models.

Effective theory for QCD

- QCD at large quark mass and strong coupling (leading center symmetric and center symmetry breaking terms):

$$S = -\tau \sum_{\vec{n}} \Big[\sum_{\nu=1}^{3} [\, \mathrm{Tr} P_{\vec{n}} \, \mathrm{Tr} P^{\dagger}_{\vec{n}+\hat{\nu}} + c.c. \,] - \kappa [\, e^{\mu} \, \mathrm{Tr} P_{\vec{n}} + e^{-\mu} \, \mathrm{Tr} P^{\dagger}_{\vec{n}} \,] \Big]$$

$P_{\vec{n}} \in \mathrm{SU}(3)$: Polyakov loop (static color charge)

τ : $\sim$ temperature

κ : decreasing function of mass; proportional to N_f

μ : chemical potential (complex action problem !!!)

- Deconfinement transition at finite temperature

$$\langle \mathrm{Tr}\, P \rangle \;\propto\; e^{-\beta F_q} \begin{cases} = 0 \ \text{(confinement)} & \text{for } T < T_c \\ \neq 0 \ \text{(deconfinement)} & \text{for } T > T_c \end{cases}$$

Dual representation

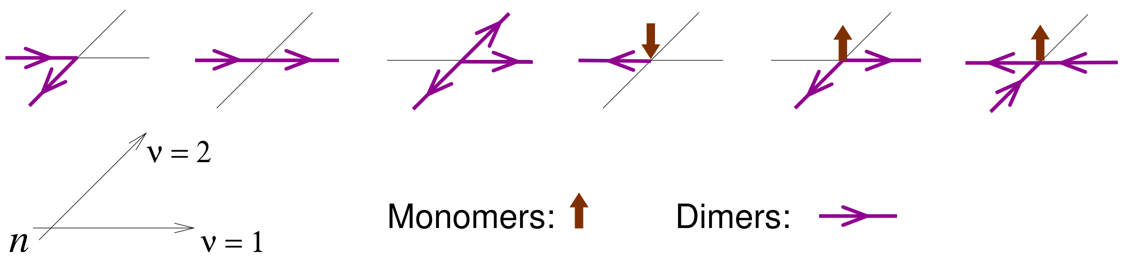
(C. Gattringer, Nucl. Phys. B, 2011)

$$Z \;=\; \sum_{\{l,\bar{l},s,\overline{s}\}} \mathcal{W}(l,\bar{l},s,\overline{s}) \; \exp\Big(\mu \sum_{\vec{n}} \overline{s}_{\vec{n}}\Big) \; \prod_{\vec{n}} T(\,\overline{f}_{\vec{n}}\,)$$

Dimer variables on links: $l_{\vec{n},\nu} \;\in\; \mathbb{N}_0 \;,\; \bar{l}_{\vec{n},\nu} \;\in\; \mathbb{Z}$

Monomer variables on sites: $s_{\vec{n}} \;\in\; \mathbb{N}_0 \;,\; \overline{s}_{\vec{n}} \;\in\; \mathbb{Z}$

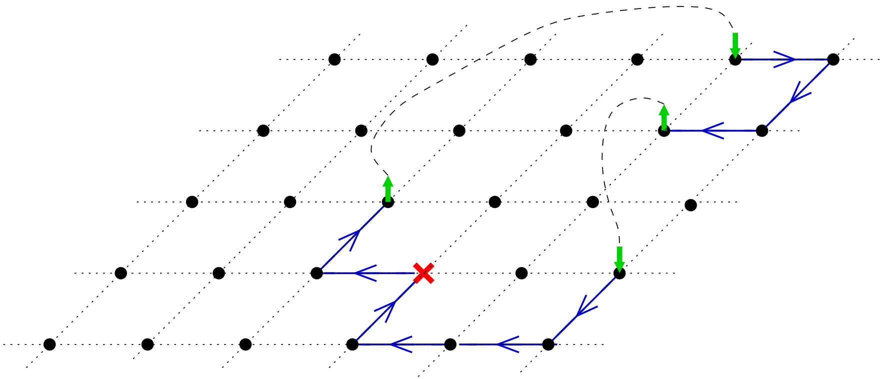
Triality function: $T(n) = \begin{cases} 1 & \text{for } \; n \bmod 3 = 0 \\ 0 & \text{for } \; n \bmod 3 \neq 0 \end{cases}$



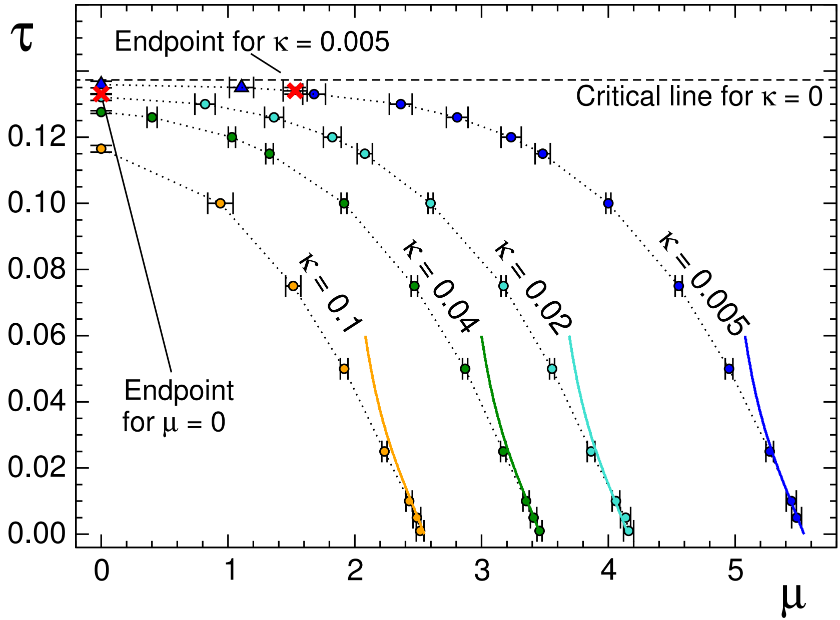
Generalized worm algorithm for systems with monomers

(Y. Delgado Mercado, H.G. Evertz, C. Gattringer, Comp. Phys. Comm., 2012)

- The worm starts at a random position on the lattice.
- The worm may decide to change dimers or insert monomers. Each change is accepted or rejected in a Metropolis step.
- Insertion of a monomer is followed by a random hop and another monomer.
- The worm closes when it reaches its starting point.



Phase diagram



"QCD phase diagram according to the center degrees of freedom"

Example 3: Z_3 gauge-Higgs model and scalar electrodynamics

C. Gattringer, A. Schmidt, arxiv:1208.6472

Y. Delgado Mercado, A. Schmidt, C. Gattringer, in preparation

Action in the conventional representation

- Gauge and Higgs field action of the Z_3 gauge-Higgs model:

$$
\begin{aligned}
S_G &= -\frac{\beta}{2}\sum_x \sum_{\sigma<\tau} [U_{x,\sigma\tau} + U^\star_{x,\sigma\tau}] \\
S_H &= -\eta \sum_x \sum_{j=1}^{3} \Big[\phi^\star_x \, U_{x,j} \, \phi_{x+\widehat{j}} + \phi^\star_x \, U^\star_{x-\widehat{j},j} \, \phi_{x-\widehat{j}}\Big] \\
&\quad -\eta \sum_x \left[e^{\mu} \, \phi^\star_x \, U_{x,4} \, \phi_{x+\widehat{4}} + e^{-\mu} \, \phi^\star_x \, U^\star_{x-\widehat{\nu},\nu} \, \phi_{x-\widehat{4}}\right]
\end{aligned}
$$

$$U_{x,\nu}, \; \phi_x \; \in \; Z_3 = \{1, e^{i2\pi/3}, e^{-i\pi/3}\}$$

- A simple lattice field theory with gauge and matter fields.

- Has a complex action problem for $\mu \neq 0$

Partition sum with dual variables

- New d.o.f.: Link and plaquette occupation numbers $l_{x,\nu},\ p_{x,\rho\sigma} \in \{-1, 0, 1\}$

- Dual partition sum:

$$Z \;=\; \sum_{\{p,l\}} \mathcal{W}_G(p)\, \mathcal{W}_H(l)\, \mathcal{C}_L(p,l)\, \mathcal{C}_S(l)$$

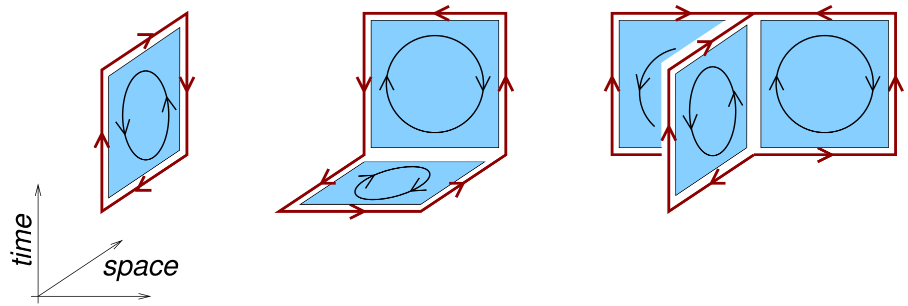
$\mathcal{W}_G(p)$: Weight factor for the plaquette occupation numbers p

$\mathcal{W}_H(l)$: Weight factor for the link occupation numbers l (contains μ)

$\mathcal{C}_L(p,l)$: At each link the flux from l and p must be a multiple of 3
$\Rightarrow$ gauge surfaces

$\mathcal{C}_S(l)$: At each site the flux from l must be a multiple of 3
$\Rightarrow$ matter loops

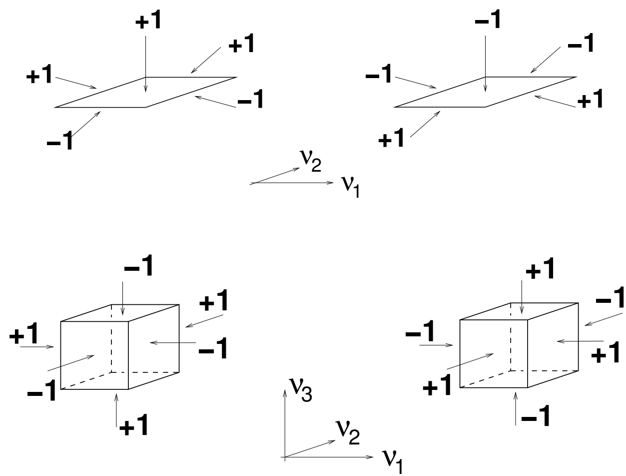
Examples of low lying excitations in the dual representation:



The baryon-type excitation (rhs.) is enhanced by the chemical potential $\propto e^{3\mu}$

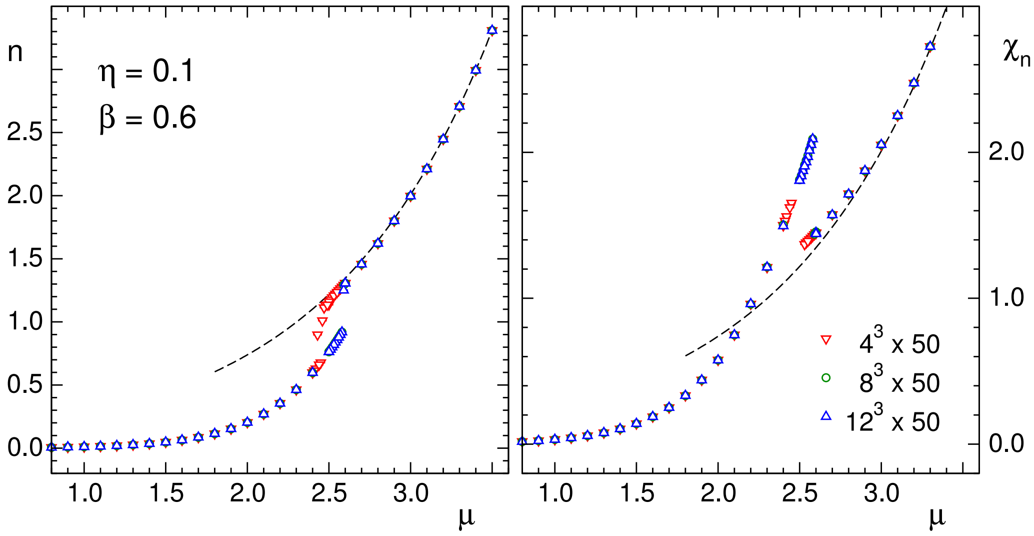
Monte Carlo update for the dual representation

- Local update based on plaquettes bounded by flux and elementary closed surfaces:



- We currently test a worm-type algorithm that breaks up the elements of the local update into smaller segments and uses them to grow filament-like clusters that are updated.

Example of a first order transition as a function of μ

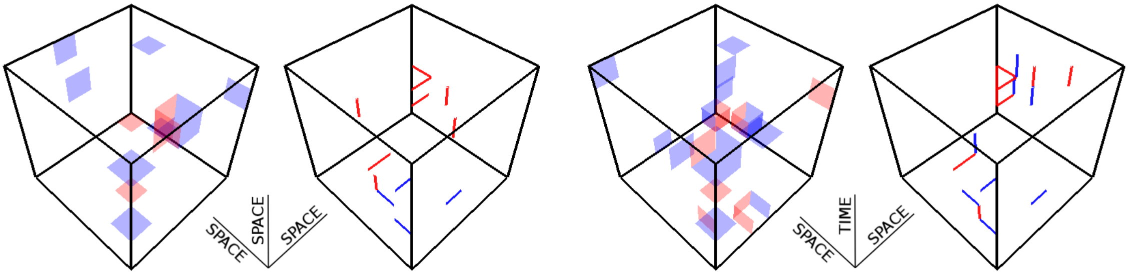


Note that some of the physics is already encoded in the weight factors.

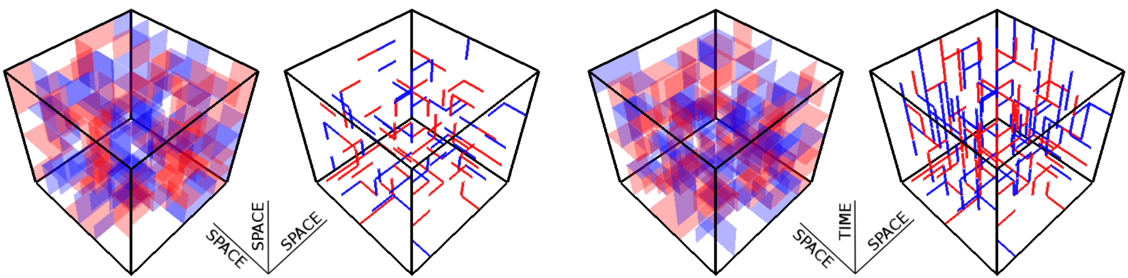
$\Rightarrow$ asymptotic curve.

Phase transition as a condensation of dual variables:

$\mu < \mu_c$:



$\mu > \mu_c$:



Scalar electrodynamics:

- We currently work on scalar QED with two flavors and chemical potential.
- Dual representation solves the complex action problem.
- Efficient surface-worm algorithm is running.
- Has a Silver Blaze phenomenon.

Summary:

- Many interesting quantum field theories have complex phase problems that spoil a direct Monte Carlo simulation on the lattice.

- Considerable progress was made towards rewriting several systems in representations where the partition sum has only real and positive terms.

- Dual degrees of freedom are surfaces for gauge fields and loops for matter.

- Constraints for dual variables can be handled with worm-type algorithms.

- Examples:
  - Silver Blaze phenomenon for charged scalar field.
  - QCD phase diagram from center degrees of freedom.
  - Abelian gauge Higgs systems.

- May serve as solved test cases for other approaches.

- I expect to see more interesting lattice results for finite μ.