GLOW Interferometry School 2012
Bielefeld

Enno Middelberg

The Measurement Equation

Astronomisches Institut
Ruhr-Universität Bochum

This is difficult matter:

- no textbooks
- involved authors use different notations and variants in the math
- no reference
- most information in memos and software

Essentially, this is only math :-(

So why bother?

- mathematically and conceptually clean representation of measurement process
- current software such as Casa implements it
- future instruments and software will use it
- concepts can help to understand the instrument
- doesn't hurt to have a look

Literature:

-The original papers:

"Understanding Radio Polarimetry: I. Mathematical foundations", Hamaker, Bregman & Sault, 1996, A&AS, 117, 137

"Understanding Radio Polarimetry: II. Instrumental calibration of an interferometer array", Sault, Hamaker & Bregman, 1996, A&AS, 117, 149

-Memos from Jan Noordam:

"The Measurement Equation of a Generic Radio Telescope", http://www.astron.nl/~noordam/aips++/185.ps

"Some practical aspects of the matrix-based measurement equation of a generic radio telescope", AIPS++ implementation note nr 182, http://www.astron.nl/aips++/docs/notes/182/182.html

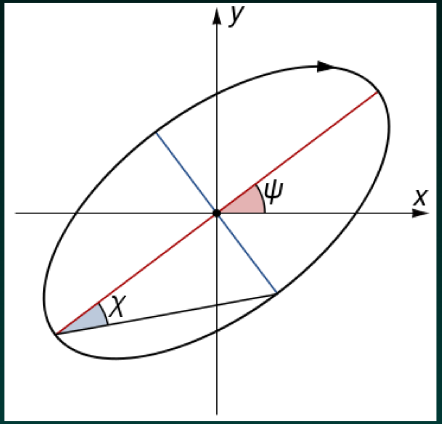
The Stokes vector

The Stokes vector describes the temporally averaged polarisation state of a wave

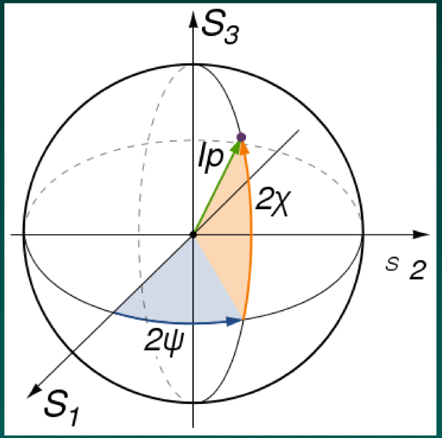
In general, emission is elliptically polarised

Stokes parameters describe the shape and orientation of the ellipse:

$$\vec{I} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I \\ Ip\, cos2\psi\, cos2\chi \\ Ip\, sin2\psi\, cos2\chi \\ Ip\, sin2\chi \end{pmatrix}$$



Parameters S_1-S_3 can be understood as axes in a spherical coordinate system (Poincaré sphere):



Relation between Stokes parameters and several basis systems:

$$\begin{aligned} I &= |E_x|^2 + |E_y|^2, \\ Q &= |E_x|^2 - |E_y|^2, \\ U &= 2\mathrm{Re}(E_x E_y^*), \\ V &= 2\mathrm{Im}(E_x E_y^*), \end{aligned}$$

$$\begin{aligned} I &= |E_l|^2 + |E_r|^2, \\ Q &= 2\mathrm{Re}(E_l^* E_r), \\ U &= -2\mathrm{Im}(E_l^* E_r), \\ V &= |E_l|^2 - |E_r|^2. \end{aligned}$$

Short reminder of Jones matrices:

Electromagnetic waves are represented by a vector:

$$\vec{E} = \begin{pmatrix} E_x(t) \\ E_y(t) \end{pmatrix}$$

External effects such as absorption, rotation of polarisation plane, conversion from linear to circular polarisation are represented by 2x2 matrices:

http://en.wikipedia.org/wiki/Jones_calculus:

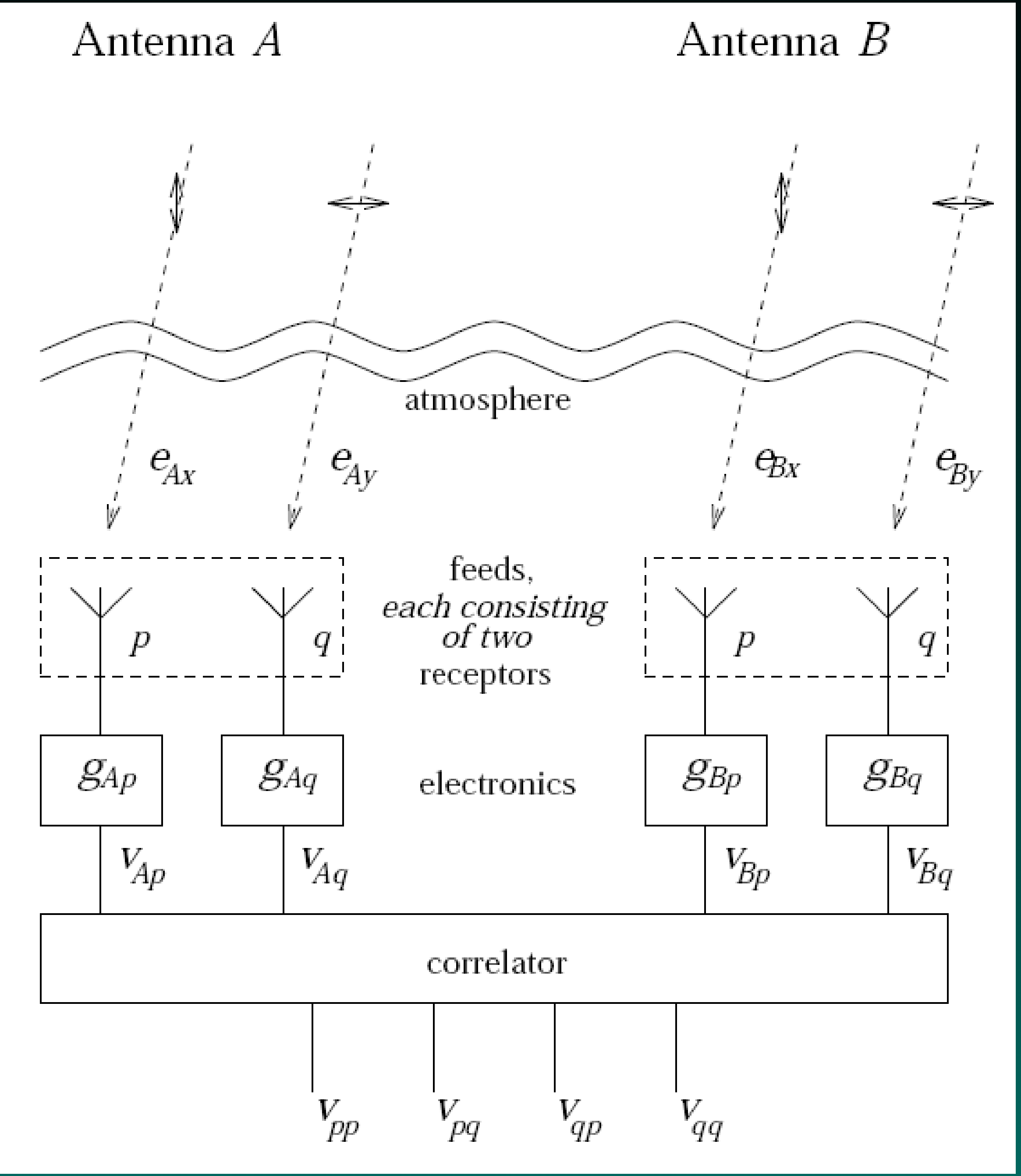
Polarization	Corresponding Jones vector
Linear polarized in the x-direction	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Linear polarized in the y-direction	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
Linear polarized at 45° from the x-axis	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
Linear polarized at -45° from the x-axis	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
Right circular polarized	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$
Left circular polarized	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

Optical element	Corresponding Jones matrix
Linear polarizer with axis of transmission horizontal	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Linear polarizer with axis of transmission vertical	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Linear polarizer with axis of transmission at 45° with the horizontal	$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
Linear polarizer with axis of transmission at -45° with the horizontal	$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
Right circular polarizer	$\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$
Left circular polarizer	$\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$
Linear polarizer with axis of transmission at angle φ with the horizontal	$\begin{pmatrix} \cos^2\varphi & \cos\varphi\sin\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi \end{pmatrix}$

Block diagram of an interferometer:

The field at each antenna requires a 2-vector, but the correlated quantity requires a 4-vector.

Here, p and q represent generalised orthogonal coordinates (XX & YY, or RCP & LCP).



Introducing the coherence vector, e:

$$\vec{e} = \langle \begin{pmatrix} e_{Ap}e^*_{Bp} \\ e_{Ap}e^*_{Bq} \\ e_{Aq}e^*_{Bp} \\ e_{Aq}e^*_{Bq} \end{pmatrix} \rangle = \langle \vec{e}_A \otimes \vec{e}^*_B \rangle$$

The coherence vector is the Kronecker product (outer product, direct product, tensor product?) of two Jones vectors:

$$C = (a_{ij} \cdot B) = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} & 2 \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \\ 3 \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} & 4 \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 5 & 6 & 10 & 12 \\ 7 & 8 & 14 & 16 \\ 15 & 18 & 20 & 24 \\ 21 & 24 & 28 & 32 \end{pmatrix}$$

(http://de.wikipedia.org/wiki/Kroneckerprodukt)

Most important property here:

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{A}' \otimes \mathbf{B}') = (\mathbf{A}\mathbf{A}') \otimes \mathbf{B}\mathbf{B}')$$

Note that indices indicate 2-vectors and 2x2 matrices, and symbols with no index represent 4-vectors and 4x4 matrices.

Now antenna elements can be expressed in the coherence vector as follows:

$$\begin{aligned}\vec{e}_{out} &= \langle \vec{e}_{A,out} \otimes \vec{e}^{*}_{B,out} \rangle \\ &= \langle \mathbf{J}_A \vec{e}_{A,in} \otimes \mathbf{J}^{*}_{B} \vec{e}^{*}_{B,in} \rangle \\ &= (\mathbf{J}_A \otimes \mathbf{J}^{*}_{B}) \langle \vec{e}_{A,in} \otimes \vec{e}^{*}_{B,in} \rangle \\ &= \mathbf{J} \vec{e}_{in}\end{aligned}$$

This is the transition from the 2-vectors and 2x2 matrices of the Jones calculus to the 4-vectors and 4x4 matrices of the Müller-calculus.

The Stokes vector and the coherence vector are related as follows:

$$\vec{e}^{S} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \mathbf{T}\vec{e}^{+}, \quad \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & -i & 0 \end{pmatrix}, \quad \vec{e}^{+} = \begin{pmatrix} e_{Ap}e^{*}_{Bp} \\ e_{Ap}e^{*}_{Bq} \\ e_{Aq}e^{*}_{Bp} \\ e_{Aq}e^{*}_{Bq} \end{pmatrix} = \mathbf{T}^{-1}\vec{e}^{S} = \mathbf{S}^{+} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}$$

This is essentially a coordinate transform from the xy system to a system which is generated by Stokes parameters.

Later on we'll need the inverse of T to go the other way:

$$\mathbf{S}^{+} = \mathbf{T}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

Matrices can be defined in different coordinate systems, and can be transformed from one to the other, depending on telescope properties:

Transformation from Stokes to linear

$$\mathbf{S}^{+} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

Transformation from Stokes to circular

$$\mathbf{S}^{\odot} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Transformation matrices:

$$\mathbf{S}^{\odot} = (\mathcal{H} \otimes \mathcal{H}^{*})\mathbf{S}^{+} \qquad \mathbf{S}^{+} = (\mathcal{H}^{-1} \otimes (\mathcal{H}^{-1})^{*})\mathbf{S}^{\odot}$$

$$\mathcal{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \qquad \mathcal{H}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

In general:

$$\mathbf{A}^{\odot} = \mathcal{H}\mathbf{A}^{+}\mathcal{H}^{-1} \qquad \mathbf{A}^{+} = \mathcal{H}^{-1}\mathbf{A}^{\odot}\mathcal{H}$$

An interferometer measures the coherence of emission by converting an EM wave's amplitude to a voltage, and by calculating the coherence vector of these voltages:

$$\vec{V}_{AB} = \langle \begin{pmatrix} v_{Ap}v^*_{Bp} \\ v_{Ap}v^*_{Bq} \\ v_{Aq}v^*_{Bp} \\ v_{Aq}v^*_{Bq} \end{pmatrix} \rangle$$

Therefore a visibility can be related to the sky brightness through:

$$\vec{V}_{AB} = \begin{pmatrix} v_{ApBp} \\ v_{ApBq} \\ v_{AqBp} \\ v_{AqBq} \end{pmatrix} = (J_A \otimes J^*_B)\, S \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}_{l,m} = (J_A \otimes J^*_B)\, S\, \vec{I}(l,m)$$

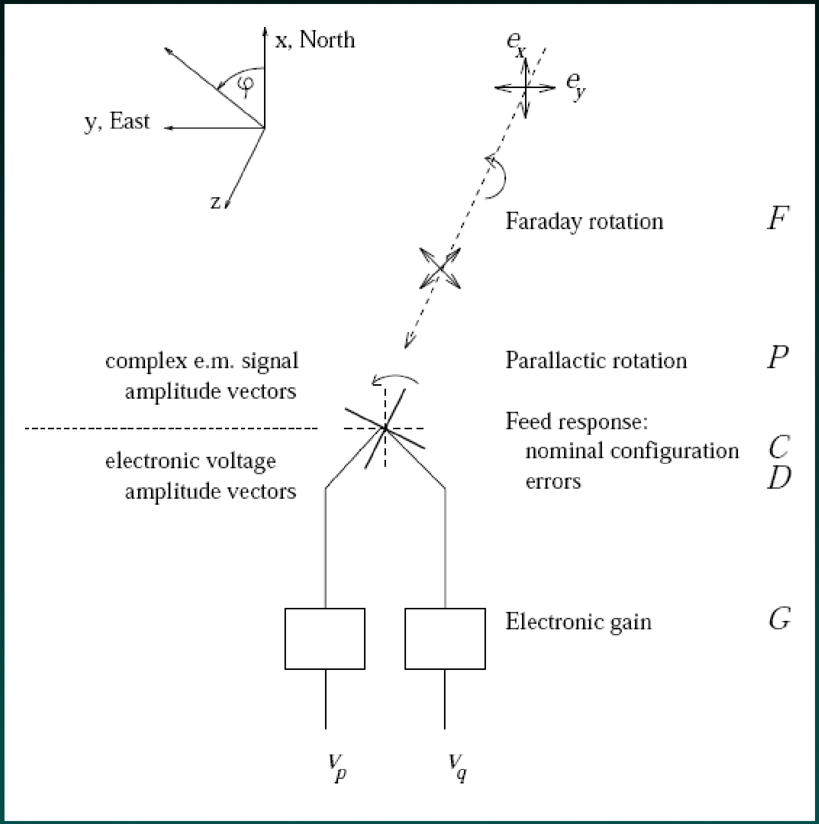
The instrument's voltage correspond to either linear or circular polarisation:

$$\vec{V}^+_A = \begin{pmatrix} v_{Ap} \\ v_{Aq} \end{pmatrix} = J^+_A \begin{pmatrix} e_x \\ e_y \end{pmatrix} \qquad \vec{V}^\odot_A = \begin{pmatrix} v_{Ap} \\ v_{Aq} \end{pmatrix} = J^\odot_A \begin{pmatrix} e_r \\ e_l \end{pmatrix}$$

The first equation on this page can now be expressed as:

$$\vec{V}_{AB} = (J^+_A \otimes J^{+*}_B)(\vec{E} \otimes \vec{E}^*) = (J^+_A \otimes J^{+*}_B) \begin{pmatrix} e_{Ax}e^*_{Bx} \\ e_{Ax}e^*_{By} \\ e_{Ay}e^*_{Bx} \\ e_{Ay}e^*_{By} \end{pmatrix} = (J^+_A \otimes J^{+*}_B)\, S^+\, \vec{I}(l,m)$$

Now the signal undergoes several modifications in its path from the sky to the final measured product (Hamaker, Bregman, Sault 1996):



F – faraday rotation

T – atmospheric attenuation (not in diagram)

P – parallactic angle

B – primary beam attenuation (not in diagram)

C – feed

D – feed error

G – electronic gain

G and D are sometimes pulled together to form a receiver term, R: $\mathbf{R}_A = \mathbf{G}_A \mathbf{D}_A$

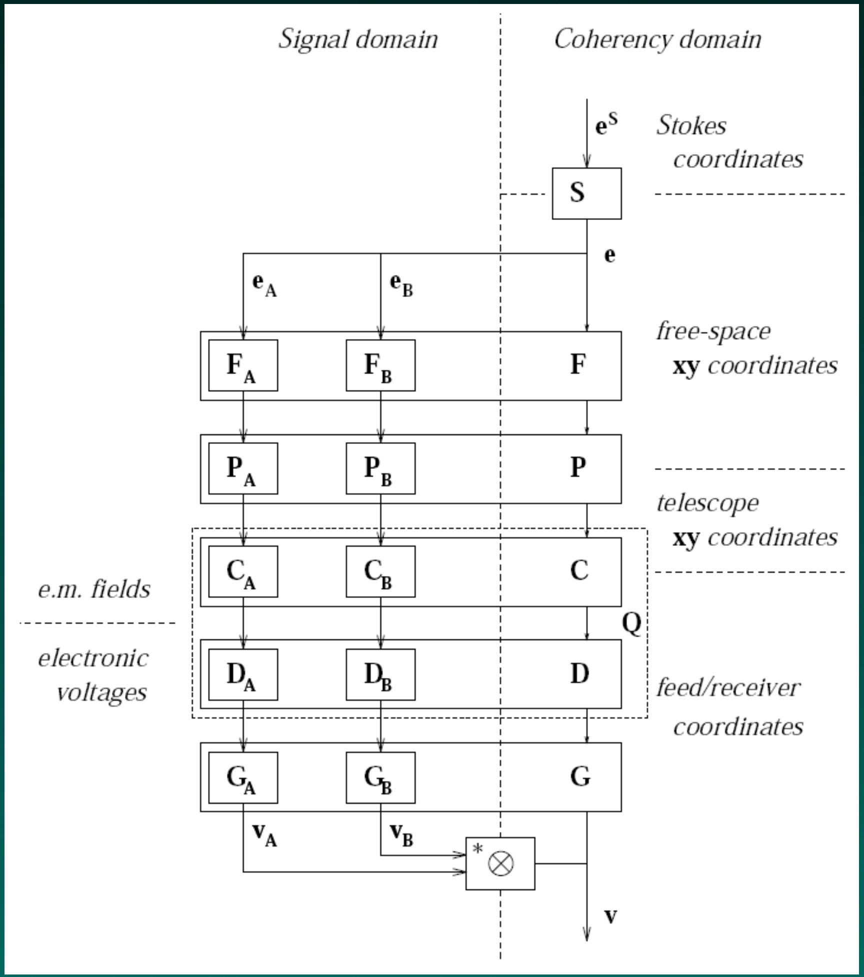
Hence, in one interferometer "arm", the signal is subject to several modifications: $\vec{v}_A = \mathbf{J}_A \vec{e}_A = \mathbf{R}_A \mathbf{C}_A \mathbf{P}_A \mathbf{F}_A \vec{e}_A$

We remember $(\mathbf{A} \otimes \mathbf{B})(\mathbf{A}' \otimes \mathbf{B}') = (\mathbf{A}\mathbf{A}') \otimes \mathbf{B}\mathbf{B}'$ and

$$
\begin{aligned}
\vec{e}_{out} &= \langle \vec{e}_{A,out} \otimes \vec{e}^{\,*}_{B,out} \rangle \\
&= (\mathbf{J}_A \otimes \mathbf{J}^*_B) \langle \vec{e}_{A,in} \otimes \vec{e}^{\,*}_{B,in} \rangle \\
&= \mathbf{J} \vec{e}_{in}
\end{aligned}
$$

And "we see immediately":

$$
\begin{aligned}
\vec{v} &= (\mathbf{J}_A \vec{e}_A) \otimes (\mathbf{J}_B \vec{e}_B) \\
&= (\mathbf{J}_A \otimes \mathbf{J}^*_B)(\vec{e}_A \otimes \vec{e}_B) \\
&= (\mathbf{J}_A \otimes \mathbf{J}^*_B)\vec{e}^{\,+} \\
&= (\mathbf{J}_A \otimes \mathbf{J}^*_B)\mathbf{T}^{-1}\vec{e}^{\,S} \\
&= \mathbf{J}\mathbf{S}\vec{e}^{\,S} \\
&= \mathbf{K}\vec{e}^{\,S}
\end{aligned}
$$



K is sometimes called "system matrix"

Disclaimer: this is only true for sources in the instrument's phase centre (no directional dependence)

Details of some of the terms:

F_A (Faraday rotation) and P_A (parallactic angle) are rotation matrices:

$$(\mathbf{F}_A \,|\, \mathbf{P}_A) = \begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix}$$

T_A (atmospheric attenuation) is a diagonal matrix with identical terms (why?)

$$\mathbf{T}_A = \begin{pmatrix} t_A & 0 \\ 0 & t_A \end{pmatrix}$$

D_A (feed errors) is a matrix with small complex values off the diagonal

$$\mathbf{D}_A = \begin{pmatrix} 1 & d_{Ap} \\ -d_{Aq} & 1 \end{pmatrix}$$

G_A (gain) is a simple diagonal matrix

$$\mathbf{G}_A = \begin{pmatrix} g_{Ap} & 0 \\ 0 & g_{Aq} \end{pmatrix}$$

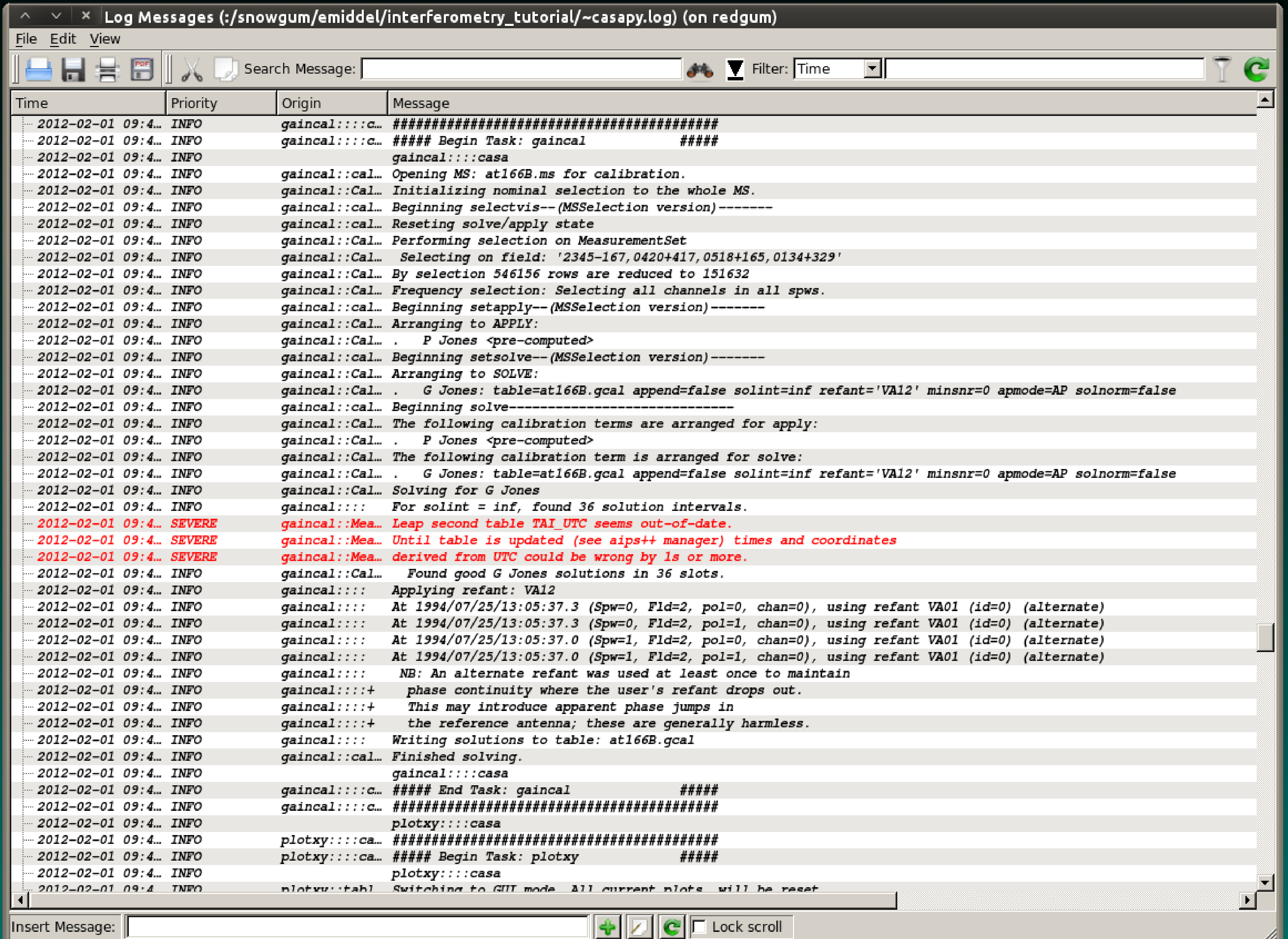
Example: point source in phase centre of ATCA (linear, on-axis feeds)

$$\mathbf{J}_A^{ATCA} = \mathbf{G}_A \mathbf{D}_A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{P}_A^{alt-az} \mathbf{F}_A$$

Example: point source in phase centre of VLA (circular, off-axis feeds)

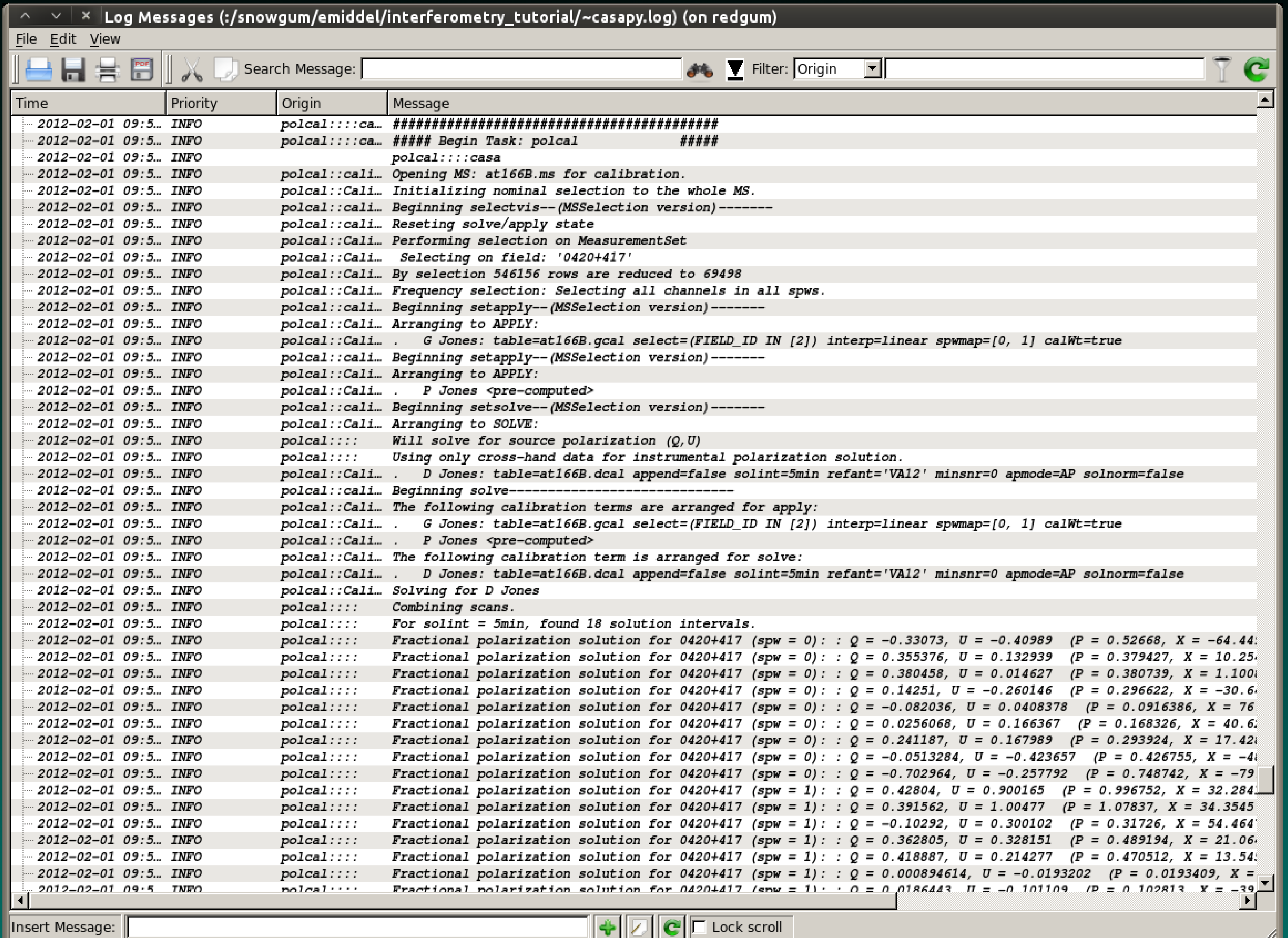
$$\mathbf{J}_A^{VLA} = \mathbf{G}_A \mathbf{D}_A \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \mathbf{B}_A \mathbf{P}_A^{alt-az} \mathbf{F}_A$$

Excerpt from Casapy logger:



Time	Priority	Origin	Message
2012-02-01 09:4...	INFO	gaincal::::c...	##
2012-02-01 09:4...	INFO	gaincal::::c...	##### Begin Task: gaincal #####
2012-02-01 09:4...	INFO		gaincal::::casa
2012-02-01 09:4...	INFO	gaincal::cal...	Opening MS: at166B.ms for calibration.
2012-02-01 09:4...	INFO	gaincal::Cal...	Initializing nominal selection to the whole MS.
2012-02-01 09:4...	INFO	gaincal::cal...	Beginning selectvis--(MSSelection version)-------
2012-02-01 09:4...	INFO	gaincal::cal...	Reseting solve/apply state
2012-02-01 09:4...	INFO	gaincal::Cal...	Performing selection on MeasurementSet
2012-02-01 09:4...	INFO	gaincal::Cal...	Selecting on field: '2345-167,0420+417,0518+165,0134+329'
2012-02-01 09:4...	INFO	gaincal::Cal...	By selection 546156 rows are reduced to 151632
2012-02-01 09:4...	INFO	gaincal::Cal...	Frequency selection: Selecting all channels in all spws.
2012-02-01 09:4...	INFO	gaincal::cal...	Beginning setapply--(MSSelection version)-------
2012-02-01 09:4...	INFO	gaincal::Cal...	Arranging to APPLY:
2012-02-01 09:4...	INFO	gaincal::Cal...	. P Jones <pre-computed>
2012-02-01 09:4...	INFO	gaincal::cal...	Beginning setsolve--(MSSelection version)-------
2012-02-01 09:4...	INFO	gaincal::Cal...	Arranging to SOLVE:
2012-02-01 09:4...	INFO	gaincal::Cal...	. G Jones: table=at166B.gcal append=false solint=inf refant='VA12' minsnr=0 apmode=AP solnorm=false
2012-02-01 09:4...	INFO	gaincal::cal...	Beginning solve------------------------------
2012-02-01 09:4...	INFO	gaincal::Cal...	The following calibration terms are arranged for apply:
2012-02-01 09:4...	INFO	gaincal::Cal...	. P Jones <pre-computed>
2012-02-01 09:4...	INFO	gaincal::Cal...	The following calibration term is arranged for solve:
2012-02-01 09:4...	INFO	gaincal::Cal...	. G Jones: table=at166B.gcal append=false solint=inf refant='VA12' minsnr=0 apmode=AP solnorm=false
2012-02-01 09:4...	INFO	gaincal::Cal...	Solving for G Jones
2012-02-01 09:4...	INFO	gaincal::::	For solint = inf, found 36 solution intervals.
2012-02-01 09:4...	SEVERE	gaincal::Mea...	Leap second table TAI_UTC seems out-of-date.
2012-02-01 09:4...	SEVERE	gaincal::Mea...	Until table is updated (see aips++ manager) times and coordinates
2012-02-01 09:4...	SEVERE	gaincal::Mea...	derived from UTC could be wrong by 1s or more.
2012-02-01 09:4...	INFO	gaincal::Cal...	Found good G Jones solutions in 36 slots.
2012-02-01 09:4...	INFO	gaincal::::	Applying refant: VA12
2012-02-01 09:4...	INFO	gaincal::::	At 1994/07/25/13:05:37.3 (Spw=0, Fld=2, pol=0, chan=0), using refant VA01 (id=0) (alternate)
2012-02-01 09:4...	INFO	gaincal::::	At 1994/07/25/13:05:37.3 (Spw=0, Fld=2, pol=1, chan=0), using refant VA01 (id=0) (alternate)
2012-02-01 09:4...	INFO	gaincal::::	At 1994/07/25/13:05:37.0 (Spw=1, Fld=2, pol=0, chan=0), using refant VA01 (id=0) (alternate)
2012-02-01 09:4...	INFO	gaincal::::	At 1994/07/25/13:05:37.0 (Spw=1, Fld=2, pol=1, chan=0), using refant VA01 (id=0) (alternate)
2012-02-01 09:4...	INFO	gaincal::::	NB: An alternate refant was used at least once to maintain
2012-02-01 09:4...	INFO	gaincal::::+	phase continuity where the user's refant drops out.
2012-02-01 09:4...	INFO	gaincal::::+	This may introduce apparent phase jumps in
2012-02-01 09:4...	INFO	gaincal::::+	the reference antenna; these are generally harmless.
2012-02-01 09:4...	INFO	gaincal::::	Writing solutions to table: at166B.gcal
2012-02-01 09:4...	INFO	gaincal::cal...	Finished solving.
2012-02-01 09:4...	INFO		gaincal::::casa
2012-02-01 09:4...	INFO	gaincal::::c...	##### End Task: gaincal #####
2012-02-01 09:4...	INFO	gaincal::::c...	##
2012-02-01 09:4...	INFO		plotxy::::casa
2012-02-01 09:4...	INFO	plotxy::::ca...	##
2012-02-01 09:4...	INFO	plotxy::::ca...	##### Begin Task: plotxy #####
2012-02-01 09:4...	INFO		plotxy::::casa
2012-02-01 09:4	INFO	plotxy::tabl	Switching to GUI mode. All current plots will be reset

Excerpt from Casapy logger:



The primary beam term B describes the conversion from an electric field to a voltage – this is somewhat more complicated:

$$\mathbf{B}_A^{+}(\vec{\rho}) = \begin{pmatrix} b_{Axa} & b_{Aya} \\ b_{Axb} & b_{Ayb} \end{pmatrix} \qquad \mathbf{B}_A^{\odot}(\vec{\rho}) = \begin{pmatrix} b_{Ara} & b_{Ala} \\ b_{Arb} & b_{Alb} \end{pmatrix}$$

The sky rotation can be separated: $\mathbf{B}_A = (\mathbf{B}_A\mathbf{P}_A^{-1})\mathbf{P}_A = \mathbf{E}'_A\mathbf{P}_A$

E_A' can further be divided into a position-dependent and position-independent part: $\mathbf{E}'_A\mathbf{P}_A = \mathbf{D}_A(\mathbf{D}_A^{-1}\mathbf{E}'_A)\mathbf{P}_A = \mathbf{D}_A\mathbf{E}_A\mathbf{P}_A$

$$\mathbf{E}'_A(\vec{\rho}) = \begin{pmatrix} e_{Aaa} & e_{Aba} + e'_{Aba} \\ e_{Aab} + e'_{Aab} & e_{Abb} \end{pmatrix} \approx \begin{pmatrix} 1 & e'_{Aba}/e_{Abb} \\ e'_{Aab}/e_{Aaa} & 1 \end{pmatrix} \begin{pmatrix} e_{Aaa} & e_{Aba} \\ e_{Aab} & e_{Abb} \end{pmatrix} = \begin{pmatrix} d_{Aaa} & d_{Aba} \\ d_{Aab} & d_{Abb} \end{pmatrix} \begin{pmatrix} e_{Aaa} & e_{Aba} \\ e_{Aab} & e_{Abb} \end{pmatrix} = \mathbf{D}_A\mathbf{E}_A(\vec{\rho})$$

Here D_A is position-independent (how useful!) and the off-diagonal terms of $E_A(\rho)$ are negligible in most cases. This assumes that the bulk of the leakage is position-independent, which is mostly true.

Then, for an idealised Gaussian beam shape (double primes indicate coordinates of the voltage beam, which may be different from the phase centre, or antenna pointing direction)

$$e_{Aaa} = \exp - [(\frac{l''_{Aa}}{\sigma_a(1+\epsilon_a)})^2 + (\frac{m''_{Aa}}{\sigma_a(1+\epsilon_a)})^2] \qquad e_{Abb} = \exp - [(\frac{l''_{Ab}}{\sigma_b(1+\epsilon_b)})^2 + (\frac{m''_{Ab}}{\sigma_b(1+\epsilon_b)})^2]$$

So far all effects were antenna-based, but there are some (small) baseline-based effects, which can be treated as follows.

-"Correlator offsets" and receiver noise are added to the signal: A_{AB}

-Decorrelation depends on baseline length and observing direction: M_{AB}

-Van Vleck correction, arising from signal quantisation: X_{AB}

Therefore $\vec{v}' = \mathbf{X}_{AB}(\mathbf{A}_{AB} + \mathbf{M}_{AB}\vec{v})$

However, the elements of XAB are equal for modern correlators, resulting in a skalar multiplication, and the elements of MAB are commonly small (but wide-field VLBI observers beware!)

$$= \begin{pmatrix} x_{ApBp} & 0 & 0 & 0 \\ 0 & x_{ApBq} & 0 & 0 \\ 0 & 0 & x_{AqBp} & 0 \\ 0 & 0 & 0 & x_{AqBq} \end{pmatrix} \left(\begin{pmatrix} a_{ApBp} \\ a_{ApBq} \\ a_{AqBp} \\ a_{AqBq} \end{pmatrix} + \begin{pmatrix} m_{ApBp} & 0 & 0 & 0 \\ 0 & m_{ApBq} & 0 & 0 \\ 0 & 0 & m_{AqBp} & 0 \\ 0 & 0 & 0 & m_{AqBq} \end{pmatrix} \begin{pmatrix} v_{ApBp} \\ v_{ApBq} \\ v_{AqBp} \\ v_{AqBq} \end{pmatrix} \right)$$

Order of matrices:

-The order is in general important and represents the sequence of effects
-Matrices do not in general commute, but:

-Purely multiplicative matrices (e.g. T) commute with all others
-Diagonal matrices with unequal elements commute with one another (G, P°)
-Purely rotational matrices (P^{+}) commute with one another

-Examples:
 -T and G can be pulled together if direction-independent
 -If E is diagonal, it commutes with the Stokes vector, and a correction can be applied in the image plane

Alternative form of the Measurement Equation
(Hamaker 2000, A&AS, 143, 515)

Antenna signals are modified by Jones matrices:

$$\vec{v} = \mathbf{J}\vec{e} = \begin{pmatrix} j_{xx} & j_{yx} \\ j_{xy} & j_{yy} \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

$$\vec{v}_p = \mathbf{J}_p \vec{e}_p \qquad \vec{v}_q = \mathbf{J}_q \vec{e}_q$$

However, the coherency vector is now a coherency matrix, leading to slightly different notation:

$$\mathbf{V}_{pq} = \langle (\mathbf{J}_p \vec{e}_p)(\mathbf{J}_q \vec{e}_q)^\dagger \rangle = \langle \mathbf{J}_p (\vec{e}_p \vec{e}_q^\dagger) \mathbf{J}_q^\dagger \rangle = \mathbf{J}_p \langle (\vec{e}_p \vec{e}_q^\dagger) \rangle \mathbf{J}_q^\dagger$$

$$\mathbf{V}_{pq} = \mathbf{J}_p \langle \begin{pmatrix} e_{px} \\ e_{py} \end{pmatrix} \begin{pmatrix} e_{qx}^* & e_{qy}^* \end{pmatrix} \rangle \mathbf{J}_q^\dagger$$

$$\mathbf{V}_{pq} = \mathbf{J}_p \begin{pmatrix} I + Q & U + iV \\ U - iV & I - Q \end{pmatrix} \mathbf{J}_q^\dagger$$

$$\mathbf{V}_{pq} = \mathbf{J}_p \mathbf{B} \mathbf{J}_q^\dagger$$

Alternative form of the Measurement Equation
(Hamaker 2000, A&AS, 143, 515)

$$\mathbf{V}_{pq} = \mathbf{J}_p \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix} \mathbf{J}_q^\dagger$$

This representation is more compact, but essentially represents the same thing. Some vocal individuals have strong opinions on which form to use.

Other properties of the Measurement Equation formalism are the same, in particular the Jones matrices.

Personally I think that mixed Stokes terms are a bit awkward.