A unitarity compatible integrand basis at two loops

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Outline

Methods

Over the last decade or so modern methods of

- on-shell recursion relations (Britto, Cachazo, Feng, Witten,...)

and

- unitarity methods (Bern, Dixon, Kosower, ..., Ossola, Pittau, Papadopoulos, ..., Badger,.....)

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Knowledge of integrand basis

$$\text{Amplitude} = \sum_{j \in \text{Basis}} c_j * \text{Integral}_j + \text{Rational}$$

is important here.

One loop

Basis is known (independently of the given one-loop process), and include (scalar) integrals: boxes, triangles, bubbles and tadpoles)

$$\int d^d q \frac{1}{D_1 D_2 ... D_n}$$

- Kallen, Toll (1965): triangles (n=3) $\rightarrow$ bubbles (in 2 dim)
- Melrose (later van Neerven and Vermaseren): pentagon (n=5) $\rightarrow$ boxes (in 4 dim), see J.Fleischer's talk
- Lorenz invariance + Passarino and Veltman: tensor n-PF $\rightarrow$ m-PF scalar integrals ($m \leq n$)

Efficient methods for finding decompositions at one loop

- *improved* tensor decomposition (Denner, Dittmaier, Fleischer, Riemann, Yundin)

Automatic packages: FeynArts, LoopTools, PJFRY

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Knowledge of (scalar) basis and their analytic structure allowed to focus and find coefficients of reductions:

- Complex integration and contour deformation (Weinzierl, Soper, Nagy,...)
- On-shell and generalised unitarity methods (OPP, Kosower, ..., Mastrolia,...), integrand reduction techniques (Ellis, Giele, Kunszt,Melnikov, Tramontano, Heinrich, Reiter)

Automatic packages: BlackHat, Golem/Samurai, GoSam, Helac,

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- It was proved that basis is finite in general (for any topology, abstract proof by A.Smirnov and Petuchov), and proved many times in practice using IBP relations (Chetyrkin-Tkachov)

$$0 = \prod_{i=1}^{L} \left(\int \frac{d^d \ell_i}{(2\pi)^d} \right) \frac{\partial}{\partial \ell_j} \cdot \left(\frac{v^{(j)}}{D_1(\ell_1, \ldots \ell_L) \cdots D_m(\ell_1, \ldots \ell_L)} \right)$$

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- Automation through a public software AIR, FIRE, Reduze, (plus IdSolver, etc)

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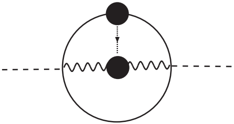
(i) IBPs create doubled ("dotted") propagators

e.g.

$$\frac{\partial}{\partial \ell_\mu} \frac{1}{(\ell - K)^2} \sim \frac{1}{[(\ell - K)^2]^2}$$



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- They can also create artificial $1/\epsilon$ factors in front of MIs (which must be then determined to higher level in ϵ)

So better if we avoid them, also in the unitarity approach.

Moreover, in the unitarity approach

(ii) Coefficients c_j are functions of the external spinors (depend on ϵ in addition)

$$\text{Amplitude} = \sum_{j \in \text{Basis}} \text{c}_j(\epsilon, \ldots) * \text{Integral}_j + \text{Rational}$$

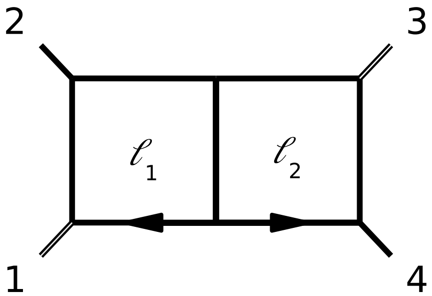
What else?

What else?

(iii) At two loops number of master integrals for a given topology often depend on the number and arrangement of external massive legs, in general

Even more: dependence on relations among masses of external legs

Integral	#MIs
$P^{**}_{2,2}$	=2
$P^{**}_{2,2}, \{1\}$	=2
$P^{**}_{2,2}, \{1,2\}$	=3
$P^{**}_{2,2}, \{1,3\}$	=2
$P^{**}_{2,2}, \{1,4\}$	=2
$P^{**}_{2,2}, \{1,2,3\}$	=3
$P^{**}_{2,2}, \{1,2,3,4\}$	=4



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(iv) bases to all orders in ϵ (d-dimensional basis)

(v) ignoring $\mathcal{O}(\epsilon)$ in amplitudes (regulated 4-dimensional basis)

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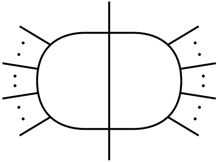
(iv) bases to all orders in ϵ (d-dimensional basis)

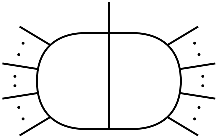
(v) ignoring $\mathcal{O}(\epsilon)$ in amplitudes (regulated 4-dimensional basis)

Our aim is to reduce any high-multiplicity two-loop integral (including numerators) to the above classes of basis integrals, which are free of higher powers of propagators.

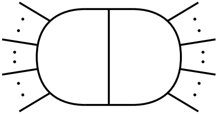
Convention, planar topologies



P_{n_1,n_2}



$P^{*}_{n_1,n_2}$



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Nonplanar cases would come with an external leg attached to the vertical, internal line.

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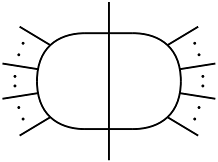

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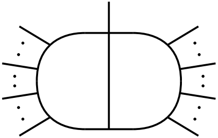

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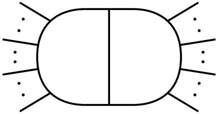

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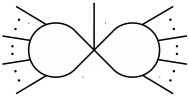
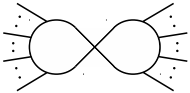


I_{n_1,n_2}



$I^{*}_{n_1,n_2}$

$$P_{n_1,n_2} = (-i)^2 \int \frac{d^d\ell_1}{(2\pi)^d}\frac{d^d\ell_2}{(2\pi)^d} \frac{1}{\ell_1^2(\ell_1-K_1)^2\cdots(\ell_1-K_{1\cdots n_1})^2(\ell_1+\ell_2+K_{n_1+n_2+2})^2}$$
$$\times \frac{1}{\ell_2^2(\ell_2-K_{n_1+n_2+1})^2\cdots(\ell_1-K_{(n_1+2)\cdots(n_1+n_2+1)})^2},$$

$$P^*_{n_1,n_2} = (-i)^2 \int \frac{d^d\ell_1}{(2\pi)^d}\frac{d^d\ell_2}{(2\pi)^d} \frac{1}{\ell_1^2(\ell_1-K_1)^2\cdots(\ell_1-K_{1\cdots n_1})^2(\ell_1+\ell_2)^2}$$
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$$
\begin{aligned}
P_{n_1,n_2} &= (-i)^2 \int \frac{d^d\ell_1}{(2\pi)^d}\frac{d^d\ell_2}{(2\pi)^d}\; \frac{1}{\ell_1^2(\ell_1-K_1)^2\cdots(\ell_1-K_{1\cdots n_1})^2(\ell_1+\ell_2+K_{n_1+n_2+2})^2} \\
&\qquad\times \frac{1}{\ell_2^2(\ell_2-K_{n_1+n_2+1})^2\cdots(\ell_1-K_{(n_1+2)\cdots(n_1+n_2+1)})^2}\,, \\
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&\qquad\times \frac{1}{\ell_2^2(\ell_2-K_{n_1+n_2})^2\cdots(\ell_1-K_{(n_1+1)\cdots(n_1+n_2)})^2}\,,
\end{aligned}
$$

$$
\begin{aligned}
P^*_{n_1,n_2}(K_1,\ldots,K_{n_1+n_2+1}) &= P_{n_1,n_2}(K_1,\ldots,K_{n_1+n_2+1},0)\,, \\
P^{**}_{n_1,n_2}(K_1,\ldots,K_{n_1+n_2}) &= P^*_{n_1,n_2}(K_1,\ldots,K_{n_1},0,K_{n_1+1},\ldots,K_{n_1+n_2})\,.
\end{aligned}
$$

At one loop

$$G\begin{pmatrix} p_1, \cdots, p_l \\ q_1, \cdots, q_l \end{pmatrix} \equiv \det_{i,j\in l\times l}(2p_i \cdot q_j)\,,$$

We can expand each of the four-dimensional vectors v_j in a basis of four chosen external momenta b_1, b_2, b_3, b_4,

$$v_j^\mu = \frac{1}{G(b_1, b_2, b_3, b_4)}\Bigg[G\begin{pmatrix} v, b_2, b_3, b_4 \\ b_1, b_2, b_3, b_4 \end{pmatrix} b_1^\mu + G\begin{pmatrix} b_1, v, b_3, b_4 \\ b_1, b_2, b_3, b_4 \end{pmatrix} b_2^\mu + G\begin{pmatrix} b_1, b_2, v, b_4 \\ b_1, b_2, b_3, b_4 \end{pmatrix} b_3^\mu + G\begin{pmatrix} b_1, b_2, b_3, v \\ b_1, b_2, b_3, b_4 \end{pmatrix} b_4^\mu\Bigg]\,.$$

We can express v_j by b_i, then $\ell \cdot b_i$ are all reducible, e.g.

$$\ell \cdot b_1 = \frac{1}{2}\left[(\ell - K)^2 - (\ell - K - b_1)^2 + (K + b_1)^2 - K^2\right]$$

I. Reduction of High-Multiplicity Integrals with Non-Trivial Numerators

talk "From tensor integral to IBP" by Mohammad Assadsolimami

At two loops, for $n_1 \geq 4$, tensor integrals ($\ell \equiv \ell_1, \ell_2$) $P_{n_1,n_2}[\ell \cdot v_1 \, \ell \cdot v_2 \cdots \ell \cdot v_n]$ can be similarly expanded with the external momenta $b_1, \ldots, b_4$ chosen amongst the first n_1 momenta. Then $\ell_1 \cdot K_j$, $1 \leq j \leq n_1$, are reducible

$$\ell_1 \cdot K_j = \frac{1}{2}\Big[\underbrace{(\ell_1 - K_{1\cdots(j-1)})^2 - (\ell_1 - K_{1\cdots j})^2}_{e.g.\ P_{n_1-1,n_2}} + \underbrace{K^2_{1\cdots j} - K^2_{1\cdots(j-1)}}_{simpler\ tensors}\Big]$$

Similarly for ℓ_2. We end up with basis containing $P^{\natural,*,**}_{n_1\leq 4, n_2 < n_1}$ and (scalar, reducible or irreducible numerators) or general (n_1, n_2) but with trivial numerators (without ℓ_i)

II. Reduction of High-Multiplicity Integrals with Trivial Numerators

Still trivial numerators but with arbitrary number of external legs, $n_1 \geq 5$.
At one loop:

$$I_n[\mathcal{P}(\ell)] \equiv -i \int \frac{d^D\ell}{(2\pi)^D} \, \frac{\mathcal{P}(\ell)}{\ell^2(\ell - K_1)^2(\ell - K_{12})^2 \cdots (\ell - K_{1\cdots(n-1)})^2} \,,$$

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$n \geq 6$. In general, (external momenta are 4 dimensional)

$$G\begin{pmatrix}\ell,1,2,3,4\\5,1,2,3,4\end{pmatrix} = 0$$

so

$$I_n\left[G\begin{pmatrix}\ell,1,2,3,4\\5,1,2,3,4\end{pmatrix}\right] = 0\,, \qquad (n \geq 6)$$

$$
\begin{aligned}
G\begin{pmatrix} \ell, 1, 2, 3, 4 \\ 5, 1, 2, 3, 4 \end{pmatrix} \quad = \quad & -\ell^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 5, 2, 3, 4 \end{pmatrix} + (\ell - K_1)^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 5, K_{12}, 3, 4 \end{pmatrix} \\
& -(\ell - K_{12})^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 5, 1, K_{23}, 4 \end{pmatrix} + (\ell - K_{123})^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 5, 1, 2, K_{34} \end{pmatrix} \\
& +(\ell - K_{1234})^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, K_{45} \end{pmatrix} - (\ell - K_{12345})^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} \\
& -K_1^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 5, K_{12}, 3, 4 \end{pmatrix} + K_{12}^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 5, 1, K_{23}, 4 \end{pmatrix} - K_{123}^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 5, 1, 2, K_{34} \end{pmatrix} \\
& -K_{1234}^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, K_{45} \end{pmatrix} + K_{12345}^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
I_n(K_1,\ldots,K_n) \quad &= \quad c_1 I_{n-1}(K_{n1},K_2,\ldots,K_{n-1}) + c_2 I_{n-1}(K_{12},K_3,\ldots,K_n) \\
&\quad + c_3 I_{n-1}(K_1,K_{23},K_4,\ldots,K_n) + c_4 I_{n-1}(K_1,K_2,K_{34},K_5,\ldots,K_n) \\
&\quad + c_5 I_{n-1}(K_1,\ldots,K_{45},\ldots,K_n) + c_6 I_{n-1}(K_1,\ldots,K_{56},\ldots,K_n)
\end{aligned}
$$

$$
\begin{aligned}
e.g.\quad c_1 \quad &= \quad \frac{1}{c_0}\, G\begin{pmatrix}1,2,3,4\\5,2,3,4\end{pmatrix}, c_2 = \ldots \\
c_0 \quad &= \quad -K_1^2\, G\begin{pmatrix}1,2,3,4\\5,K_{12},3,4\end{pmatrix} + K_{12}^2\, G\begin{pmatrix}1,2,3,4\\5,1,K_{23},4\end{pmatrix} - K_{123}^2\, G\begin{pmatrix}1,2,3,4\\5,1,2,K_{34}\end{pmatrix} \\
&\quad -K_{1234}^2\, G\begin{pmatrix}1,2,3,4\\1,2,3,K_{45}\end{pmatrix} + K_{12345}^2\, G\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix},
\end{aligned}
$$

Similarly, at two loops:

$$
\begin{aligned}
&P_{n_1,n_2}(K_1,\ldots,K_{n_1+n_2+2}) \quad = \\
&c_1 P_{n_1-1,n_2}(K_2,\ldots,K_{(n_1+n_2+2)1}) + c_2 P_{n_1-1,n_2}(K_{12},K_3,\ldots,K_{n_1+n_2+2}) \\
&+c_3 P_{n_1-1,n_2}(K_1,K_{23},K_4,\ldots,K_{n_1+n_2+2}) + c_4 P_{n_1-1,n_2}(K_1,K_2,K_{34},K_5,\ldots,K_{n_1+n_2+2}) \\
&+c_5 P_{n_1-1,n_2}(K_1,\ldots,K_{45},\ldots,K_{n_1+n_2+2}) + c_6 P_{n_1-1,n_2}(K_1,\ldots,K_{56},\ldots,K_{n_1+n_2+2}),
\end{aligned}
$$

We arrived at: P_{n_1,n_2} with $n_2 \le n_1 \le 4$

III. Truly Irreducible Numerators and IBPs.

Avoiding dotted propagators.

For $P_{n_1<4, n_2\leq n_1}$, which can still include truly irreducible numerators, the IBP machinery has to be used.

As already discussed, we want to avoid simultanously appearance of doubled propagators in the basis.

$$\int \frac{d^d\ell_1}{(2\pi)^d} \int \frac{d^d\ell_2}{(2\pi)^d} \frac{\partial}{\partial \ell_{\mu j}} \frac{v^\mu}{D(\ell_1,\ell_2,\{K_i\})},$$

$$\frac{\partial}{\partial \ell_\mu} \frac{1}{(\ell-K)^2} = 2\frac{(\ell-K)^\mu}{[(\ell-K)^2]^2}$$

First idea: we can choose vectors whose dot product with the numerator resulting from differentiating any propagator vanishes

$$v \cdot (\ell - K) = 0$$

However, it is a too strong constraint, it is sufficient to require that

$$v \cdot (\ell - K) \propto (\ell - K)^2$$

We impose this constraint for every propagator ($\sigma_j = \pm 1, 0$)

$$\left[\sigma_{j1} v_1 + \sigma_{j2} v_2\right] \cdot (\sigma_{j1}\ell_1 + \sigma_{j2}\ell_2 - K_j) + u_j \, (\sigma_{j1}\ell_1 + \sigma_{j2}\ell_2 - K_j)^2 = 0$$

$$u_j = Polyn\{\ell \cdot b\}$$

IBP-generating vectors

$$\left[\sigma_{j1} v_1 + \sigma_{j2} v_2\right] \cdot (\sigma_{j1}\ell_1 + \sigma_{j2}\ell_2 - K_j) + u_j (\sigma_{j1}\ell_1 + \sigma_{j2}\ell_2 - K_j)^2 = 0$$

$$v_i^\mu = c_i^{(\ell_1)} \ell_1^\mu + c_i^{(\ell_2)} \ell_2^\mu + \sum_{b \in B} c_i^{(b)} b^\mu$$

Each of the coefficients $c_i^{(x)}$ is again a polynomial in the various independent Lorentz invariants
$V = \{\ell_1^2, \ell_1 \cdot \ell_2, \ell_2^2, \{\ell_1 \cdot b\}_{b \in B}, \{\ell_2 \cdot b\}_{b \in B}, s_{12}\}$, e.g. for dim. 2

$$c_i^{(p)} = c_{i,1}^{(p)} s_{12} + \sum_{b \in B} c_{i,b1}^{(p)} \ell_1 \cdot b + \sum_{b \in B} c_{i,b2}^{(p)} \ell_2 \cdot b + c_{i,2}^{(p)} \ell_1^2 + c_{i,3}^{(p)} \ell_1 \cdot \ell_2 + c_{i,4}^{(p)} \ell_2^2$$

where $c_{i,1}^{(p)}$ depends on $\chi_{ij} = \frac{s_{ij}}{s_{12}}, \chi_{i\cdots j} = \frac{s_{i\cdots j}}{s_{12}}, \mu_i = \frac{m_i^2}{s_{12}}$, We can assemble the set of equations into a single matrix equation

$$\tilde{c}E = 0$$

where $\tilde{c}$ (rows) gathers all coefficients $(c_1^{\ell_1}, ..., c_1^{b_4}, c_2^{\ell_1}, ..., c_2^{b_4}, u_1, ..., u_n)$ and E is $(2n_B + 4 + n_d) \times n_d$ matrix, which depends on chosen topology [propagators]
For the planar double box

$$v_i^\mu = c_i^{(\ell_1)} \ell_1^\mu + c_i^{(\ell_2)} \ell_2^\mu + c_i^{(1)} k_1^\mu + c_i^{(2)} k_2^\mu + c_i^{(4)} k_4^\mu$$

where e.g.

$$c^{(\ell_1)}(\underbrace{\{\ell_1^2\,, \ell_1 \cdot \ell_2, \ell_2^2\,, \ell_1 \cdot k_1\,, \ell_1 \cdot k_2\,, \ell_1 \cdot k_4\,, \ell_2 \cdot k_1\,, \ell_2 \cdot k_3\,, \ell_2 \cdot k_4\,, s_{12}\}}_{symbols})$$

vector:

$$\tilde{c} = \left(c_1^{(\ell_1)}\, c_1^{(\ell_2)}\, c_1^{(1)}\, c_1^{(2)}\, c_1^{(4)}\, c_2^{(\ell_1)}\, c_2^{(\ell_2)}\, c_2^{(1)}\, c_2^{(2)}\, c_2^{(4)}\, u_{1\ldots7}\right)$$

$$
E=8\left(\begin{array}{cccc}
\ell_1^2 & -k_1\cdot\ell_1+\ell_1^2 & -k_1\cdot\ell_1-k_2\cdot\ell_1+\ell_1^2 & 0 \\
\ell_1\cdot\ell_2 & -k_1\cdot\ell_2+\ell_1\cdot\ell_2 & k_3\cdot\ell_2+k_4\cdot\ell_2+\ell_1\cdot\ell_2 & 0 \\
k_1\cdot\ell_1 & k_1\cdot\ell_1 & k_1\cdot\ell_1-s_{12}/2 & 0 \\
k_2\cdot\ell_1 & k_2\cdot\ell_1-s_{12}/2 & k_2\cdot\ell_1-s_{12}/2 & 0 \\
k_4\cdot\ell_1 & k_4\cdot\ell_1-\chi_{14}s_{12}/2 & k_4\cdot\ell_1+s_{12}/2 & 0 \\
0 & 0 & 0 & \ell_1\cdot\ell_2 \\
0 & 0 & 0 & \ell_2^2 \\
0 & 0 & 0 & k_1\cdot\ell_2 \\
0 & 0 & 0 & -k_1\cdot\ell_2-k_3\cdot\ell_2-k_4\cdot\ell_2 \\
0 & 0 & 0 & k_4\cdot\ell_2 \\
\ell_1^2/4 & 0 & 0 & 0 \\
0 & \ell_1^2/4-k_1\cdot\ell_1/2 & 0 & 0 \\
0 & 0 & \ell_1^2/4+s_{12}/4-k_1\cdot\ell_1/2-k_2\cdot\ell_1/2 & 0 \\
0 & 0 & 0 & \ell_2^2/4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right.
$$

$$
\left.\begin{array}{ccc}
0 & 0 & \ell_1^2+\ell_1\cdot\ell_2 \\
0 & 0 & \ell_1\cdot\ell_2+\ell_2^2 \\
0 & 0 & k_1\cdot\ell_1+k_1\cdot\ell_2 \\
0 & 0 & k_2\cdot\ell_1-k_1\cdot\ell_2-k_3\cdot\ell_2-k_4\cdot\ell_2 \\
0 & 0 & k_4\cdot\ell_1+k_4\cdot\ell_2 \\
-k_4\cdot\ell_1+\ell_1\cdot\ell_2 & k_1\cdot\ell_1+k_2\cdot\ell_1+\ell_1\cdot\ell_2 & \ell_1^2+\ell_1\cdot\ell_2 \\
-k_4\cdot\ell_2+\ell_2^2 & -k_3\cdot\ell_2-k_4\cdot\ell_2+\ell_2^2 & \ell_1\cdot\ell_2+\ell_2^2 \\
k_1\cdot\ell_2-\chi_{14}s_{12}/2 & k_1\cdot\ell_2+s_{12}/2 & k_1\cdot\ell_1+k_1\cdot\ell_2 \\
(1+\chi_{14})s_{12}/2-k_1\cdot\ell_2-k_3\cdot\ell_2-k_4\cdot\ell_2 & s_{12}/2-k_1\cdot\ell_2-k_3\cdot\ell_2-k_4\cdot\ell_2 & k_2\cdot\ell_1-k_1\cdot\ell_2-k_3\cdot\ell_2-k_4\cdot\ell_2 \\
k_4\cdot\ell_2 & k_4\cdot\ell_2-s_{12}/2 & k_4\cdot\ell_1+k_4\cdot\ell_2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-k_4\cdot\ell_2/2+\ell_2^2/4 & 0 & 0 \\
0 & \ell_2^2/4+s_{12}/4-k_3\cdot\ell_2/2-k_4\cdot\ell_2/2 & 0 \\
0 & 0 & \ell_1^2/4+\ell_1\cdot\ell_2/2+\ell_2^2/4
\end{array}\right).
$$

Coefficients found (syzygies) using Gröbner basis (another algorithm by Robert Schabinger)
In this way, e.g. for $P^{**}_{2,2}$, the IBP generating vectors (of dim. 2) are:

$$\begin{aligned} v_{1;1} &= -2(k_4\cdot\ell_1 + \ell_1^2)k_1^\mu - \ell_1^2 k_2^\mu + (2k_1\cdot\ell_1 - \ell_1^2)k_4^\mu \\ &+ (4k_1\cdot\ell_1 + 2k_2\cdot\ell_1 + 2k_4\cdot\ell_1 - s_{12})\ell_1^\mu\,, \\ v_{1;2} &= 2(\ell_2^2 - k_4\cdot\ell_2)k_1^\mu + \ell_2^2 k_2^\mu + (2k_1\cdot\ell_2 + \ell_2^2)k_4^\mu \\ &+ (2k_3\cdot\ell_2 - 2k_1\cdot\ell_2 - s_{12})\ell_2^\mu\,; \end{aligned}$$

There are another two pairs of solutions of dim. 4.

$$
\begin{aligned}
&\frac{\partial}{\partial \ell_1^\mu}\left[\frac{v_{1;1}}{\ell_1^2(\ell_1-K_{12})^2(\ell_1+\ell_2)^2\ell_2^2(\ell_1-k_1)^2(\ell_2-K_{34})^2}\right] \\
= \quad & \frac{1}{\ell_1^2(\ell_1-K_{12})^2(\ell_1+\ell_2)^2\ell_2^2(\ell_1-k_1)^2(\ell_2-K_{34})^2} \\
\times \quad & (2dk_1\cdot\ell_1-2k_3\cdot\ell_1-s_{12})-(8k_1\cdot\ell_1-8k_3\cdot\ell_1-4s_{12}+s_{14}) \\
+ \quad & \frac{4}{(\ell_1+\ell_2)^2}(2k_1\cdot\ell_2k_1\cdot\ell_1-2k_1\cdot\ell_1k_4\cdot\ell_2+k_1\cdot\ell_2\ell_1^2 \\
- \quad & k_3\cdot\ell_2\ell_1^2+2k_1\cdot\ell_1\ell_2^2+k_2\cdot\ell_1\ell_2^2+k_4\cdot\ell_1\ell_2^2+(\ell_1^2-\ell_2^2)s_{12}/2)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial \ell_1^\mu}\left[\frac{v_{1;1}}{\ell_1^2(\ell_1-K_{12})^2(\ell_1+\ell_2)^2\ell_2^2(\ell_1-k_1)^2(\ell_2-K_{34})^2}\right]\\
= &\frac{1}{\ell_1^2(\ell_1-K_{12})^2(\ell_1+\ell_2)^2\ell_2^2(\ell_1-k_1)^2(\ell_2-K_{34})^2}\\
\times &(2dk_1\cdot\ell_1-2k_3\cdot\ell_1-s_{12})-(8k_1\cdot\ell_1-8k_3\cdot\ell_1-4s_{12}+s_{14})\\
+ &\frac{4}{(\ell_1+\ell_2)^2}(2k_1\cdot\ell_2k_1\cdot\ell_1-2k_1\cdot\ell_1k_4\cdot\ell_2+k_1\cdot\ell_2\ell_1^2\\
- &k_3\cdot\ell_2\ell_1^2+2k_1\cdot\ell_1\ell_2^2+k_2\cdot\ell_1\ell_2^2+k_4\cdot\ell_1\ell_2^2+(\ell_1^2-\ell_2^2)s_{12}/2))
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial \ell_2^\mu}\left[\frac{v_{1;2}}{\ell_1^2(\ell_1-K_{12})^2(\ell_1+\ell_2)^2\ell_2^2(\ell_1-k_1)^2(\ell_2-K_{34})^2}\right]\\
= &\frac{1}{\ell_1^2(\ell_1-K_{12})^2(\ell_1+\ell_2)^2\ell_2^2(\ell_1-k_1)^2(\ell_2-K_{34})^2}\\
\times &(2dk_1\cdot\ell_2-2k_3\cdot\ell_2+s_{12})-(8k_1\cdot\ell_2-8k_3\cdot\ell_2+4s_{12}+s_{14})\\
- &\frac{4}{(\ell_1+\ell_2)^2}(2k_1\cdot\ell_2k_3\cdot\ell_2-2k_1\cdot\ell_1k_4\cdot\ell_2+k_1\cdot\ell_2\ell_1^2\\
- &k_3\cdot\ell_2\ell_1^2+2k_1\cdot\ell_1\ell_2^2+k_2\cdot\ell_1\ell_2^2+k_4\cdot\ell_1\ell_2^2+(\ell_1^2-\ell_2^2)s_{12}/2))
\end{aligned}
$$

Final operations

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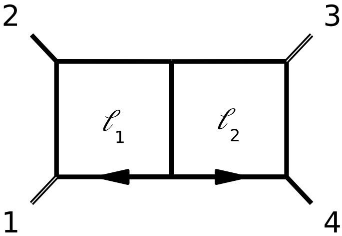
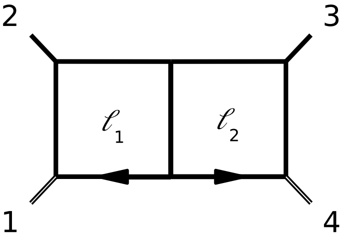
- Vectors will depend on number and pattern of external masses
- Number of truly-irreducible integrals ("master integrals") also depends on number and pattern of external masses

Some solutions (in d dimensions)

- Massless, one-mass, diagonal two-mass, long-side two-mass double boxes (here five IBP-generating vectors of dim. 4): two integrals

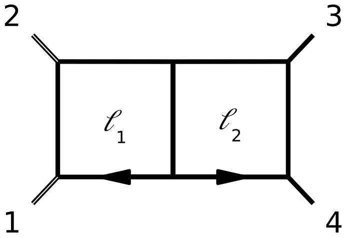


$P^{**}[1], \quad P^{**}_{2,2}[k_1 \cdot \ell_2]$



Some solutions (in d dimensions)

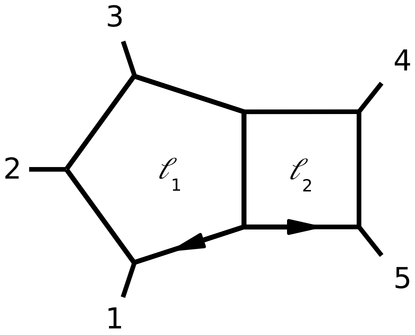
- Short-side two-mass (four IBP-generating vectors, of dim. 4), three-mass double boxes: 3 IBP-gen. vectors of dim 4, 2 of dim 6,fixed numerically): three integrals



$$P^{**}[1], \quad P^{**}_{2,2}[k_1 \cdot \ell_2], \quad P^{**}_{2,2}[k_4 \cdot \ell_2]$$

Some solutions (in d dimensions)

- Four-mass double box: four integrals: e.g. $P^{**}_{2,2}[1]\,, P^{**}_{2,2}[k_1\cdot\ell_2]\,, P^{**}_{2,2}[k_4\cdot\ell_1]\,, P^{**}_{2,2}[k_1\cdot\ell_2 k_4\cdot\ell_1]$
- massless pentabox (six IBP-gen. vectors of dim. 4, three of dim. 6, fixed numerically): three integrals



$$P^{**}_{3,2}[1], \quad P^{**}_{3,2}[k_1\cdot\ell_2], \quad P^{**}_{3,2}[k_5\cdot\ell_1]$$

d-dimensional and 4-dim basis, exploring Gram determinants. I. Pentagons

At one loop, in d dimensions, they are independent basis elements.
Expanding in $d = 4 - 2\epsilon$, only the $\mathcal{O}(\epsilon)$ terms are independent, so that the integral can be eliminated from the basis.

$$G\begin{pmatrix} \ell_1, 1, 2, 3, 4 \\ \ell_1, 1, 2, 3, 4 \end{pmatrix} = \mathcal{O}(\epsilon)$$

then also

$$I_5[G(\ell, 1, 2, 3, 4)]\mathcal{O}(\epsilon)$$

[Integral itself is UV finite by power counting and vanishes in all regions that give rise to soft and collinear singularities, where also Gram determinant vanishes]

$$G\begin{pmatrix} \ell_1, 1, 2, 3, 4 \\ \ell_1, 1, 2, 3, 4 \end{pmatrix} =$$

$$\underbrace{d_0}_{pentagon} + \underbrace{d_1 \ell^2 + d_2(\ell - K_1)^2 + d_3(\ell - K_{12})^2 + d_4(\ell - K_{123})^2 + d_5(\ell - K_{1234})^2}_{boxes}$$

$$-\ell^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ \ell, 2, 3, 4 \end{pmatrix} + (\ell - K_1)^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ \ell, K_{12}, 3, 4 \end{pmatrix} - (\ell - K_{12})^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ \ell, 1, K_{23}, 4 \end{pmatrix}$$

$$+(\ell - K_{123})^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ \ell, 1, 2, K_{34} \end{pmatrix} - (\ell - K_{1234})^2\, G\begin{pmatrix} 1, 2, 3, 4 \\ \ell, 1, 2, 3 \end{pmatrix},$$

rest (two last rows) is proportional to odd powers of ℓ and vanishes in d-dimensions

Insert this into the numerator of a five-point integral to obtain a relation relating it to five box integrals, up to terms of $\mathcal{O}(\epsilon)$

Vanishing Gram determinants at two loops, example

For $P^{**}_{2,2}$ integrals we haven't found any useful, additional relations.

Pentagonbox: $3 \to 1$ MIs.

Two additional relations from considering the following two integrals:

$$P^{**}_{3,2}\left[G\begin{pmatrix}\ell_1,1,2,3,5\\ \ell_2,1,2,3,5\end{pmatrix}\right] \text{ and } P^{**}_{3,2}\left[k_5\cdot\ell_1\, G\begin{pmatrix}\ell_1,1,2,3,5\\ \ell_2,1,2,3,5\end{pmatrix}\right]$$

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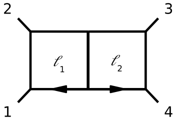
These kind of Gram determinants all vanish when either loop momentum approaches a potential (on-shell) collinear or soft configuration, thereby removing the corresponding divergences from the integral, and rendering it finite. In addition, the Gram determinants vanish when both loop momenta are four-dimensional, so that the integrals are of $\mathcal{O}(\epsilon)$.

Procedure

We first solve all d-dimensional IBP equations, and use the solutions of those equations (in analytical or numerical form) to reduce the integrals obtained from inserting Gram determinants into the numerator; this will provide additional identities to $\mathcal{O}(\epsilon^0)$ between the independent master integrals.

- $\mathcal{O}(\epsilon)$ Gram dets give no new equations for double boxes $P^{**}_{2,2}$



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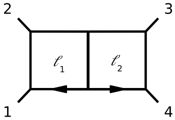
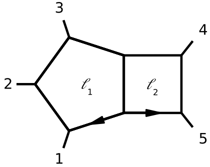


- Reduce three integrals for the pentabox $P^{**}_{3,2}$ to one



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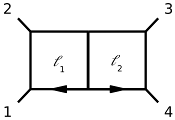

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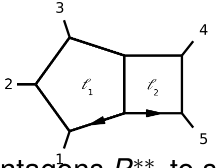
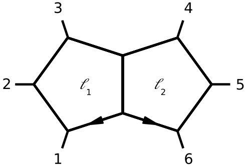


- Reduce all double pentagons $P^{**}_{3,3}$ to simpler integrals



Application: maximal generalized unitarity approach[1]

Kosower, Larsen, PRD2012,
Caron-Huot, Larsen, JHEP2012,
Johansson, Kosower, Larsen, 1208.1754

Basis is needed to ensure unique solutions to the coefficients of the MIs.

[1] different approaches based on OPP generalization by Ossola, Mastrolia, see also Simon Badger's talk

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- There are many places for improvements and new ideas
- One example: chiral integrals in any gauge theory (to build a basis with as many IR finite MIs as possible)

$$A^{(2)} = \sum_i c_i(\epsilon)\,\mathrm{Int}_i + \mathit{Rational}$$

Chiral double boxes as basis at two loops (Caron-Huot, Larsen, 1205.0801)

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- Knowledge of an integral basis plays an important role in modern unitarity calculations
- Two kinds of bases
- To be done: non-planar topologies; massive propagators
- To be done: maximally generalized unitarity cuts with (ϵ) (and higher terms), master contours: 1208.1754
- Beyond two loops? (Zhang, Badger)

Backup slides

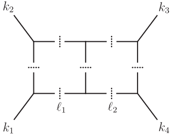
Kosower, Larsen, PRD2012,
Caron-Huot, Larsen, JHEP2012,
Johansson, Kosower, Larsen, 1208.1754

$$\mathbb{R}^{1,3} \to \mathbb{C}^4,$$

$$\int \frac{d^4\ell}{(2\pi)^4}\, \mathrm{N}_F \delta\big(\ell^2\big)\delta\big((\ell-k_1)^2\big)\delta\big((\ell-k_1-k_2)^2\big)\delta\big((\ell+k_4)^2\big) \equiv$$
$$\oint_{T_Q} \frac{d^4\ell}{(2\pi)^4}\, \frac{\mathrm{N}_F(\ell,\cdots)}{\ell^2(\ell-k_1)^2(\ell-k_1-k_2)^2(\ell+k_4)^2}\,,$$

T_Q: four-torus encircling the solutions to the on-shell eqns.

E.g. at two loops



$$\int \frac{d^4\ell_1}{(2\pi)^4}\,\frac{d^4\ell_2}{(2\pi)^4}\,\delta\big(\ell_1^2\big)\delta\big((\ell_1-k_1)^2\big)\delta\big((\ell_1-K_{12})^2\big)\delta\big((\ell_1+\ell_2)^2\big)$$
$$\times\delta\big(\ell_2^2\big)\delta\big((\ell_2-k_4)^2\big)\delta\big((\ell_2-K_{34})^2\big)\,,$$

On-shell constraints:

$$\ell_1^2=0\,(\ell_1-k_1)^2=0\,,(\ell_1-K_{12})^2=0\,\ell_2^2=0\,,(\ell_2-k_4)^2=0\,,$$
$$(\ell_2-K_{34})^2=0\,,(\ell_1+\ell_2)^2=0\,.$$

- On-shell constraints allow by choosing the integration contours to encircle poles unique to each MI in the basis decomposition, their coeff. can be extracted, so amplitude can be determined

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- Comparing # constraints (cuts) with dimensionality of the integral: 1 degree of freedom remains (not so at 1-loop), there is a Jacobian arising from solving the δ-functions which helps to identify poles at specific locations
- Applied in the recent paper 1208.1754: uniqueness of contours on Riemann spheres