Inflationary Correlation Functions without Infrared Divergences

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**IR divergences in $\delta N$ - formalism**

Starobinsky '85, Sasaki/Stewart '95, Wands/Malik/Lyth/Liddle '00
Lyth/Malik/Sasaki '04, Lyth/Rodriguez '05

- Consider some late constant energy-density surface (reheating surface)

$$ ds^2 = e^{2\zeta(x)} dx \, dx $$

- The curvature perturbation $\zeta$ can be interpreted as perturbation in the local number of e-foldings $N(\phi(x))$

$$ \zeta(x) = \delta N(x) = N(\phi_0 + \delta\phi(x)) - N(\phi_0) $$

$$ = N_\phi \delta\phi(x) + \frac{1}{2} N_{\phi\phi} \delta\phi(x)^2 + \ldots $$

- Yields 2-point correlator in Fourier space:

$$ \langle \zeta_k \zeta_p \rangle \sim N_\phi^2 \langle \delta\phi_k \delta\phi_p \rangle + \frac{1}{4} N_{\phi\phi}^2 \langle (\delta\phi^2)_k (\delta\phi^2)_p \rangle + N_\phi N_{\phi\phi\phi} \ldots $$
Infrared Divergences in Inflation

Focus on the second term

\[ N_{\phi\phi}^2 \langle (\delta \phi^2)_k (\delta \phi^2)_p \rangle \sim N_{\phi\phi}^2 \int_{q,l} \langle \delta \phi_q \delta \phi_{k-q} \delta \phi_l \delta \phi_{p-l} \rangle . \]

use

\[ \delta \phi_k \sim \frac{H}{k^{3/2}} a_k \]

to yield leading-log contribution from \( q, l \ll k, p \):

\[ N_{\phi\phi}^2 H^4 \int \frac{d^3 q}{q^3} \sim N_{\phi\phi}^2 H^4 \ln(kL) , \]

with cut-off \( 1/L \).

\( L \approx \) side-length of observable patch

Lyth '07
Origin of IR effects is the dependence of $N(\phi(x))$ on $\delta\phi_q$ with $q \ll k$.

A similar dependence appears in the Hubble function $H(\phi(x))$.

The Hubble scale $H(\phi(x))$ should be modified analogously!

This has not been taken into account in (higher-order) $\delta N$-calculations.
Fluctuations of the Hubble scale $H$

- Define local background of the scalar field:

$$\delta \bar{\phi}(x) = \int \limits_{\substack{q\ll k}} e^{-iqx} \delta \phi_q$$

- Hubble function should be evaluated at horizon exit of mode $k$.
  Local scalar field value at horizon exit: $\phi_0 + \delta \bar{\phi}(x)$

$$\delta \phi(x) \sim \int_k \frac{e^{-ikx}}{k^{3/2}} H(\phi_0 + \delta \bar{\phi}(x)) a_k$$

- Using this modified $\delta \phi$ in $\zeta = N(\phi_0 + \delta \phi) - N(\phi_0)$ and expanding in both, $\delta \phi$ and $\delta \bar{\phi}$, yields

$$\langle \zeta_k \zeta_p \rangle \sim \frac{\delta^{(3)}(k + p)}{k^3} \left[ N^2_\phi H^2 + \frac{1}{2} (H^2 \ln kL) \frac{d^2}{d\phi^2}(N^2_\phi H^2) + \ldots \right]$$
Geometry of the Reheating Surface

With $H^2 \ln kL \sim \langle \delta \phi^2 \rangle$ and $\mathcal{P}_\zeta^{(0)} \sim N^2_\phi H^2$ this gives

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta^{(0)}(k) + \frac{1}{2} \langle \delta \phi^2 \rangle \frac{d^2}{d\phi^2} \mathcal{P}_\zeta^{(0)} \ldots$$

Replace $\delta \phi$ by $\bar{\zeta} \sim \int_{q \ll k} e^{-iqx} \zeta_q$ and use $\frac{d}{d\phi} = N_\phi \frac{d}{d\ln k}$ to write this as

$$\mathcal{P}_\zeta(k) = \left(1 - \langle \bar{\zeta} \rangle \frac{d}{d\ln k} + \frac{1}{2} \langle \bar{\zeta}^2 \rangle \frac{d^2}{(d\ln k)^2}\right) \mathcal{P}_\zeta^{(0)}(k)$$

see also Giddings/Sloth ’10

Obviously, these are the first terms of the Taylor expansion of

$$\mathcal{P}_\zeta(k) = \langle \mathcal{P}_\zeta^{(0)}(ke^{-\bar{\zeta}}) \rangle$$

where $\langle \ldots \rangle$ is the average in $\bar{\zeta}$ over a box of size $L$
Infrared-safe Correlator

According to its definition

$$\langle \zeta_k \zeta_p \rangle \sim \frac{\delta(k + p)}{k^3} \mathcal{P}_\zeta(k)$$

the power spectrum may be written as

$$\mathcal{P}_\zeta(k) \sim k^3 \int_y e^{iky} \langle \zeta(x) \zeta(x + y) \rangle$$

The average $$\langle \ldots \rangle$$ is over pairs of points separated by a coordinate vector $$y$$.

In other words, we are averaging over the location $$x$$ of such pairs.
Infrared-safe Correlator

At each location there is a local background due to long-wavelength modes

\[ \bar{\zeta}(x) = \int_{q \ll k \sim 1/y} dq \; e^{-iqx} \zeta_q \]

The physical separation \( z = e^{\bar{\zeta}(x)} y \) of the pairs is \( x \)-dependent and therefore different for each pair

Moreover, this mismatch between \( y \) and the true distance \( z \) grows with \( L \). More precise: \( \langle \bar{\zeta}^2 \rangle \sim \mathcal{P}_\zeta^{(0)} \ln(kL) \).

⇒ Select pairs separated by same physical distance \( z \) \( \Leftarrow \)
**IR-safe Correlator**

- The correlator

\[ \langle \zeta(x) \, \zeta(x + e^{-\zeta(x)} \, z) \rangle \]

averages over pairs of points **ALL** separated by the same physical distance \( z \).

(Note: Now the coordinate vector \( y(x) = e^{-\zeta(x)} \, z \) is different for each pair.)

- The \( z \)-dependence of this correlator is then a **background-independent** and, hence, **IR-safe** object.

- Consequently, the Fourier transform is the desired **IR-safe** power spectrum

\[ \mathcal{P}_{\zeta}^{(0)}(k) \sim k^3 \int_{z} e^{ikz} \langle \zeta(x) \, \zeta(x + ze^{-\zeta}) \rangle \]

related to Urakawa/Tanaka ’10 ?

see also Giddings/Sloth ’11
The conventional IR-sensitive power spectrum is obtained via

\[ P_\zeta(k) \sim k^3 \int_y e^{i ky} \langle \zeta(x) \zeta(x + y) \rangle \]

\[ \sim k^3 \int_y e^{i ky} \langle \zeta(x) \zeta(x + (y e^{\bar{\zeta}}) e^{-\bar{\zeta}}) \rangle \]

\[ \sim \langle (ke^{-\bar{\zeta}})^3 \int_z \exp(ike^{-\bar{\zeta}} z) \zeta(x) \zeta(x + ze^{-\bar{\zeta}}) \rangle \]

\[ \sim \langle P_\zeta^{(0)}(ke^{-\bar{\zeta}}) \rangle \]

i.e. in a resummed, all-orders form.

This is in absolute agreement with the previous result!!!
Tensor modes $\gamma$

- Include tensor modes in the metric ($\gamma$: transverse, traceless)

$$ds^2 = e^{2\zeta(x)} \left( e^{\gamma(x)} \right)_{ij} dx^i dx^j$$

- Generalization yields ($\hat{k}$: unit vector in $k$-direction)

$$\mathcal{P}_\zeta(k) = \langle (e^{-\tilde{\gamma}/2} \hat{k})^{-3} \mathcal{P}_\zeta^{(0)} (e^{-\tilde{\zeta}-\tilde{\gamma}/2} k) \rangle$$

- Expanding to first non-trivial order yields:

$$\mathcal{P}_\zeta(k) = \left( 1 - \frac{1}{20} \langle \text{tr} \tilde{\gamma}^2 \rangle \frac{d}{d \ln k} + \frac{1}{2} \langle \tilde{\zeta}^2 \rangle \frac{d^2}{(d \ln k)^2} \right) \mathcal{P}_\zeta^{(0)}(k)$$

in agreement with Giddings/Sloth ’10
Conclusions

- A wide class of inflationary IR divergences comes from long-wavelength background modes.

- This can be seen in an (appropriately modified) $\delta N$ formalism as well as from the ‘geometry of the reheating surface’.

- It is possible to define IR-safe correlators.

- Conventional correlators (with their IR-dependence) can be easily calculated.

- Inclusion of tensor modes is straightforward.

Beyond this talk: $n$-point correlators, all-orders evaluation, convergence of perturbative expansion