Gauge theories, graph polynomials and graph homologies

Dirk Kreimer, Humboldt University, Berlin

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based on joint work with F.Brown, M.Sars, W.van Suijlekom, K.Yeats

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Hopf algebra of graphs $H = \mathbb{Q}1 \oplus \bigoplus_{j=1}^{\infty} H^j$

- The coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \overbrace{\sum_{\gamma = \cup_i \gamma_i, \omega_4(\gamma_i) \geq 0} \gamma \otimes \Gamma/\gamma}^{\Delta'(\Gamma)} \tag{1}$$

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- The antipode

$$S(\Gamma) = -\Gamma - \sum S(\gamma)\Gamma/\gamma = -m(S \otimes \mathrm{P})\Delta \quad (2)$$

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$$G_V^H \ni \Phi \Leftrightarrow \Phi : H \to V, \Phi(h_1 \cup h_2) = \Phi(h_1)\Phi(h_2) \tag{3}$$

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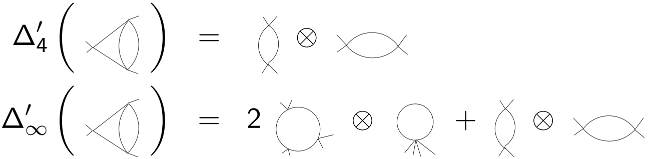
- The renormalized Feynman rules

$$\Phi_R = m(S_R^{\Phi} \otimes \Phi)\Delta \qquad (5)$$

BCFW and the core Hopf algebra

- a sequence of quotient Hopf algebras by looking at short distance singularities in $2k$ dimensions

$$H_0 \subset H_2 \subset H_4 \subset \cdots \subset H_{2k} \subset \cdots H_\infty \tag{6}$$



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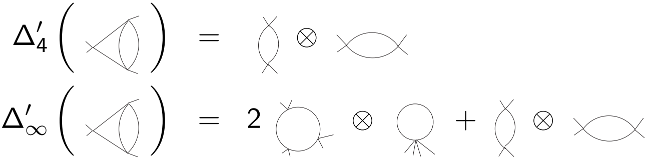

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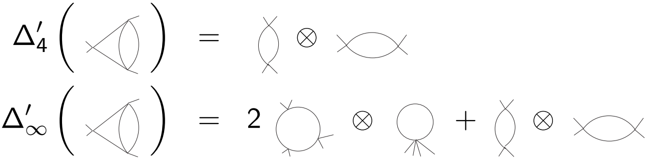

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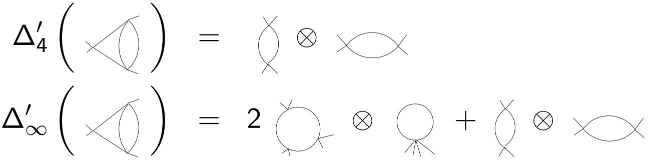

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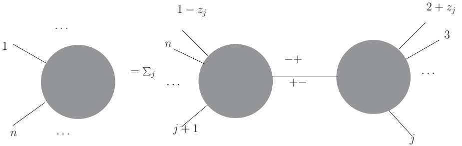

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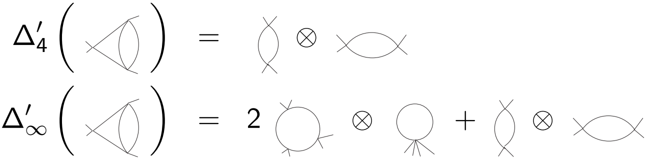

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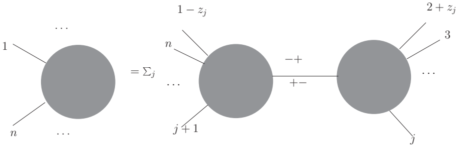

- KLT relations or kinematic STU $\leftrightarrow$ co-ideal respected

Kinematics and Cohomology

- Exact co-cycles
$$[B_+^{r,j}] = B_+^{r;j} + b\phi^{r;j} \tag{8}$$
with $\phi^{r;j} : H \to \mathbb{C}$

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- Exact co-cycles
$$[B_+^{r;j}] = B_+^{r;j} + b\phi^{r;j} \tag{8}$$
with $\phi^{r;j} : H \to \mathbb{C}$
- Variation of momenta
$$G^R(\{g\}, \ln s, \{\Theta\}) = 1 \pm \Phi^R_{\ln s, \{\Theta\}}(X^r(\{g\})) \tag{9}$$
with $X^r = 1 \pm \sum_j g^j B_+^{r;j}(X^r Q^j(g))$, $bB_+^{r;j} = 0$. Note:
$\beta(g) = 0 \Leftrightarrow Q(g) = \mathrm{constant}$.
Then, for kinematic renormalization schemes:
$\{\Theta\} \to \{\Theta'\} \Leftrightarrow B_+^{r;j} \to B_+^{r;j} + b\phi^{r;j}$.
$\Phi^R_{L_1+L_2,\{\Theta\}} = \Phi^R_{L_1,\{\Theta\}} \star \Phi^R_{L_2,\{\Theta\}}$.
$\Phi^R(\ln s, \{\Theta\}, \{\Theta_0\}) = \Phi^{-1}_{\mathrm{fin}}(\{\Theta_0\} \star \Phi^R_{1-\mathrm{scale}}(\ln s) \star \Phi_{\mathrm{fin}}(\{\Theta\})$.

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Graph Polynomials

- $$0 \to H^1(\Gamma) \to \mathbb{Q}^{E_\Gamma} \to \mathbb{Q}^{V_\Gamma,0} \to 0. \tag{10}$$

  $\{h_i\}$ basis of homology (loops!)

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- $$N := \begin{pmatrix} N_0 := \left(\sum_{e \in h_i \cap h_j} A_e\right)_{ij} \mathbb{I} & \sum_{e \in h_j} \mu_e A_e \\ \sum_{e \in h_j} \bar{\mu}_e A_e & \sum_{e \in \Gamma^{[1]}} \bar{\mu}_e \mu_e A_e \end{pmatrix}$$

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- $|N_0| = \psi(\Gamma)$,
  $|N| = \phi(\Gamma) := -\sum_{T_1\cup T_2} \sum_{e\notin T_1\cup T_2} (\sigma(e) q_e)^2 \prod_{e\notin T_1\cup T_2} A_e.$

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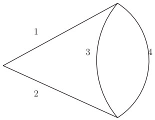
Example

$\Gamma =$ 

$$N_\Gamma = \begin{pmatrix} A_1 + A_2 + A_3 & A_1 + A_2 & A_1\mu_1 + A_2\mu_2 + A_3\mu_3 \\ A_1 + A_2 & A_1 + A_2 + A_4 & A_1\mu_1 + A_2\mu_2 + A_4\mu_4 \\ A_1\bar{\mu}_1 + A_2\bar{\mu}_2 + A_3\bar{\mu}_3 & A_1\bar{\mu}_1 + A_2\bar{\mu}_2 + A_4\bar{\mu}_4 & \sum_{i:=1}^4 A_i\bar{\mu}_i\mu_i \end{pmatrix}$$

$$\psi_\Gamma = (A_1 + A_2)(A_3 + A_4) + A_3A_4 = \sum_{\mathrm{sp.Tr.}\,T} \prod_{e\notin T} A_e$$

$$\phi_\Gamma = (A_3 + A_4)A_1A_2p_a^2 + A_2A_3A_4p_b^2 + A_1A_3A_4p_c^2 = \sum_{\mathrm{sp.2-Tr.}\,T_1\cup T_2} Q(T_1)\cdot Q(T_2) \prod_{e\notin T_1\cup T_2} A_e.$$

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The Feynman rules in projective space

First, $\phi_\Gamma \to \phi_\Gamma + \psi_\Gamma(\sum_e m_e^2 A_e)$.

$$\Phi_\Gamma^R(S, S_0, \{\Theta, \Theta_0\}) = \int_{\mathbb{P}^{E-1}(\mathbb{R}_+)} \overbrace{\sum_f}^{\text{forestsum}} (-1)^{|f|}$$

$$\frac{\ln \frac{\frac{S}{S_0}\phi_{\Gamma/f}\psi_f + \phi_f^0 \psi_{\Gamma/f}}{\phi^0_{\Gamma/f}\psi_f + \phi_f^0\psi_{\Gamma/f}}}{\psi^2_{\Gamma/f}\psi_f^2} \underbrace{\Omega_\Gamma}_{(E-1)-\text{form}}$$

Note: for 1-scale graphs, $\phi_\Gamma = \psi_\Gamma^\bullet$.

The polylog as a Hodge structure

Iterated integrals: obvious Hopf algebra structure

$$\begin{pmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2\pi i & 0 \\ -Li_2(z) & 2\pi i \ln(z) & (2\pi i)^2 \end{pmatrix} = (C_1, C_2, C_3) \qquad (11)$$

$$\mathrm{Var}(\Im Li_2(z) - \ln|z|\,\Im Li_1(z)) = 0 \qquad (12)$$

Hodge sructure from Hopf algebra structure: branch cut ambiguities columnwise

Griffith transversality $\Leftrightarrow$ differential equation

Limiting mixed Hodge structures

- Hopf algebra from flags

$$f := \gamma_1 \subset \gamma_2 \subset \ldots \subset \Gamma, \; \Delta'(\gamma_{i+1}/\gamma_i) = 0 \tag{13}$$

The set of all such flags $F_\Gamma \ni f$ determines Hopf algebra structure, $|F_\Gamma|$ is the length of the flag.

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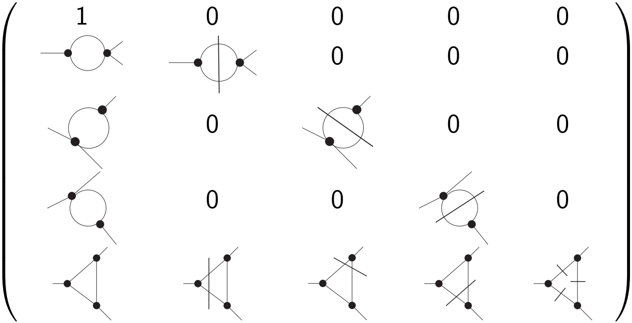
- It also determines a column vector $v = v(F_\Gamma)$ and a nilpotent matrix $(N) = (N(|F_\Gamma|))$, $(N)^{k+1} = 0$, $k = corad(\Gamma)$ such that

$$\lim_{t \to 0} (e^{-\ln t(N)})\Phi_R(v(F_\Gamma)) = (c_1^\Gamma(\Theta)\ln s, c_2^\Gamma(\Theta), c_k^\Gamma(\Theta)\ln^k s)^T \tag{14}$$

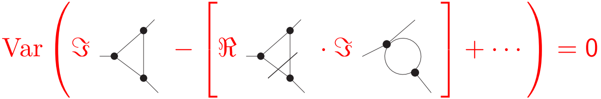
where k is determined from the co-radical filtration and t is a regulator say for the lower boundary in the parametric representation.

The Feynman graph as a Hodge structure

Hopf algebra structure as above



$= (C_1, C_2, C_3, C_4, C_5)$



Hodge structure: cut-reconstructability: from Hopf algebra structure:
branch cut ambiguities columnwise

Griffith transversality $\Leftrightarrow$ differential equation?

$\zeta(s_1, \cdots, s_k) = \sum_{n_i<n_{i+1}} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$

- counting over $\mathbb{Q}$

$$1 - \frac{x^3 y}{1-x^2} + \frac{x^{12} y^2 (1-y^2)}{(1-x^4)(1-x^6)} = \prod_{n \geq 3} \prod_{k \geq 1} (1 - x^n y^k)^{D_{n,k}} \quad (15)$$

$\rightarrow$ first irreducible MZV from planar graphs at 7 loops in scalar field theory (integrability???)

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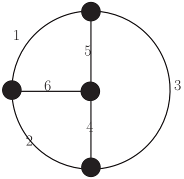
- When is a graph redicible to MZVs? Francis Brown: when it has vertex width three.
- Caution! Non-MZVs at eight loops from non-planar graphs, at nine loops from planar graphs ('A $K3$ in ϕ^4', Brown and Schnetz). Proof from counting points $[X_\Gamma]$ on graph hypersurfaces X_Γ over $\mathbb{F}_q$, defined by vanishing of the first Symanzik polynomial. If the graph gives a MZV, $[X_\Gamma]$ better is polynomial in the prime power $q = p^n$. Alas, it is not in general, with counterexamples relating graphs to elliptic curves with complex multiplication, and point-counting function a modular form.

Decomposing scales and angles

Consider

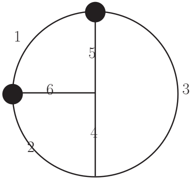
$\Gamma =$



, (16)

and

$\Gamma^2 =$



. (17)

We let $S = p_1^2 + p_2^2 + p_3^2 + 2p_1 \cdot p_2 + 2p_2 \cdot p_3 + 2p_3 \cdot p_1$ (which defines the variable angles $\Theta^{ij} = p_i \cdot p_j / S, \Theta^e = m_e^2/S$) and subtract symmetrically say at $S_0, \Theta_0^{ij} = \frac{1}{3}(4\delta_{ij} - 1)$ and $\Theta_0^e = m_e^2/S_0$, which specifies the fixed angles Θ_0.

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$$\Phi^R_\Gamma = \frac{\ln \frac{\frac{S}{S_0}\phi_\Gamma(\Theta)}{\phi_\Gamma(\Theta_0)}}{\psi^2_\Gamma}\Omega_\Gamma. \tag{18}$$

To find the desired decomposition, we use

$$\Delta^2(\Gamma) = \Gamma \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \Gamma. \tag{19}$$

We then have

$$\Phi^R_\Gamma = \Phi^{-1}_{\mathrm{fin}}(\Theta_0)(\Gamma) + \Phi^R_{1-\mathrm{s}}(S/S_0)(\Gamma) + \Phi_{\mathrm{fin}}(\Theta)(\Gamma). \tag{20}$$

We have

$$\Phi_{\mathrm{fin}}^{-1}(\Theta_0)(\Gamma) = -\frac{\ln\frac{\phi_\Gamma(\Theta_0)}{\psi_{\Gamma^2\bullet}}}{\psi_\Gamma^2}\Omega_\Gamma, \tag{21}$$

$$\Phi_{1-\mathrm{s}}^R(S/S_0)(\Gamma) = \frac{\ln\frac{S}{S_0}}{\psi_{\Gamma^2}^2}\Omega_\Gamma, \tag{22}$$

which integrates to the renormalized value $\Phi_{1-\mathrm{s}}^R(S/S_0)(\Gamma) = 6\zeta(3)\ln\frac{S}{S_0}$. Finally,

$$\Phi_{\mathrm{fin}}(\Theta)(\Gamma) = \frac{\ln\frac{\phi_\Gamma(\Theta)}{\psi_{\Gamma^2\bullet}}}{\psi_\Gamma^2}\Omega_\Gamma. \tag{23}$$

These integrands indeed all converge, which is synonymous for us to say that they can be integrated against $\mathbb{P}^{E-1}(\mathbb{R}_+)$.

From scalar 3-regular graphs to gauge theory amplitudes

Needed:

- The corolla polynomial and differentials

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- Previous set-up: Kirchhoff polynomials for 3-regular scalar graphs
- Then: The renormalized Feynman integrand of gauge theory from the sum of all 3-regular connected graphs.

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The corolla polynomial

We now consider a polynomial based on half-edges. We need the following definitions

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- For $i \geq 0$ let

$$C^i_{3g} = \sum_{\substack{C_1, C_2, \ldots C_i \in \mathcal{C} \\ C_j \text{pairwise disjoint}}} \left(\left(\prod_{j=1}^{i} \prod_{v \in C_j} a_{v, v_C} \right) \prod_{v \notin C_1 \cup C_2 \cup \cdots \cup C_i} D_v \right)$$

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- For a vertex $v \in V$ let $n(v)$ be the set of edges incident to v (internal or external).
- For a vertex $v \in V$ let $D_v = \sum_{j \in n(v)} a_{v,j}$.
- Let $\mathcal{C}$ be the set of all cycles of G (cycles, not circuits). This is a finite set.
- For C a cycle and v a vertex in V, since G is 3-regular, there is a unique edge of G incident to v and not in C, let v_C be this edge.
- For $i \geq 0$ let

$$C^i_{3g} = \sum_{\substack{C_1, C_2, \ldots C_i \in \mathcal{C} \\ C_j \text{pairwise disjoint}}} \left(\left(\prod_{j=1}^{i} \prod_{v \in C_j} a_{v,v_C} \right) \prod_{v \notin C_1 \cup C_2 \cup \cdots \cup C_i} D_v \right)$$

- Let

$$C_{3g} = \sum_{j \geq 0} (-1)^j C^j_{3g}$$

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This is a polynomial -the corolla polynomial- because $C^i_{3g} = 0$ for $i > |\mathcal{C}|$.
We write C^Γ_{3g} for the corolla polynomial of a 3-regular connected graph Γ.

Theorem

Let $\mathcal{T}$ be the set of sets T of half edges of G with the property that

- *every vertex of G is incident to exactly one half edge of T*
- *$G \setminus T$ has no cycles*

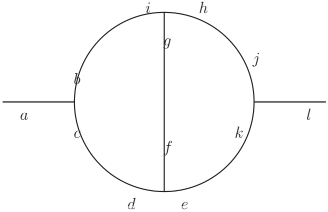
Then

$$C_{3g} = \sum_{T \in \mathcal{T}} \prod_{h \in T} a_h$$

More properties joint with Karen Yeats.

Example

Look at

$\Gamma =$ 

$$
\begin{aligned}
C_{3g}(\Gamma) \quad = \quad & (a+b+c)(d+e+f)(i+g+h)(j+k+l) \\
- \quad & (aeh)(j+k+l) \\
- \quad & (lid)(a+b+c) \\
- \quad & (algf) \qquad (24)
\end{aligned}
$$

The corolla differentials

Our main use of the corolla polynomial is to construct differential operators with it.
These operators differentiate wrt momenta $\xi(e)$ assigned to edges e of a graph, and act on the second Kirchhoff polynomial written for generic edge momenta $\xi(e)$.
Only at the end of the computation will we determine the $\xi(e)$ so that the agree with external momenta.

For a half edge $h \equiv (w, f) \in H^\Gamma$, we let $e(h) = f$ and $v(h) = w$.
h_+ and h_- are the successor and the precursor of h in the oriented corolla at $v(h)$.
We assign to a graph Γ:

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We assign to a graph Γ:

- i) to each (possibly external) edge e, a variable $A(e)$ and a 4-vector $\xi(e)$;
- ii) to each half edge h, a Lorentz index $\mu(h)$;
- iii) to each corolla, a representation of the Lie algebra of the gauge group.

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Gluon-ghost and fermion differentials

Furthermore, we assign to each $h \in H^\Gamma$ two types of differential operators: either the differential operator $D_g(h)$,

$$D_g(h) := \left(\frac{1}{A_{e(h_+)}} \frac{\partial}{\partial \xi(e(h_+))_{\mu(h)}} - \frac{1}{A_{e(h_-)}} \frac{\partial}{\partial \xi(e(h_-))_{\mu(h)}} \right) g_{\mu(h_+)\mu(h_-)}.$$

Or the differential operator

$$D_f(h) := \left(\frac{1}{A_{e(h_+)}} \frac{\partial}{\partial \xi(e(h_+))_{\mu(h_+)}} \gamma_{\mu(h_+)} - \frac{1}{A_{e(h_-)}} \frac{\partial}{\partial \xi(e(h_-))_{\mu(h_-)}} \gamma_{\mu(h_-)} \right) \gamma_{\mu(h)}.$$

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Graph differentials

The corolla polynomial is an alternating sum $(-1)^j C_{3g}^j$.
It depends on half-edge variables a_h.
For a collection of cycles $C_1, \cdots, C_j$ contributing to C_{3g}^j, consider partitions of this set into two subsets I_k, I_l containing $k+l=j$ cycles.
Replace $a_{v,v_C} \to b_{v,v_C}$ for each $C \in I_l$. This defines $C_{3g}^{I_k,I_l}(\Gamma)(a_h, b_h)$.
Sum over all possible partitions I_k, I_l of the cycles for each j. This gives a further corolla polynomial for which we write in slight abuse of notation $C_{3g}(\Gamma)(a_h, b_h)$.
Assign a differential operator as follows:

$$C_{3g}(\Gamma)(a_h, b_h) \to D^{\mathrm{gauge}}(\Gamma) = \mathrm{colour}^{I_k,I_l}(\Gamma) \sum_{j \geq 0} \sum_{|I_k|+|I_l|=j} \tilde{C}_{3g}^{I_k,I_l}(\Gamma)(D_g(h), D_f(h)). \quad (25)$$

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cont'd

Note that the restriction to $I_l = \emptyset$ gives the corresponding operator for Yang-Mills theory.
$C_{3g}^{I_k,I_l}(\Gamma)(D_g(h), D_f(h))$ is a homogeneous differential operator of degree $|V^\Gamma|$ which is at most quadratic in each derivative $\partial_{\xi(e)}$, $e \in E^\Gamma$.
For non-empty $I_k \cup I_l$, let $\tilde{C}_{3g}^{I_k,I_l}$ be the part of $C_{3g}^{I_k,I_l}$ which is linear in each variable $1/A_e$. Set $\tilde{C}_{3g}^{\emptyset,\emptyset} = C_{3g}^{\emptyset,\emptyset}$ else.
For k open ghost lines and j open fermion lines there is a similar definition available.

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3-regular graphs

This is joint with Matthias Sars and Walter van Suijlekom.

- We start with connected 3-regular graphs. To a graph with k external edges, we assign a powercounting weight $\omega_\Gamma = 4 - k$.
  Γ is convergent for $k > 4$.
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- Similar if we allow for 4-valent vertices, we also filter by the number of such vertices.

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Cycle Homology

The corolla polynomial is a sum of half-edge variables such that the variables not in any contributing monomial do not correspond to a cycle in the graph.
This allows to consider a 3-regular graph together with its set of (disjoint) cycles.

$$\Gamma \to (\Gamma, C_{i_1} \cdots C_{i_k}).$$

Disjointness makes this interesting.
The corolla polynomial eliminates all pairs but $(\Gamma, \emptyset)$ (ghost loops eliminate gaugeons).
This suggests $t := \sum_i \partial_{C_i}$ where we can treat the C_i as formal Grassmann variables. Together with graph homology this leads to a double complex which ensures gauge invariant amplitudes.

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Graph Homology

For an edge e in a graph Γ, let Γ_e be the graph where e shrinks to zero length.
Its orientation is obtained as follows:
we permute vertex labels collecting signs until the edge e connects vertex 1, $s(e) = 1$, to vertex 2, $t(e) = 2$.
Let σ be the sign of the necessary permutations.
Then we shrink edge e and the so-obtained vertex is labelled 1.
We inherit all remaining edge orientations and the ordering of vertices remains unchanged, with vertices $3, 4, \ldots, |V^\Gamma|$ relabelled to $2, 3, \ldots, |V^\Gamma| - 1$.
This defines an orientation of Γ_e.
If σ is negative, we change the orientation by en edge swap.

cont'd

For an oriented graph Γ, let

$$s\Gamma = \sum_{e \in E_I} \Gamma_e, \tag{26}$$

be a sum of graphs obtained by shrinking edge e and assigning the orientation as above. Graph homology comes from the classical result

Theorem

(graph homology) $s \circ s = 0$.

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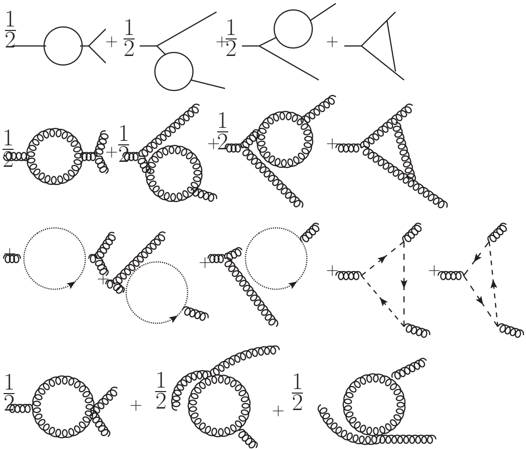
Graph homology and the residue

Note that we integrate against the simplex σ_Γ with boundary $\prod_e A_e = 0$.
We have co-dimension k-hypersurfaces given by $A_{i_1} = \cdots = A_{i_k} = 0$.
The Feynman integrand we have constructed above comes from a regular parts, and from residues along these hypersurfaces.
It can be described by the following commutative diagram.

$$\begin{array}{ccc}
\Gamma & \xrightarrow{s=\sum_e s_e} & s\Gamma \\
\downarrow \Phi & & \downarrow \Phi \\
\Phi(\Gamma) & \xrightarrow{\sum_e \mathrm{Res}_e} & \Phi(s\Gamma)
\end{array}$$

Example



The renormalized result

Theorem

The unrenormalized Feynman integrand at n loops for the sum of all Feynman graphs contributing to the connected k-loop amplitude is $\Phi(\Gamma^k) =$

$$\sum_{|\Gamma|=n,|E_E(\Gamma)|=k} e^{-\sum_e \oint_{c_e}} \left(\prod_{e\in E^\Gamma} g_{\mu(v_1(e))\mu(v_2(e))} D^{\mathrm{gauge}}_{\mathrm{hom}}(\Gamma)\right) \frac{e^{-\frac{\phi_\Gamma}{\psi_\Gamma}}}{\psi_\Gamma^2} \prod_{e\in E^\Gamma} dA_e.$$

The renormalized result is obtained as

$$D^{\mathrm{gauge}}_{\mathrm{hom}} \sum_{f\in\mathcal{F}} (-1)^{|f|} \frac{e^{-\frac{\phi_{\Gamma/f}}{\psi_{\Gamma/f}}}}{\psi_{\Gamma/f}^2} \frac{e^{-\frac{\phi_f}{\psi_f}}}{\psi_f^2}$$

with the graph differential in front of the forest sum.

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- Physical amplitudes are closed in graph and cycle homology.
- Spin $1/2$, spin 1 from scalar 3-regular graphs and restricted graph homology. What about spin 2, gravity?

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