

# *Higher order corrections in quantum field theory*

**Sven-Olaf Moch**

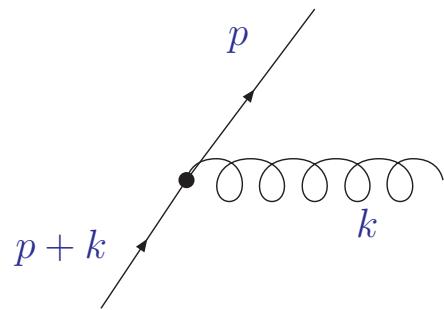
*Universität Hamburg & DESY, Zeuthen*

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*International Workshop on Frontiers in Perturbative Quantum Field Theory, Bielefeld, Sep 10, 2012*

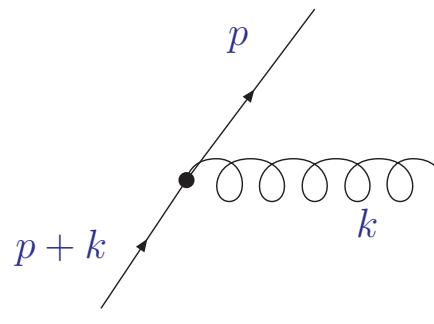
## Soft and collinear singularities

- Soft/collinear regions of phase space
  - massless partons

$$\frac{1}{(p - k)^2} = \frac{1}{2p \cdot k} = \frac{1}{2E_q E_g (1 - \cos \theta_{qg})}$$


# Soft and collinear singularities

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  - massless partons



Feynman diagram showing a quark line with momentum  $p$  and a gluon line with momentum  $k$ . A gluon loop is attached to the quark line, with a factor of  $\alpha_s$  and a loop integral  $\int d^4 k \frac{1}{(p-k)^2}$ .

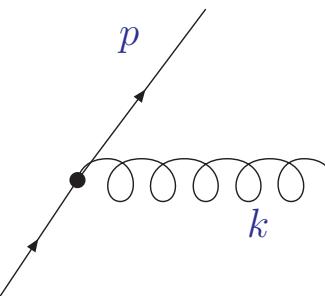
$$\frac{1}{(p-k)^2} = \frac{1}{2p \cdot k} = \frac{1}{2E_q E_g (1 - \cos \theta_{qg})}$$

$$\alpha_s \int d^4 k \frac{1}{(p-k)^2} \rightarrow \alpha_s \int dE_g d\sin \theta_{qg} \frac{1}{2E_q E_g (1 - \cos \theta_{qg})}$$

$$\rightarrow \alpha_s \frac{1}{\epsilon^2} \times (\dots) \quad \text{in dim. reg.} \quad D = 4 - 2\epsilon$$

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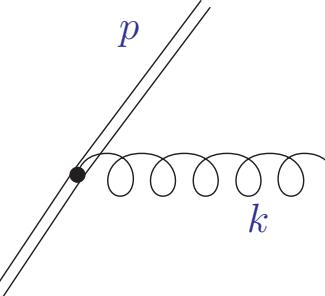
Feynman diagram showing a gluon-gluon fusion process. Two incoming gluons with momenta  $p$  and  $p+k$  interact via a gluon-gluon vertex to produce two outgoing gluons with momenta  $k$ . The interaction is represented by a coiled line.

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- Parton masses regulate collinear singularity



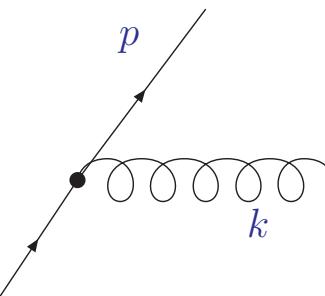
Feynman diagram showing a gluon-gluon fusion process where the incoming gluons  $p$  and  $p+k$  are now parallel. The outgoing gluons  $k$  are produced at a small angle from each other. The interaction is represented by a coiled line.

$$\frac{1}{(p-k)^2 - m_q^2} = \frac{1}{2p \cdot k} = \frac{1}{2E_q E_g (1 - \beta \cos \theta_{qg})}$$

with  $\beta = \left(1 - \frac{m_q^2}{E_q^2}\right)^{1/2} < 1$

# Soft and collinear singularities

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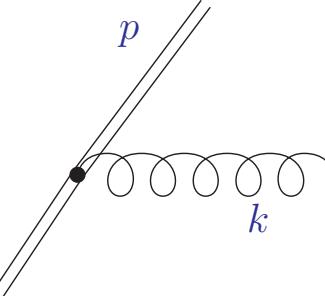
Feynman diagram showing a gluon line (curly line) with momentum  $k$  emitting a gluon (curly line) with momentum  $p - k$ . The incoming gluon has momentum  $p$  and the final state has momentum  $p + k$ .

$$\frac{1}{(p - k)^2} = \frac{1}{2p \cdot k} = \frac{1}{2E_q E_g (1 - \cos \theta_{qg})}$$

$$\alpha_s \int d^4 k \frac{1}{(p - k)^2} \rightarrow \alpha_s \int dE_g d\sin \theta_{qg} \frac{1}{2E_q E_g (1 - \cos \theta_{qg})}$$

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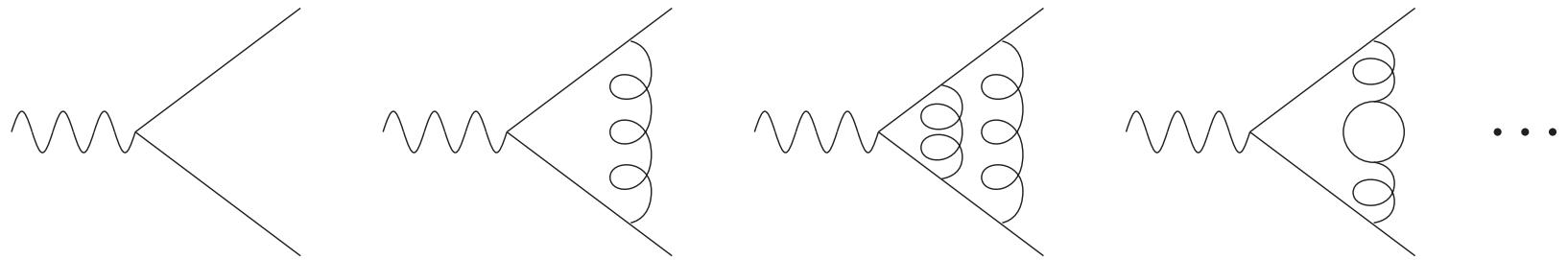
Feynman diagram showing a gluon line (curly line) with momentum  $k$  emitting a gluon (curly line) with momentum  $p - k$ . The incoming gluon has momentum  $p$  and the final state has momentum  $p + k$ . The outgoing gluon is shown as two parallel lines.

$$\frac{1}{(p - k)^2 - m_q^2} = \frac{1}{2p \cdot k} = \frac{1}{2E_q E_g (1 - \beta \cos \theta_{qg})}$$

with  $\beta = \left(1 - \frac{m_q^2}{E_q^2}\right)^{1/2} < 1$

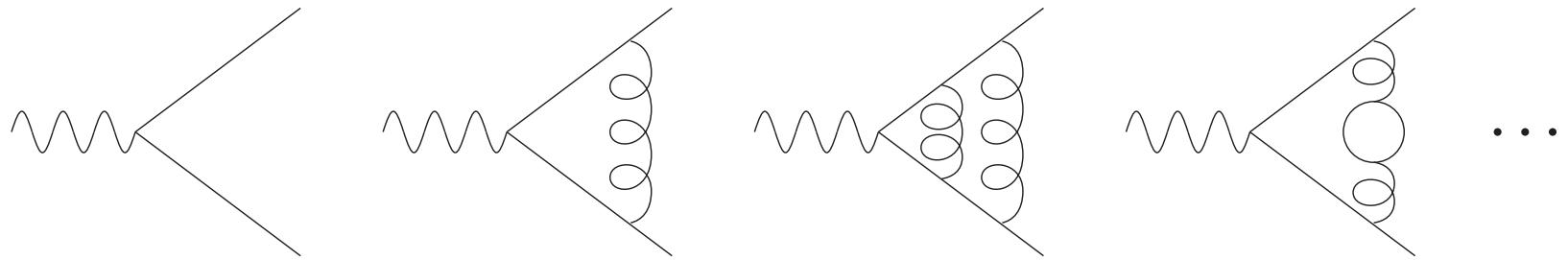
$$\alpha_s \int d^4 k \frac{1}{(p - k)^2 - m_q^2} \rightarrow \alpha_s \frac{1}{\epsilon} \ln(m_q^2) \times (\dots)$$

# Form factor in QCD



- Quark form factor (on-shell,  $m = 0$ )
  - QCD corrections to vertex  $\gamma^* q \bar{q}$ , i.e.  $\Gamma_\mu = ie_q (\bar{u} \gamma_\mu u) \mathcal{F}_q(Q^2, \alpha_s)$
  - gauge invariant quantity
  - infrared divergent (dimensional regularization  $D = 4 - 2\epsilon$ )

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  - infrared divergent (dimensional regularization  $D = 4 - 2\epsilon$ )
- Gluon form factor  $\mathcal{F}_g$  in effective gg Higgs vertex  
$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} C_H H G_{\mu\nu}^a G^{a\mu\nu}$$
  - coefficient  $C_H$  known to N<sup>3</sup>LO Chetyrkin, Kniehl, Steinhauser '97

# Exponentiation

- Form factor  $\mathcal{F}(Q^2, \alpha_s)$  exponentiates
  - long history for massless case  
Collins '80; Sen '81; Korchemsky '88; Magnea, Sterman '90; Contopanagos, Laenen, Sterman '97; Magnea '00

$$Q^2 \frac{\partial}{\partial Q^2} \ln \mathcal{F}(Q^2, \alpha_s, \epsilon) = \frac{1}{2} K(\alpha_s, \epsilon) + \frac{1}{2} G\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right).$$

- Renormalization group equations for functions  $G$  and  $K$ 
  - all  $Q^2$ -scale dependence in  $G$  (finite in  $\epsilon$ )
  - pure counter term function  $K$  (contains poles in  $\frac{1}{\epsilon}$ )
- Cusp anomalous dimension  $A$ 
  - $A$  governs evolution for  $G$  and  $K$
  - splitting functions  $P_{\text{pp}}^{(n)}$  in large  $x$ -limit  $x \rightarrow 1$ 
$$P_{\text{pp}}^{(n-1)}(x) = \frac{A_n^{\text{p}}}{(1-x)_+} + B_n^{\text{p}} \delta(1-x) + C_n^{\text{p}} \ln(1-x) + \mathcal{O}(1)$$

$$A_1^q = 4 C_F$$

$$A_2^q = 8 C_F C_A \left( \frac{67}{18} - \zeta_2 \right) - \frac{5}{9} C_F n_f \quad \text{Kodaira, Trentadue '82}$$

$$\begin{aligned} A_3^q &= 16 C_F C_A^2 \left( \frac{245}{24} - \frac{67}{9} \zeta_2 + \frac{11}{6} \zeta_3 + \frac{11}{5} \zeta_2^2 \right) + 16 C_F^2 n_f \left( -\frac{55}{24} + 2 \zeta_3 \right) \\ &\quad + 16 C_F C_A n_f \left( -\frac{209}{108} + \frac{10}{9} \zeta_2 - \frac{7}{3} \zeta_3 \right) + 16 C_F n_f^2 \left( -\frac{1}{27} \right) \end{aligned}$$

S.M., Vogt, Vermaseren '04

- Maximally non-Abelian colour structure (Casimir scaling),  $A_3^g = \frac{C_A}{C_F} A_3^q$   
Korchemsky '89
  - simple replacement for gluon form factor  $\mathcal{F}_g$  in effective gg Higgs vertex  $\mathcal{L}_{\text{eff}} = -\frac{1}{4} C_H H G_{\mu\nu}^a G^{a\mu\nu}$
- Conjecture Ahrens, Neubert, Vernazza '12
  - Casimir scaling of cusp anomalous dimension  $A$  holds at all loops

## Solution

- Solution for  $\ln \mathcal{F}(Q^2, \alpha_s, \epsilon)$  in  $D$ -dimensions

- boundary condition  $\mathcal{F}(0, \alpha_s, \epsilon) = 1$

$$\ln \mathcal{F}\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right) =$$

$$\frac{1}{2} \int_0^{Q^2/\mu^2} \frac{d\xi}{\xi} \left( K(\alpha_s, \epsilon) + G(1, \bar{a}(\xi\mu^2, \alpha_s, \epsilon), \epsilon) + \int_{\xi}^1 \frac{d\lambda}{\lambda} A(\bar{a}(\lambda\mu^2, \epsilon)) \right)$$

- use running coupling in  $D$ -dimensions from

$$\lambda \frac{\partial}{\partial \lambda} \bar{a}(\lambda, \alpha_s, \epsilon) = -\epsilon \bar{a}(\lambda, \alpha_s, \epsilon) - \beta_0 \bar{a}^2(\lambda, \alpha_s, \epsilon) - \dots$$

- boundary condition  $\bar{a}(1, \alpha_s, \epsilon) = \alpha_s$
  - universality of subleading infrared poles (function  $G$ )

Dixon, Magnea, Sterman '08

## Upshot

- Generating functional for Laurent-series in  $\epsilon$  to all orders

# Result

- Result up to three loops in terms of expansion coefficients of  $A$  and  $G$

$$\mathcal{F}_1 = -\frac{1}{2} \frac{1}{\epsilon^2} A_1 - \frac{1}{2} \frac{1}{\epsilon} G_1$$

$$\mathcal{F}_2 = \frac{1}{8} \frac{1}{\epsilon^4} A_1^2 + \frac{1}{8} \frac{1}{\epsilon^3} A_1 (2G_1 - \beta_0) + \frac{1}{8} \frac{1}{\epsilon^2} (G_1^2 - A_2 - 2\beta_0 G_1) - \frac{1}{4} \frac{1}{\epsilon} G_2$$

$$\begin{aligned} \mathcal{F}_3 = & -\frac{1}{48} \frac{1}{\epsilon^6} A_1^3 - \frac{1}{16} \frac{1}{\epsilon^5} A_1^2 (G_1 - \beta_0) - \frac{1}{144} \frac{1}{\epsilon^4} A_1 (9G_1^2 - 9A_2 - 27\beta_0 G_1 + 8\beta_0^2) \\ & - \frac{1}{144} \frac{1}{\epsilon^3} (3G_1^3 - 9A_2 G_1 - 18A_1 G_2 + 4\beta_1 A_1 - 18\beta_0 G_1^2 + 16\beta_0 A_2 + 24\beta_0^2 G_1) \\ & + \frac{1}{72} \frac{1}{\epsilon^2} (9G_1 G_2 - 4A_3 - 6\beta_1 G_1 - 24\beta_0 G_2) - \frac{1}{6} \frac{1}{\epsilon} G_3 \end{aligned}$$

- Expansion in terms of bare coupling  $a_s^b = \alpha_s^b / (4\pi)$

$$\mathcal{F}(Q^2, \alpha_s^b) = 1 + \sum_{l=1} \left( a_s^b \right)^l \left( \frac{Q^2}{\mu^2} \right)^{-l\epsilon} \mathcal{F}_l$$

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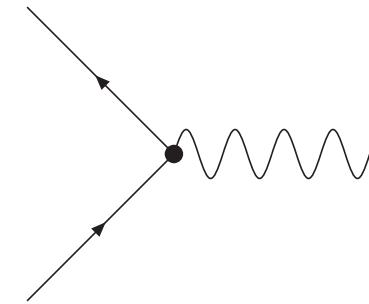
$\mathcal{F}_2$ : Hamberg, van Neerven, Matsuura '88; Harlander '00; Gehrmann, Huber, Maitre '05; S.M. Vermaseren, Vogt '05

$\mathcal{F}_3$ : S.M. Vermaseren, Vogt '05; Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser '09; Gehrmann, Glover, Huber, Ikizlerli, Studerus '10

$\mathcal{F}_4$ : prediction for  $\epsilon$ -poles (all terms  $\epsilon^{-8} \dots \epsilon^{-3}$ )

# Massive form factor

- Form factor for massive quarks
  - simplest amplitude:  $q\bar{q} \rightarrow \gamma$



$$\Gamma_\mu(k_1, k_2) =$$

$$ie_q \bar{u}(k_1) \left( \gamma_\mu \mathcal{F}_1(Q^2, m^2, \alpha_s) + \frac{1}{2m} \sigma_{\mu\nu} q^\nu \mathcal{F}_2(Q^2, m^2, \alpha_s) \right) u(k_2)$$

- gauge invariant quantity
- infrared divergent (dimensional regularization  $D = 4 - 2\epsilon$ )
- QCD corrections to vertex  $\gamma^* q\bar{q}$ 
  - one-scale problem (dependent on ratio  $Q^2/m^2$ )
  - radiative corrections expressible in terms of harmonic polylogarithms  $H_{m_1 \dots m_k}$  Remiddi, Vermaseren '99
- Current status
  - exact (fully analytic) QCD corrections to two-loops  
Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi '04; Gluza, Mitov, S.M., Riemann '09

## Massive form factor (cont'd)

- Form factor for massive quarks in limit  $m^2 \ll Q^2$ 
  - logarithms  $\ln(m)$  from parton mass and poles in  $\frac{1}{\epsilon}$
- Massive form factor  $\mathcal{F}(Q^2, m^2, \alpha_s)$  exponentiates  
Collins '80; Korchemsky '89; Mitov, S.M.'06
- Renormalization group equations factorizes into functions  $G$  and  $K$ 
  - again all  $Q^2$ -scale dependence in finite function  $G$
  - function  $K$  dependent on infrared sector (parton mass  $m$ )

$$Q^2 \frac{\partial}{\partial Q^2} \ln \mathcal{F}\left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \alpha_s, \epsilon\right) = \frac{1}{2} K\left(\frac{m^2}{\mu^2}, \alpha_s, \epsilon\right) + \frac{1}{2} G\left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \alpha_s, \epsilon\right)$$

- Solution for evolution equation

$$\ln \mathcal{F}\left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \alpha_s, \epsilon\right) = -\frac{1}{2} \int_0^{Q^2/\mu^2} \frac{d\xi}{\xi} \left\{ G(\bar{a}(\xi\mu^2, \epsilon)) + K(\bar{a}(\xi\mu^2 m^2/Q^2, \epsilon)) + \int_{\xi m^2/Q^2}^{\xi} \frac{d\lambda}{\lambda} A(\bar{a}(\lambda\mu^2, \epsilon)) \right\}$$

- Perturbative expansion of  $A$  and  $G$ 
  - *same* functions as in massless calculation
- Matching to fixed order results determines function  $K$ 
  - different infrared regularization due to parton mass  $m^2$

# Result

- Massive form factor with logarithms  $L = \ln(Q^2/m^2)$ 
  - expansion up to two loops in terms of coefficients of  $A, G, K$   
Berreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi '04; Mitov, S.M. '06

- constant terms  $C$  finite in  $m^2$  and  $\epsilon$

$$\mathcal{F}_1 = \frac{1}{\epsilon} \left\{ \frac{1}{2} A_1 L + \frac{1}{2} (G_1 + K_1) \right\} - \frac{1}{4} A_1 L^2 - \frac{1}{2} G_1 L + C_1$$

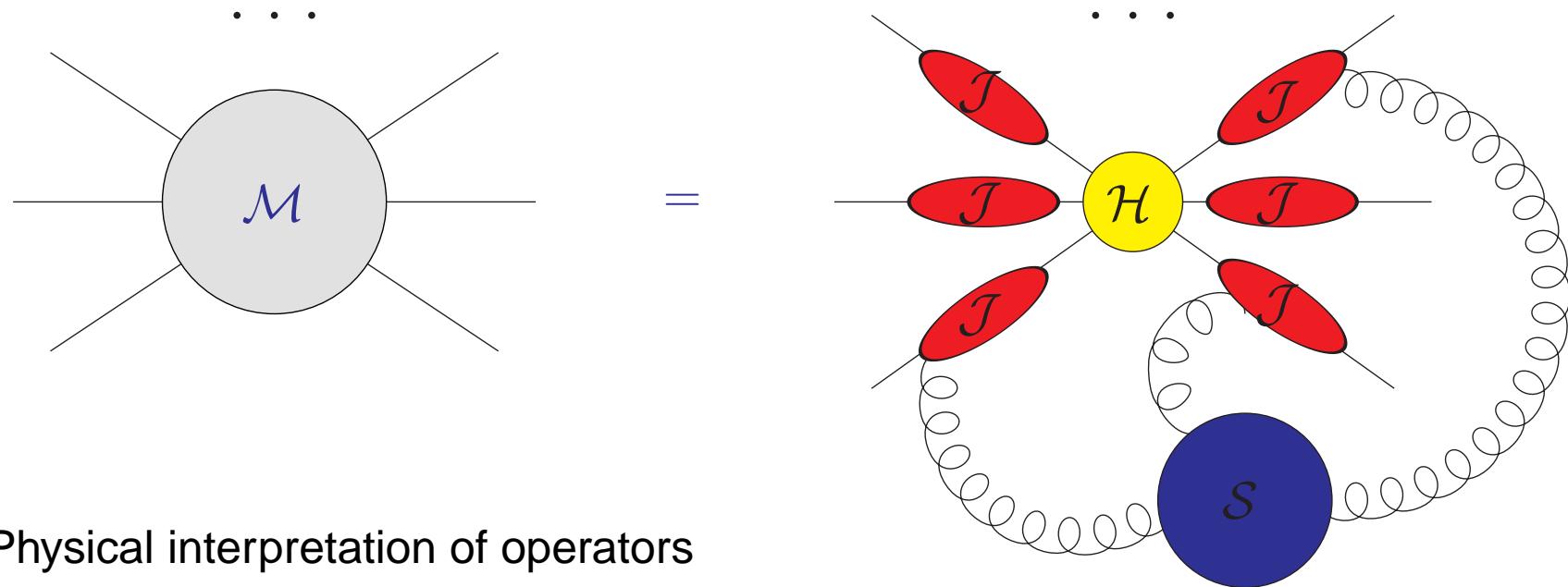
$$\begin{aligned} \mathcal{F}_2 = & \frac{1}{\epsilon^2} \left\{ \frac{1}{8} A_1^2 L^2 + \frac{1}{4} A_1 (G_1 + K_1 - \beta_0) L + \frac{1}{8} (G_1 + K_1)(G_1 + K_1 - 2\beta_0) \right\} \\ & + \frac{1}{\epsilon} \left\{ -\frac{1}{8} A_1^2 L^3 - \frac{1}{8} A_1 (3G_1 + K_1) L^2 + \frac{1}{4} (A_2 - G_1^2 - K_1 G_1 + 2A_1 C_1) L \right. \\ & \left. + \frac{1}{4} (G_2 + K_2) + \frac{1}{2} C_1 (G_1 + K_1) \right\} + \frac{7}{96} A_1^2 L^4 \\ & + \frac{1}{24} A_1 (7G_1 + K_1 + 2\beta_0) L^3 + \frac{1}{8} G_1 (2G_1 + K_1 + 2\beta_0) L^2 \\ & - \frac{1}{4} (A_2 + A_1 C_1) L^2 - \frac{1}{2} (G_2 + G_1 C_1) L + C_2 \end{aligned}$$

- Prediction of singularities in  $\mathcal{F}_3$  at three loops

## Soft and collinear factorization

- Amplitude  $\mathcal{M}$  factorizes into various functions  $\mathcal{J}^{[p]}$ ,  $\mathcal{S}^{[p]}$  and  $\mathcal{H}^{[p]}$

$$|\mathcal{M}_p\rangle = \mathcal{J}^{[p]}(Q^2, \alpha_s, \epsilon) \mathcal{S}^{[p]}(\{k_i\}, \alpha_s, \epsilon) |\mathcal{H}_p\rangle$$



- Physical interpretation of operators
  - $\mathcal{J}^{[p]}$  → collinear partons near the light cone
  - $\mathcal{S}^{[p]}$  → soft partons of long wave-length at large angle  
(matrix in colour space)
  - $\mathcal{H}^{[p]}$  → hard off-shell partons at short distances  
(vector in colour space)

## Jet function

- Define jet function  $\mathcal{J}^{[i]}$  for external parton as square root of form factor

$$\mathcal{J}^{[i]}(Q^2, \alpha_s) = \left( \mathcal{F}^{[i]}(Q^2, \alpha_s) \right)^{\frac{1}{2}}, \quad i = q, g$$

- factorization for  $\mathcal{M}^{[i\bar{i}\rightarrow 1]}$  with trivial color dependence

$$\mathcal{M}^{[i\bar{i}\rightarrow 1]}(Q^2, \alpha_s) = \left( \mathcal{J}^{[i]}(Q^2, \alpha_s) \right)^2 \mathcal{H}^{[i\bar{i}\rightarrow 1]}(Q^2, \alpha_s)$$

- Exponentiation of form factor acts as generating functional to all orders

# Soft function

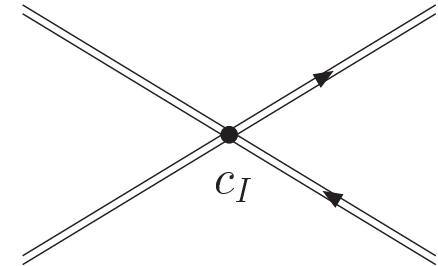
- Sensitivity to color structure of scattering process
  - color mixing through soft gluon exchange

- Construction of  $\mathcal{S}$  as composite operator

Korchemsky, Korchemskaya '94; Contopanagos, Laenen, Sterman '97;

Aybat, Dixon, Sterman '06; ...

- coupling to Wilson-lines  
→ partons in eikonal approximation



- Renormalization group equations  
with soft anomalous dimension  $\Gamma^{[p]}$

$$\mu^2 \frac{d}{d\mu^2} \mathcal{S}_{IJ}^{[p]} = - \left( \Gamma^{[p]} \right)_{IK} \mathcal{S}_{KJ}^{[p]}$$

- $\Gamma^{[p]}$  with smooth limit  $m \rightarrow 0$
- Solution (matrix in color space)

$$\mathcal{S}^{[p]} (\{k_i\}, Q^2, \alpha_s, \epsilon) = P \exp \left[ -\frac{1}{2} \int_0^{Q^2} \frac{dk^2}{k^2} \Gamma^{[p]} (\bar{a}(k^2, \epsilon)) \right]$$

# Soft anomalous dimension

- Soft anomalous dimension  $\Gamma^{[p]}$
- Massless partons
  - explicit calculations up to two-loops Aybat, Dixon, Sterman '06
  - functional form (color dipole) conjectured to all orders Becher, Neubert '09; Gardi, Magnea '09

$$\Gamma^{[p]} = \sum_{i,j} \mathbf{T}_i \mathbf{T}_j A(\alpha_s) \ln \frac{\mu}{-s_{ij}}$$

- consistent with all-order collinear limits and two-loop splitting amplitudes Badger, Glover '04
- possible extensions by four- and five particle correlations/functions of conformal cross ratios under study Ahrens, Neubert, Vernazza '12
- Massive partons
  - all order ansatz with additional contributions from three-parton correlations  
Becher, Neubert '09
  - complete calculation of  $\Gamma^{[p]}$  to two loops  
Beneke, Falgari, Schwinn '09; Mitov, Sterman, Sung '09; Czakon, Mitov, Sterman '09; Ferroglia, Neubert, Pecjak, Yang '09

# Massless amplitudes

- Singularity structure of massless amplitudes  $|\mathcal{M}_p\rangle$ 
  - process  $p$  for  $2 \rightarrow n$  parton scattering
  - poles in  $\frac{1}{\epsilon}$  in terms of universal anomalous dimensions Catani '98
  - soft and collinear divergences exhibit exponentiation to all orders Tejeda-Yeomans, Sterman '02

$$|\mathcal{M}_p^{(0)}\rangle = |\mathcal{H}_p^{(0)}\rangle$$

$$|\mathcal{M}_p^{(1)}\rangle = \frac{1}{2} \sum_{i \in \{\text{all legs}\}} \mathcal{F}_1^{[i]} |\mathcal{H}_p^{(0)}\rangle + \mathcal{S}_1^{[p]} |\mathcal{H}_p^{(0)}\rangle + |\mathcal{H}_p^{(1)}\rangle$$

$$\begin{aligned} |\mathcal{M}_p^{(2)}\rangle &= \frac{1}{2} \sum_{i \in \{\text{all legs}\}} \left( \mathcal{F}_2^{[i]} - \frac{1}{4} (\mathcal{F}_1^{[i]})^2 + \frac{1}{2} \mathcal{F}_1^{[i]} \mathcal{S}_1^{[p]} \right) |\mathcal{H}_p^{(0)}\rangle \\ &\quad + \frac{1}{2} \sum_{i \in \{\text{all legs}\}} \mathcal{F}_1^{[i]} |\mathcal{H}_p^{(1)}\rangle + \mathcal{S}_2^{[p]} |\mathcal{H}_p^{(0)}\rangle + \mathcal{S}_1^{[p]} |\mathcal{H}_p^{(1)}\rangle + |\mathcal{H}_p^{(2)}\rangle \end{aligned}$$

- Checks with explicit calculations of NNLO QCD  $2 \rightarrow 2$  amplitudes  
Anastasiou, Bern, v.d.Bij, De Freitas, Dixon, Garland, Gehrmann, Ghinculov, Glover,  
Koukoutsakis, S.M., Oleari, Remiddi, Schmidt, Tejeda-Yeomans, Uwer, Weinzierl, Wong  
'01-'04

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$$|\mathcal{M}_p^{(0)}\rangle = |\mathcal{H}_p^{(0)}\rangle$$

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$$|\mathcal{M}_p^{(2)}\rangle = \frac{1}{2} \sum_{i \in \{\text{all legs}\}} \left( \mathcal{F}_2^{[i]} - \frac{1}{4} \left( \mathcal{F}_1^{[i]} \right)^2 + \frac{1}{2} \mathcal{F}_1^{[i]} \mathcal{S}_1^{[p]} \right) |\mathcal{H}_p^{(0)}\rangle$$

$$+ \frac{1}{2} \sum_{i \in \{\text{all legs}\}} \left( \mathcal{F}_1^{[i]} |\mathcal{H}_p^{(1)}\rangle + \mathcal{S}_2^{[p]} |\mathcal{H}_p^{(0)}\rangle + \mathcal{S}_1^{[p]} |\mathcal{H}_p^{(1)}\rangle + |\mathcal{H}_p^{(2)}\rangle \right)$$

massless form factor

- Checks with explicit calculations of NNLO QCD  $2 \rightarrow 2$  amplitudes  
 Anastasiou, Bern, v.d.Bij, De Freitas, Dixon, Garland, Gehrmann, Ghinculov, Glover,  
 Koukoutsakis, S.M., Oleari, Remiddi, Schmidt, Tejeda-Yeomans, Uwer, Weinzierl, Wong  
 '01-'04

# Massive amplitudes

- Singularity structure of massive amplitudes  $|\mathcal{M}_{p,\{m_i\}}\rangle$ 
  - process  $p$  for  $2 \rightarrow n$  parton scattering
  - generalization of Catani's massless formula and one-loop massive results [Catani, Dittmaier, Trocsanyi '00](#)
  - amplitude factorizes in terms of three functions  $\mathcal{F}$ ,  $\mathcal{S}_p$  and  $|\mathcal{H}_p\rangle$  ( $\mathcal{S}_p$  and  $|\mathcal{H}_p\rangle$  largely same as in massless case) [Mitov S.M. '06](#)

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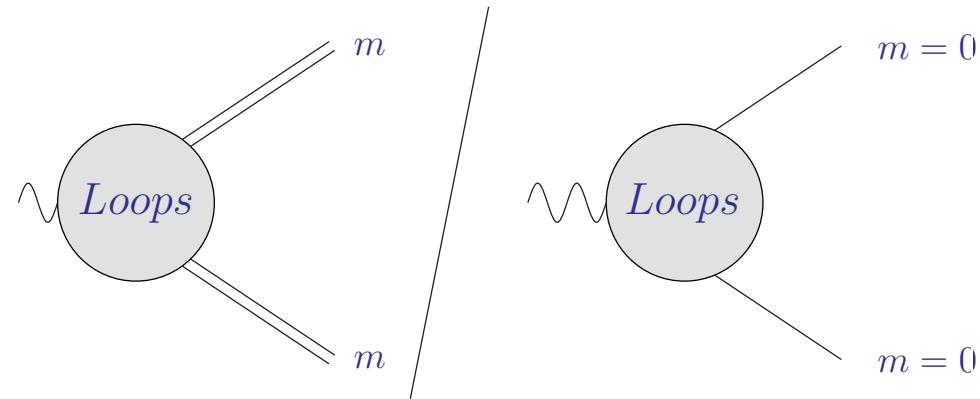
massive form factor

## Upshot

- Simple multiplicative relation between massless and massive amplitudes to all orders Mitov S.M. '06
  - hierarchy of scales  $m^2 \ll s, t, u$
  - relation correct up to power suppressed terms  $\mathcal{O}(m)$

$$\mathcal{M}^{[p],(m)} \left( \{k_i\}, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = \prod_{i \in \{\text{all legs}\}} \left( Z_{[i]}^{(m|0)} \left( \frac{m^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right)^{\frac{1}{2}} \times \mathcal{M}^{[p],(m=0)} \left( \{k_i\}, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right)$$

- factor  $Z_{[i]}^{(m|0)}$  with  $i = q, g$ 
  - determined by ratio of massless and massive form factor



# $N = 4$ Super Yang-Mills theory

- Interacting QFT in  $d = 4$  with highest degree of symmetry
- Interest from: integrability in AdS/CFT, scattering amplitudes in maximally susy Yang-Mills, finiteness of  $N = 8$  supergravity in  $d = 4$

# Form factor in $N=4$ SYM

- Scattering amplitudes of  $N = 4$  SYM
  - dimensional regularization in  $D = 4 - 2\epsilon$
  - massive (or Higgs) regularization  
Alday, Henn, Plefka, Schuster '09; Henn, Naculich, Schnitzer, Spradlin '10
- Structure of color-ordered  $n$ -particle amplitude  $A_n = A_n^{\text{tree}} M_n$   
Catani '98; ...; Anastasiou, Bern, Dixon, Kosower '03; Bern, Dixon, Smirnov '05; Aybat, Dixon, Sterman '06; ... [many people]

$$M_n = S_n \times J_n \times H_n$$

- Study regulator independence (planar limit) Henn, S.M., Naculich '11
  - definition of unambiguous, regulator-independent finite part of  $M_n$
- Factorization of a product of IR-divergent "wedge" functions  $W(s_{i-1,i})$ 
  - wedge  $W(s_{i-1,i})$  for two adjacent particles  $i-1$  and  $i$  of color-ordered amplitude

$$M_n = H_n \times \prod_{i=1}^n W(s_{i-1,i}), \quad s_{i-1,i} = (p_{i-1} + p_i)^2$$

- regularization independence of amplitude  $M_n$  through two loops
- extensions: general differential-mass Higgs regulator

# *Operator product expansion in $N=4$ SYM*

- Correlation functions of composite operators  
 $\mathcal{O}_n = \text{Tr}(\mathcal{W}_1 \mathcal{W}_2 \dots \mathcal{W}_n)$  and  $\mathcal{W}_i \in \{\mathcal{D}\Phi, \mathcal{D}\Psi, \mathcal{D}F\}$

$$\langle \mathcal{O}_a(x_1) \mathcal{O}_b(x_2) \rangle = \frac{\delta_{ab}}{(x_1 - x_2)^{2\Delta_a(\lambda)}}$$

- Spectrum of scaling dimensions:  $\Delta(\lambda) = \Delta_0 + \gamma(\lambda)$  governed by anomalous dimension
  - weak coupling expansion in limit:  $N_c \rightarrow \infty$  and  $\lambda = g^2 N_c$  fixed
- Dilatation operator corresponds to Heisenberg spin chain
  - solution with asymptotic Bethe ansatz
  - wrapping corrections when loop order  $l \geq 2L$

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## *Correspondence with QCD*

- QCD results carry over to  $N = 4$  SYM
  - principle of “leading transcendentality”  
(keep only highest weight in  $\zeta$ -function / harmonic sums)
- color coefficients  $C_A = C_F = N_c$

## Anomalous dimension of $N=4$ SYM

- Universal anomalous dimension in  $N = 4$  SYM to three loops  
Kotikov, Lipatov, Onishchenko, Velizhanin '04
  - at loops  $l$ -loops harmonic sums of weight  $w = 2l - 1$

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is

$$\gamma(j) \equiv \gamma_{uni}(j) = \hat{a} \gamma_{uni}^{(0)}(j) + \hat{a}^2 \gamma_{uni}^{(1)}(j) + \hat{a}^3 \gamma_{uni}^{(2)}(j) + \dots, \quad \hat{a} = \frac{\alpha N_c}{4\pi}, \quad (9)$$

where

$$\frac{1}{4} \gamma_{uni}^{(0)}(j+2) = -S_1, \quad (10)$$

$$\frac{1}{8} \gamma_{uni}^{(1)}(j+2) = (S_3 + \bar{S}_{-3}) - 2\bar{S}_{-2,1} + 2S_1(S_2 + \bar{S}_{-2}), \quad (11)$$

$$\begin{aligned} \frac{1}{32} \gamma_{uni}^{(2)}(j+2) = & 2\bar{S}_{-3}S_2 - S_5 - 2\bar{S}_{-2}S_3 - 3\bar{S}_{-5} + 24\bar{S}_{-2,1,1,1} \\ & + 6(\bar{S}_{-4,1} + \bar{S}_{-3,2} + \bar{S}_{-2,3}) - 12(\bar{S}_{-3,1,1} + \bar{S}_{-2,1,2} + \bar{S}_{-2,2,1}) \\ & - (S_2 + 2S_1^2)(3\bar{S}_{-3} + S_3 - 2\bar{S}_{-2,1}) - S_1(8\bar{S}_{-4} + \bar{S}_{-2}^2 \\ & + 4S_2\bar{S}_{-2} + 2S_2^2 + 3S_4 - 12\bar{S}_{-3,1} - 10\bar{S}_{-2,2} + 16\bar{S}_{-2,1,1}) \end{aligned} \quad (12)$$

and  $S_a \equiv S_a(j)$ ,  $S_{a,b} \equiv S_{a,b}(j)$ ,  $S_{a,b,c} \equiv S_{a,b,c}(j)$  are harmonic sums

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Kotikov, Lipatov, Onishchenko, Velizhanin '04
  - at loops  $l$ -loops harmonic sums of weight  $w = 2l - 1$
- Result agrees with predictions based on integrability for planar three-loop contribution to dilatation operator  
Beisert, Kristjansen, Staudacher '03, Staudacher '04; ... [many people]

# Anomalous dimension beyond three loops

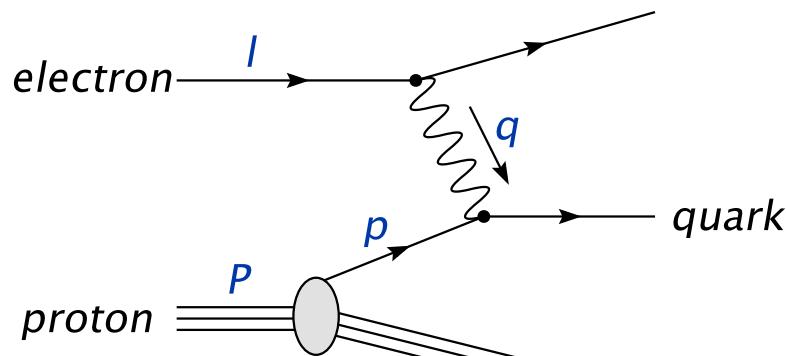
- Properties of  $\gamma(N)$ 
  - dependence only on integer negative powers of product  $N(N + 1)$   
Basso, Korchemsky '07
  - all-order relation (angular ordering of gluon radiation)  
Dokshitzer, Marchesini, Salam '05; Dokshitzer, Marchesini '06

$$\gamma(N) = f\left(N + \frac{1}{2}\gamma(N)\right)$$

- Wrapping corrections to complement asymptotic Bethe ansatz
  - control high-energy behaviour ( $\sim 1/N^l$ )
  - four-loop Bajnok, Janik, Lukowski '08
  - five-loop Lukowski, Rej, Velizhanin '09
- $\gamma(N)^{\text{wrap},(4)} \simeq S_1(N)^2 f^{\text{wrap}}(N)$

$$f^{\text{wrap}}(N) = 5\zeta_5 - 2S_{-5} + 4S_{-2}\zeta_3 - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5$$

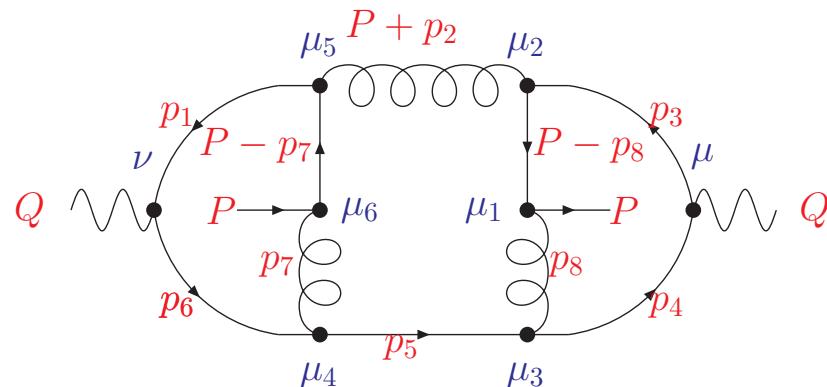
# Deep-inelastic scattering



- Kinematic variables
  - momentum transfer  $Q^2 = -q^2$  (space-like)
  - Bjorken variable  $x = Q^2/(2p \cdot q)$
- Parton distributions  $PDF$ 
  - scale evolution governed by splitting functions  $P_{ij}$  (anomalous dimensions  $\gamma_{ij}$ )
- Hard scattering described by Wilson coefficients  $C_i$ 
  - include radiative corrections

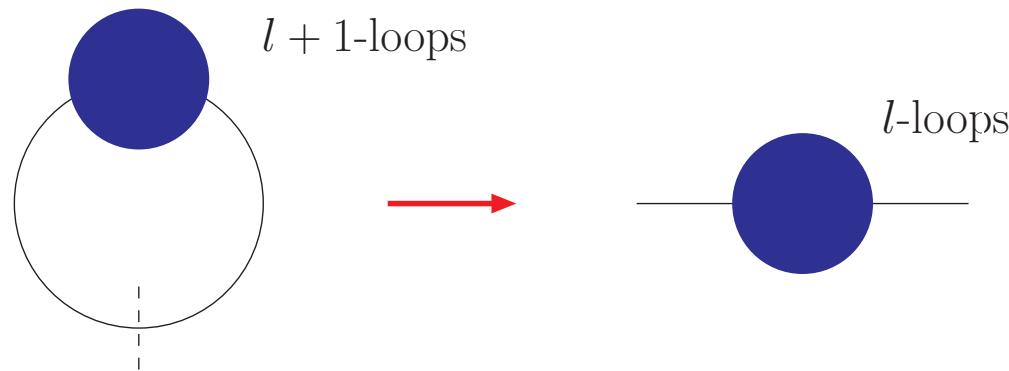
# DIS coefficient functions

- Wilson coefficient  $C(N)$  in Mellin space
  - finite part in OPE
- Three-loop Wilson coefficient  $C^{(3)}(N)$  knows about  $f^{\text{wrap}}(N)$   
S.M., Vermaseren, Vogt '05
  - $C^{(3)}(N) \simeq C_F \left( C_F - \frac{C_A}{2} \right)^2 \{N(N+1) f^{\text{wrap}}(N)\}$
- $f^{\text{wrap}}(x)$  exhibits Gribov-Lipatov reciprocity
 
$$f(x) = -x f\left(\frac{1}{x}\right)$$
- Feynman diagram contributing at three loops to  $f^{\text{wrap}}(N)$  (sample)
  - exhibits very peculiar symmetry: momenta  $P \leftrightarrow Q$



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- Cut and glue Chetyrkin, Tkachov '81; ... ; Baikov, Chetyrkin '10
  - relation between  $l$ -loop coefficient functions and  $l+1$ -loop anomalous dimensions
  - parts of  $\gamma^{(4)}(N)$  in full QCD within reach



# *Summary*

## *QCD*

- QCD radiative corrections known to higher orders
  - wealth of information in perturbation theory
- Form factor exponentiates in massless and massive case for  $m^2 \ll Q^2$
- Factorization of amplitudes in soft and collinear limits
- Cancellation of soft and collinear divergences in observables
- QCD results in space- and time-like kinematics

## *N=4 SYM*

- Relations of entire building blocks to QCD
- Systematic expansion beyond planar and conformal limit