

Gravitational Lensing by Accelerating Black Holes with Cosmological Constant

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Outline of the Talk

- 1 Motivation
- 2 The C-Metric with Cosmological Constant
- 3 Gravitational Lensing
- 4 Implications for Astronomical Observations

Motivation

- 2019: Event Horizon Telescope Collaboration [2019] published the first observations of the shadow of the supermassive black hole in the centre of the galaxy M87
⇒ allows to test strong gravity using gravitational lensing
- general relativity: different models for spacetimes
 - ▶ some are considered to be of astrophysical relevance such as Schwarzschild/Kerr
 - ▶ for others, such as the C-metric, the physical relevance is still unclear

Motivation for Gravitational Lensing in the C-Metric with Cosmological Constant

- C-metric [Griffiths et al., 2006]:
 - ▶ introduces acceleration parameter α
 - ▶ describes spacetime of accelerating black hole
 - ▶ acceleration caused by conical singularities on the axes; interpretation of the singularities still unclear but usually considered unphysical
 - ▶ α usually considered to be small
- astrophysics: cosmological constant also present
⇒ leads to C-metric with cosmological constant
- focus of this talk:
 - ▶ How does the acceleration parameter affect lensing variables?
 - ▶ How can we distinguish effects from the acceleration parameter and from the cosmological constant?

The C-Metric with Cosmological Constant

- line element in Boyer-Lindquist-like coordinates Griffiths and Podolský [2009], pp. 258–290 ($c = G = 1$):

$$g_{\mu\nu}(x)dx^\mu dx^\nu = \frac{1}{\Omega(r, \vartheta)^2} \left(-Q(r)dt^2 + \frac{dr^2}{Q(r)} + \frac{r^2d\vartheta^2}{P(\vartheta)} + r^2 P(\vartheta) \sin^2 \vartheta d\varphi^2 \right)$$

where

$$Q(r) = \left(1 - \alpha^2 r^2\right) \left(1 - \frac{2m}{r}\right) - \frac{\Lambda}{3} r^2,$$

$$P(\vartheta) = 1 - 2\alpha m \cos \vartheta, \quad \Omega(r, \vartheta) = 1 - \alpha r \cos \vartheta$$

m : mass parameter

Λ : cosmological constant (here $\Lambda > 0$)

α : acceleration parameter

Properties

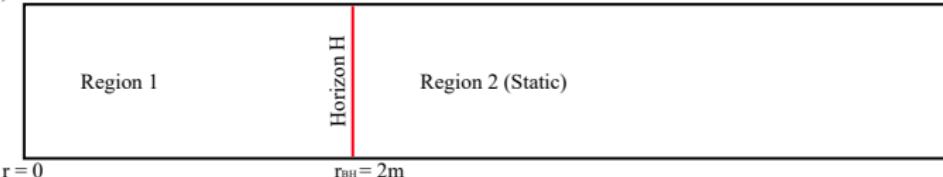
- solution of Einstein's vacuum field equations with cosmological constant:

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

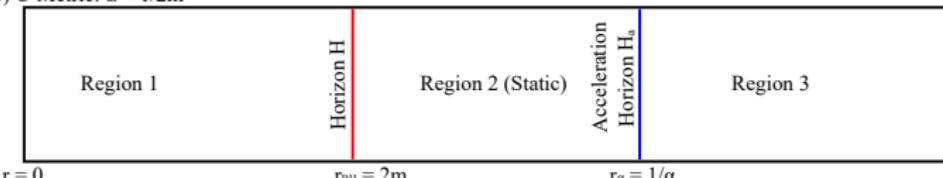
- axisymmetric and static
- curvature singularity at $r = 0$
- $\alpha \rightarrow -\alpha$ corresponds to mirroring at $\vartheta = \frac{\pi}{2}$
⇒ restriction to $\alpha > 0$
- r coordinate: $\Omega(r, \vartheta) = 0$ corresponds to conformal infinity
- ϑ, φ : coordinates on the two sphere S^2
- conical singularities on infinitessimally thin line elements on the axes
- deficit angle for $\vartheta < \frac{\pi}{2}$ and surplus angle for $\frac{\pi}{2} < \vartheta$ (increases with ϑ)

Horizon Structure: C-Metric

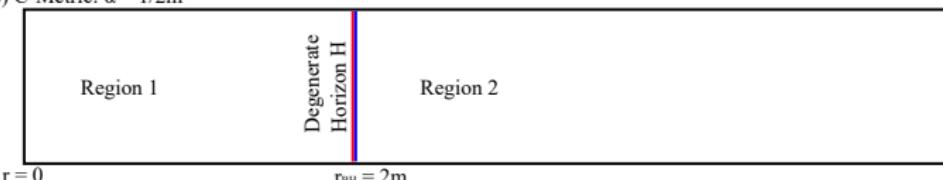
a) Schwarzschild Metric



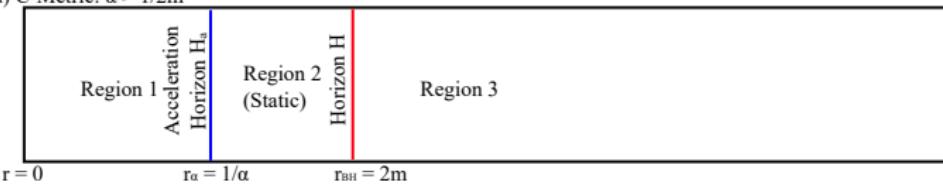
b) C-Metric: $\alpha < 1/2m$



c) C-Metric: $\alpha = 1/2m$

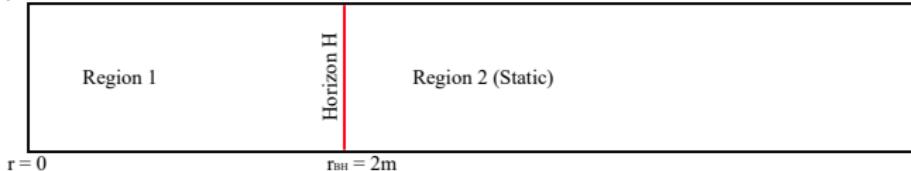


d) C-Metric: $\alpha > 1/2m$

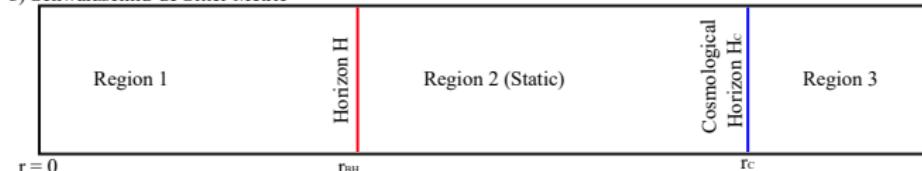


Horizon Structure: Comparison

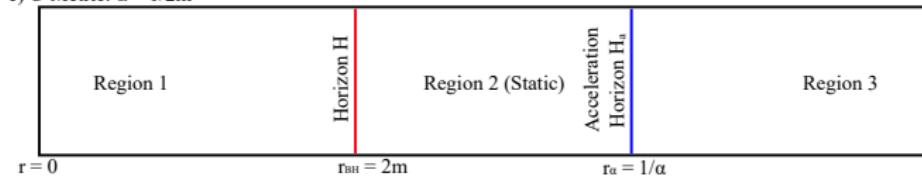
a) Schwarzschild Metric



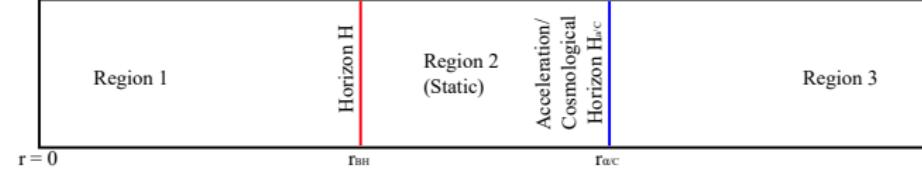
b) Schwarzschild-de Sitter Metric



c) C-Metric: $\alpha < 1/2m$



d) C-Metric with Cosmological Constant: $\alpha < 1/2m$



Equations of Motion

- equations of motion only separable for lightlike geodesics
- introducing energy E , angular momentum L_z and the Carter-constant K the EoM read (see, e.g., Grenzebach et al. [2015]):

$$\frac{dt}{d\lambda} = \frac{r^2 E}{Q(r)} \quad \left(\frac{dr}{d\lambda} \right)^2 = r^4 E^2 - r^2 Q(r) K$$

$$\left(\frac{d\vartheta}{d\lambda} \right)^2 = P(\vartheta) K - \frac{L_z^2}{\sin^2 \vartheta} \quad \frac{d\varphi}{d\lambda} = \frac{L_z}{P(\vartheta) \sin^2 \vartheta}$$

- λ : Mino-parameter (Mino [2003]; s : affine parameter)

$$\frac{d\lambda}{ds} = \frac{\Omega(r, \vartheta)^2}{r^2}$$

Instable Geodesics and Turning Points: r Motion

- instable photon orbits: $\frac{dr}{d\lambda} = \frac{d^2r}{d\lambda^2} = 0$
- Schwarzschild-de Sitter-metric: photon sphere at $r_{ph} = 3m$
- C-de Sitter-metric:

$$r_{ph} = \frac{6m}{1 + \sqrt{1 + 12\alpha^2 m^2}}$$

- cosmological constant Λ has no influence
- contained in the results of Grenzebach et al. [2015], see also Alraies Alawadi et al. [2020]
- turning points of the r motion:

$$r_{BH} < r_{max} < r_{ph} \quad r_{ph} < r_{min} < r_{\alpha/C}$$

Instable Geodesics and Turning Points: ϑ Motion

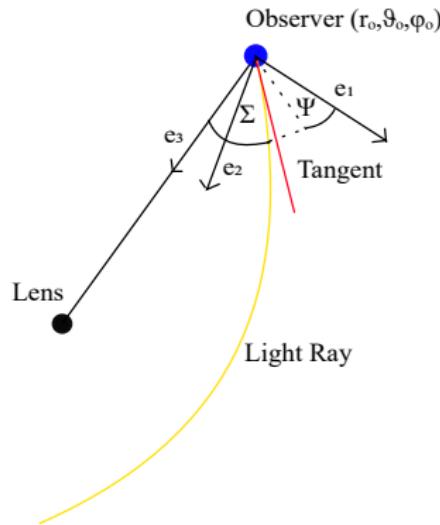
- Schwarzschild-de Sitter-metric: equatorial plane at $\vartheta = \frac{\pi}{2}$
- instable photon orbits: $\frac{d\vartheta}{d\lambda} = \frac{d^2\vartheta}{d\lambda^2} = 0$
- C-de Sitter-metric:

$$\vartheta_{ph} = \arccos \left(-\frac{2\alpha m}{1 + \sqrt{1 + 12\alpha^2 m^2}} \right)$$

- single cone with constant angle ϑ_{ph} !
- first published by Alrais Alawadi et al. [2020]
- ϑ motion bound between turning points:

$$0 \leq \vartheta_{min} \leq \vartheta \leq \vartheta_{max} \leq \pi$$

Gravitational Lensing: Observer's Sky



- astronomical convention: introduce coordinates on the celestial sphere with lens as centre (Σ : latitude; Ψ : longitude), after Fig. 4 in Grenzebach et al. [2015]

Orthonormal Tetrad

- fix observer at $(x_O^\mu) = (t_O, r_O, \vartheta_O, \varphi_O)$ between photon sphere and acceleration/cosmological horizon
- introduce orthonormal tetrad following Grenzebach et al. [2015]:

$$e_0 = \frac{\Omega(r, \vartheta)}{\sqrt{Q(r)}} \partial_t \Big|_{(x_O^\mu)} \quad e_1 = \frac{\Omega(r, \vartheta) \sqrt{P(\vartheta)}}{r} \partial_\vartheta \Big|_{(x_O^\mu)}$$

$$e_2 = - \frac{\Omega(r, \vartheta)}{r \sin \vartheta \sqrt{P(\vartheta)}} \partial_\varphi \Big|_{(x_O^\mu)} \quad e_3 = - \Omega(r, \vartheta) \sqrt{Q(r)} \partial_r \Big|_{(x_O^\mu)}$$

Relations between Constants of Motion and Angles on the Observer's Celestial Sphere

- tangent vector of the light ray in Mino parameterisation:

$$\frac{d\eta}{d\lambda} = \frac{dt}{d\lambda}\partial_t + \frac{dr}{d\lambda}\partial_r + \frac{d\vartheta}{d\lambda}\partial_\vartheta + \frac{d\varphi}{d\lambda}\partial_\varphi$$

- (x_O^μ) : use Σ and Ψ to write the tangent vector as

$$\frac{d\eta}{d\lambda} = \sigma (-e_0 + \sin \Sigma \cos \Psi e_1 + \sin \Sigma \sin \Psi e_2 + \cos \Sigma e_3)$$

- σ : normalisation constant

$$\sigma = g \left(\frac{d\eta}{d\lambda}, e_0 \right)$$

- without loss of generality: $\sigma = -r_O^2/\Omega(r_O, \vartheta_O)^2$
- relations between constants of motion and celestial angles Σ and Ψ

$$E = \frac{\sqrt{Q(r_O)}}{\Omega(r_O, \vartheta_O)}, \quad L_z = \frac{r_O \sqrt{P(\vartheta_O)} \sin \vartheta_O \sin \Sigma \sin \Psi}{\Omega(r_O, \vartheta_O)}, \quad K = \frac{r_O^2 \sin^2 \Sigma}{\Omega(r_O, \vartheta_O)^2}$$

Shadow of the Black Hole

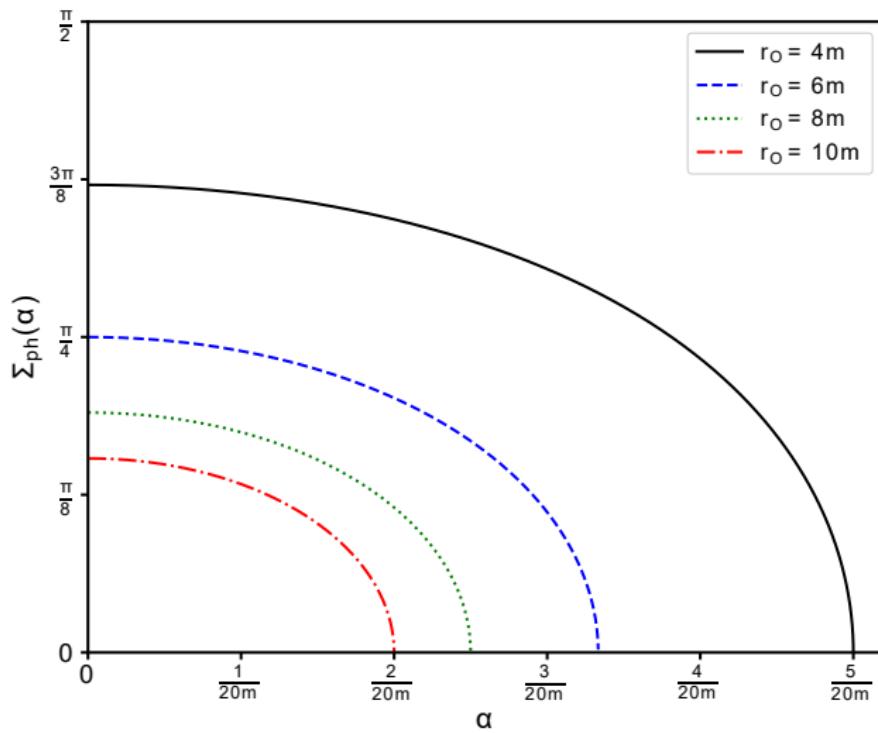
- shadow of black hole:
 - ▶ assumption 1: observer at radius coordinate $r_{ph} < r_O < r_{\alpha,C}$
 - ▶ assumption 2: light sources everywhere except between black hole and observer
- light rays asymptotically coming from the photon sphere:

$$\frac{dr}{d\lambda} \Big|_{r=r_{ph}} = 0$$

$$\Sigma_{ph} = \arcsin \left(\frac{r_{ph}}{r_O} \sqrt{\frac{Q(r_O)}{Q(r_{ph})}} \right)$$

- shadow is circular (confirms results from Grenzebach et al. [2015])

Shadow of the Black Hole: C-Metric



Lens Equation

- lens equation defines map from the celestial sphere of the observer at (x_O^μ) to the two sphere S_L^2 with radius r_L (Frittelli and Newman [1999], Perlick [2004])

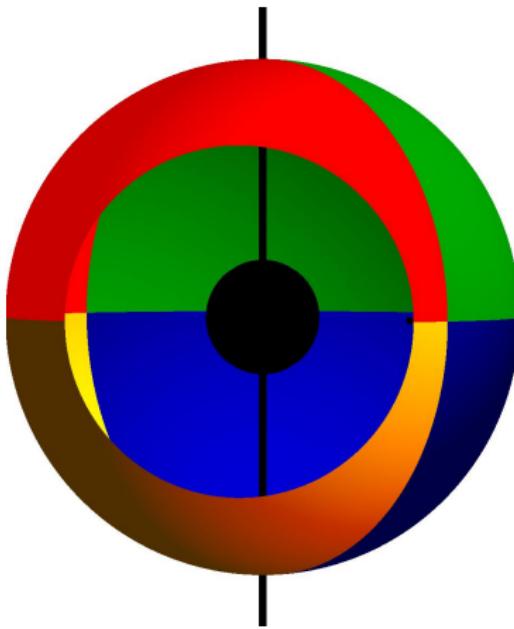
$$(\Sigma, \Psi) \rightarrow (\vartheta_L(\Sigma, \Psi), \varphi_L(\Sigma, \Psi))$$

- follow light ray ending at observer into the past until it intersects sphere of light sources at $\lambda = \lambda_L$
- choose $t_O = 0$, $\varphi_O = 0$, and $\lambda_L < \lambda_O = 0$
- step 1: calculate Mino parameter λ_L from r_O and r_L :

$$\lambda_L = \int_{r_O..}^{..r_L} \frac{\Omega(r_O, \vartheta_O) dr'}{\sqrt{Q(r_O)r'^4 - r_O^2 \sin^2 \Sigma r'^2 Q(r')}}$$

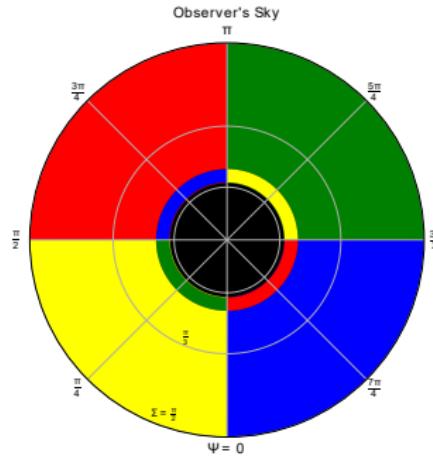
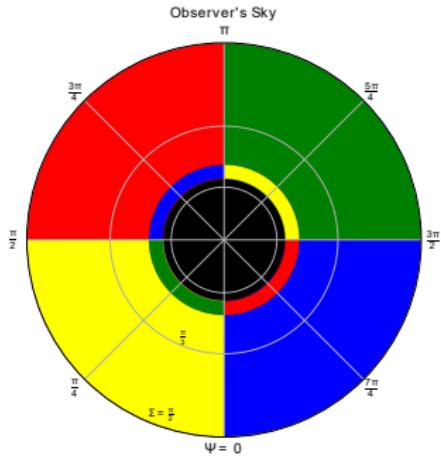
- step 2: determine number of turning points of the ϑ motion
- step 3: calculate $\vartheta_L(\Sigma, \Psi)$ and $\varphi_L(\Sigma, \Psi)$ using elliptic integrals and elliptic functions

Lens Equation for the Schwarzschild and Schwarzschild-de Sitter Metric



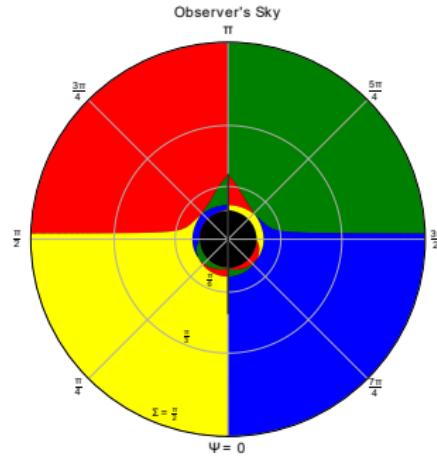
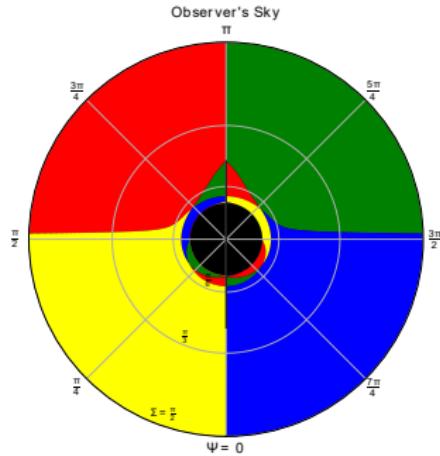
- produced by V. Perlick, colour convention after Bohn et al. [2015]
- $0 \leq \vartheta_L \leq \frac{\pi}{2}$: red/green; $\frac{\pi}{2} < \vartheta_L \leq \pi$: yellow/blue
- $0 \leq \varphi_L < \pi$: green/blue; $\pi \leq \varphi_L < 2\pi$: red/yellow

Lens Equation for the Schwarzschild and Schwarzschild-de Sitter Metric



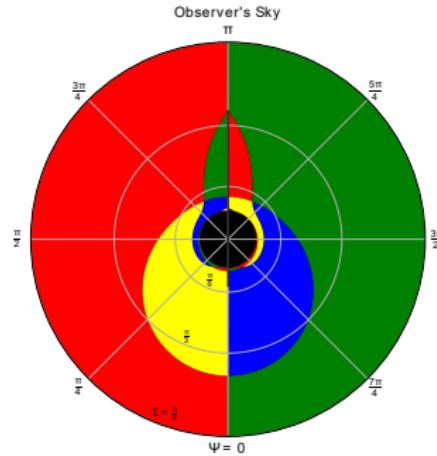
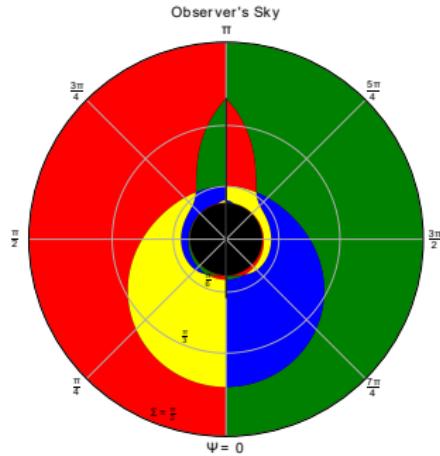
- Schwarzschild metric (left) and Schwarzschild-de Sitter metric with $\Lambda = \frac{1}{200m^2}$ (right)
- $r_O = 8m$ and $r_L = 9m$

Lens Equation for the C-Metric and the C-de Sitter-Metric for $\vartheta_O = \frac{\pi}{2}$



- C-metric (left) and C-de Sitter-metric with $\Lambda = \frac{1}{200m^2}$ (right);
 $\alpha = \frac{1}{10m}$
- $r_O = 8m$ and $r_L = 9m$

Lens Equation for the C-Metric and the C-de Sitter-Metric for $\vartheta_O = \frac{\pi}{4}$



- C-metric (left) and C-de Sitter-metric with $\Lambda = \frac{1}{200m^2}$ (right);
 $\alpha = \frac{1}{10m}$
- $r_O = 8m$ and $r_L = 9m$

Redshift Factor

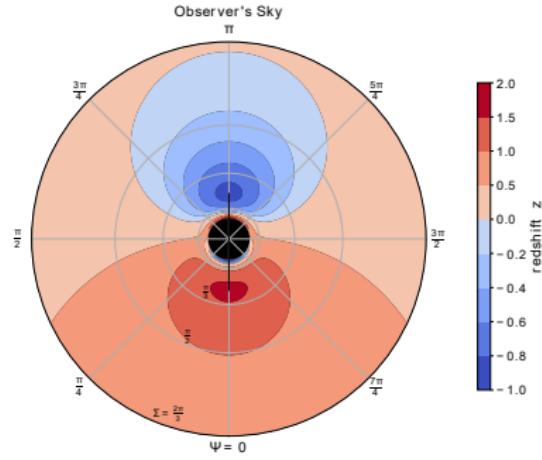
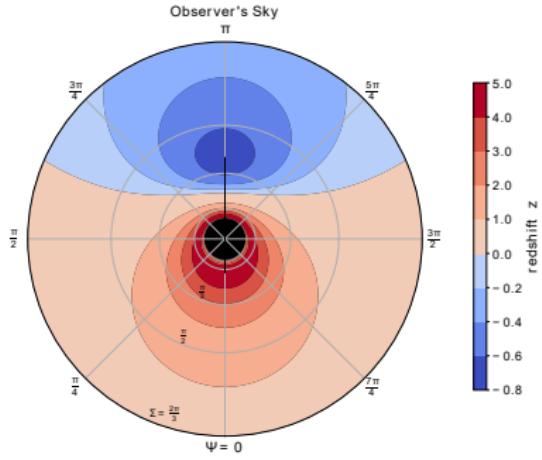
- relates energy of light ray at the position of the light source to energy measured by observer
- general redshift formula according to Straumann [2013], pp. 45:

$$z = \sqrt{\frac{g_{tt}|_{x_O}}{g_{tt}|_{x_L}}} - 1$$

- for the C-de Sitter-metric the redshift reads:

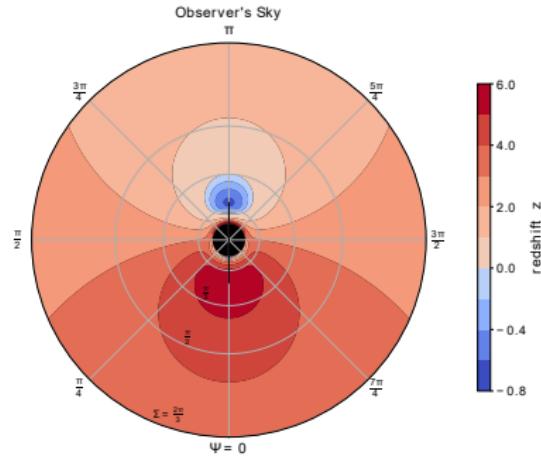
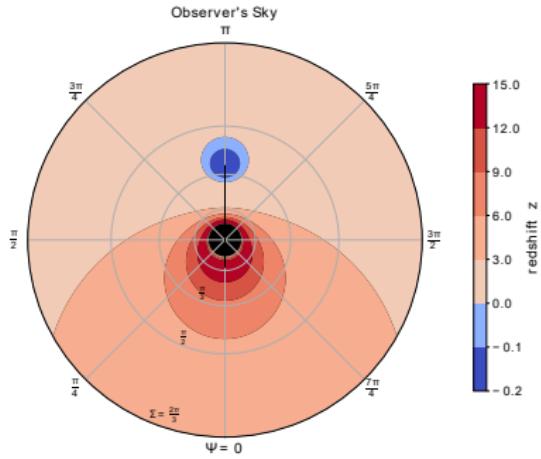
$$z = \sqrt{\frac{Q(r_O)}{Q(r_L)}} \frac{\Omega(r_L, \vartheta_L(\Sigma, \Psi))}{\Omega(r_O, \vartheta_O)} - 1$$

Redshift Factor for the C-Metric



- $\vartheta_O = \frac{\pi}{4}$ (left) and $\vartheta_O = \frac{\pi}{2}$ (right), $r_O = 8m$ and $r_L = 9m$
- $\alpha = \frac{1}{10m}$
- $z_S = -0.018$

Redshift Factor for the C-de Sitter-Metric



- $\vartheta_O = \frac{\pi}{4}$ (left) and $\vartheta_O = \frac{\pi}{2}$ (right), $r_O = 8m$ and $r_L = 9m$
- $\alpha = \frac{1}{10m}$ and $\Lambda = \frac{1}{200m^2}$
- $z_{SdS} = 0.0004$

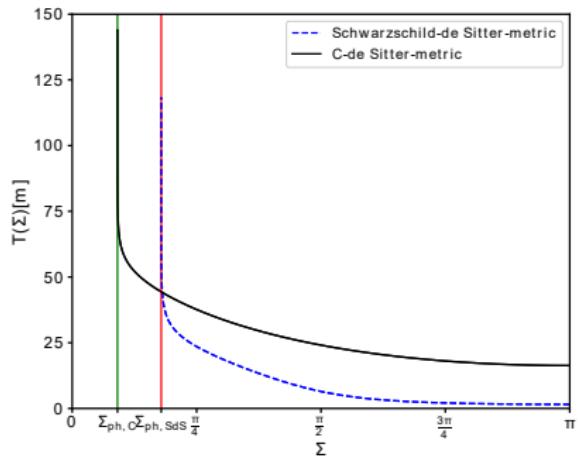
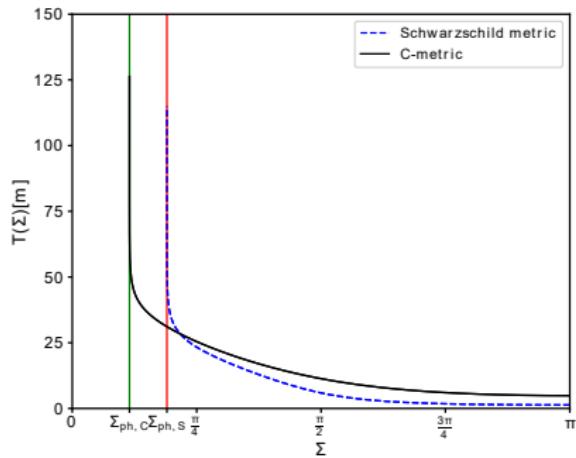
Travel Time

- measures in terms of the time coordinate time a light ray needs to travel from the source to the observer:

$$T(\Sigma) = \int_{r_O}^{r_L} \frac{\sqrt{Q(r_O)r'^2} dr'}{Q(r')\sqrt{Q(r_O)r'^4 - r_O^2 \sin^2(\Sigma)r'^2 Q(r')}}$$

- calculation using elliptic integrals of first and third kind

Travel Time in the C-de Sitter-Metric



- $r_O = 8m$ and $r_L = 9m$
- $\Lambda = \frac{1}{200m^2}$ and $\alpha = \frac{1}{10m}$

Summary and Implications for Astronomical Observations

- shadow of the black hole:
 - ▶ Λ and α both lead to a decrease of the angular diameter
 - ▶ always circular
- lens equation in presence of $\alpha > 0$:
 - ▶ axisymmetry
 - ▶ asymmetry with respect to $\vartheta = \frac{\pi}{2}$
 - ▶ qualitatively the same when $\Lambda > 0$
- redshift:
 - ▶ $\alpha > 0$: z depends on celestial coordinates
 - ▶ problem: effects very small; so might not be detectable in presence of microlensing
- travel time:
 - ▶ Λ and α both lead to an increase of the travel time
 - ▶ regularly measured: travel time differences
 - ▶ needs to be converted to proper time of the observer
- extra distance measures needed

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