Gravitational lensing and geometric optics in general relativity

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What is geometric optics?

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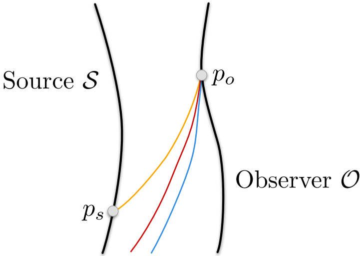
- PDEs are hard.
- ODEs aren't.
- Geometric optics turns PDEs into ODEs, at high frequencies.

Applications of geometric optics in GR

- **Gravitational lensing**: light deflection, shape distortion, intensity modulation, time delays, redshifts/blueshifts, ...
- **Gravitational wave detection**: Circulating light in interferometers, timing of radio signals from pulsars.

Typical application

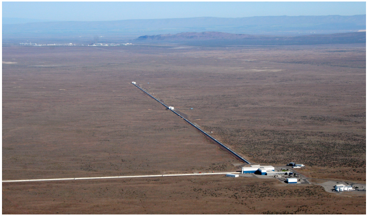
Look at future-directed null geodesics from a source to an observer.



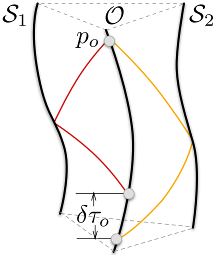
Measure: Angles, intensities, frequency shifts, etc.

Another application: interferometry

Null geodesics now circulate between three timelike worldlines.



$\Leftrightarrow$



Phase difference at p_o determines the time delay $\delta\tau_o$.

Laws of geometric optics

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How does this come out of the Maxwell's or Einstein's equations?

The high-frequency method

1. As $\omega \to \infty$, suppose that the vector potential is

$$A_a(x;\omega) = \underbrace{\mathrm{Re}\Big(e^{i\omega\varphi(x)}\big[\mathcal{A}^0_a(x)}_{\text{geometric optics}} + \omega^{-1}\mathcal{A}^1_a(x) + \mathcal{O}(\omega^{-2})\big]\Big).$$

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3. Equate powers of ω to constrain

$$(\varphi, \mathcal{A}^0_a, \ldots) \leftrightarrow A_a.$$

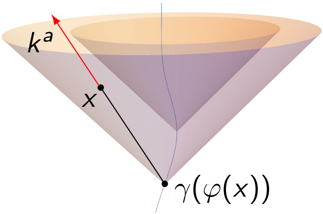
Example: Radiation from a compact source

With $A_a = \mathrm{Re}(e^{i\omega\varphi}\mathcal{A}^0_a + \ldots)$ and a point source on worldline $\gamma(\tau)$,

Eikonal $\varphi = \tau_{\mathrm{ret}}$,

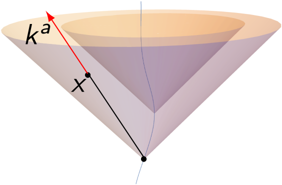
Wavevector $k_a = -\nabla_a \varphi$,

Amplitude $\mathcal{A}^0_a = \frac{1}{r} g_a{}^{a'} f_{a'}(\theta, \phi, \tau_{\mathrm{ret}})$.



1st law of geometric optics

Substituting $A_a = \mathrm{Re}(e^{i\omega\varphi}\mathcal{A}_a^0 + \ldots)$ into Maxwell's equations,



The wavevector $k_a = -\nabla_a \varphi$ is null

Rays are tangent to k^a. They are null geodesics.

The rest of geometric optics

Substituting $A_a = \mathrm{Re}(e^{i\omega\varphi}\mathcal{A}_a^0 + \ldots)$ into Maxwell's equations,

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This implies...

1. the area-intensity law,
2. that phases are constant on each ray,
3. that polarization is parallel transported.

Physically, what are the rays? I.

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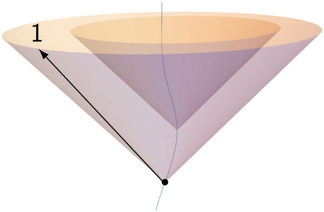

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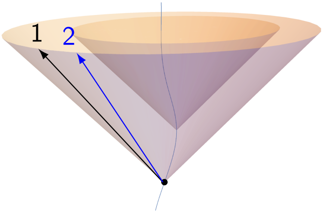

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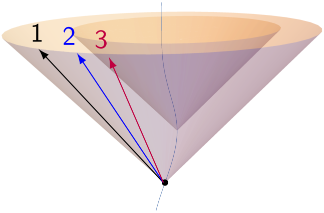

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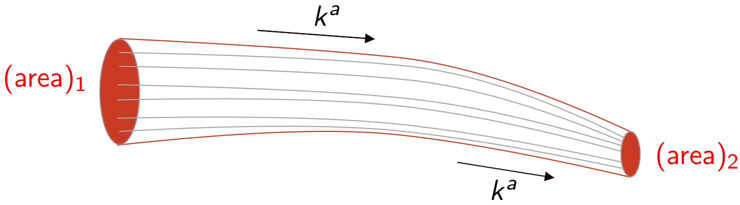
$\Rightarrow$ Any observer sees momentum coming from the ray direction:

$$(\text{momentum density})_a = T_{ab} u^b \propto k_a$$

Intensity-area law

$J^a = |\mathcal{A}_0|^2 k^a$ is conserved: $\nabla \cdot J = 0$.



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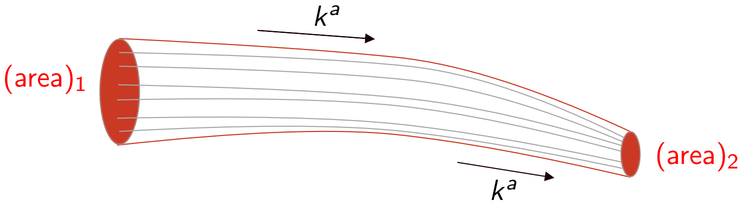

$$0 = \int_B (\nabla \cdot J) dV = \oint_{\partial B} |\mathcal{A}_0|^2 k \cdot dS \sim \Delta \left[|\mathcal{A}_0|^2 (\text{area}) \right]$$

When is geometric optics valid?

$$F_{ab} = \mathrm{Re}\big[k_{[a}\mathcal{A}^0_{b]}e^{i\omega\varphi} + \mathcal{O}(\omega^{-1})\big]$$

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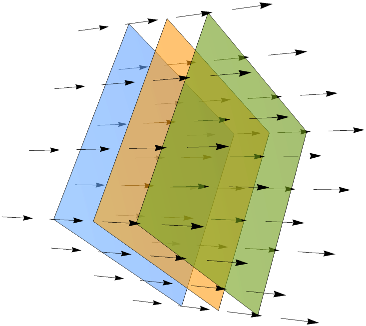
1. Lengthscales in spacetime curvature
2. Radius of curvature of wavefront

Example I: Plane waves

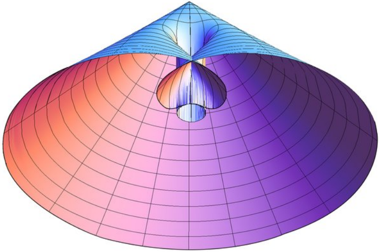
There are no finite lengthscales. . .



Every ω is large. *Geometric optics is exact.*

Example II: Caustics

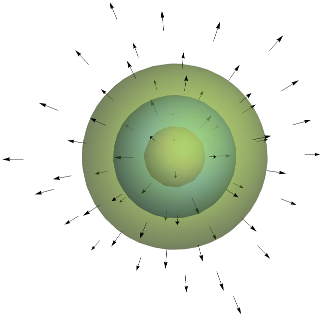
At a caustic, $\ell \to 0$.



[Source: Wardell, arXiv:0910.2634]

No ω is large. *Geometric optics breaks down.*

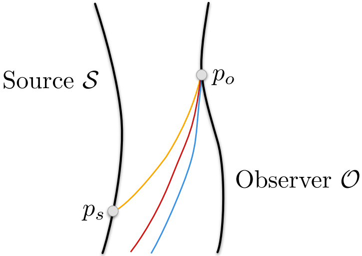
Only lengthscale is $\ell \sim r$.



Geometric optics holds when $r \gg \omega^{-1}$.

Example IV: Accelerating source

With acceleration a, there are lengthscales r and $1/a$.



Corrections are $\mathcal{O}(1/\omega r)$ and $\mathcal{O}(a/\omega)$. The latter *does not decay*!

Beyond geometric optics

$$A_a = \mathrm{Re}\big(e^{i\omega\varphi}\big[\mathcal{A}^0_a + \omega^{-1}\mathcal{A}^1_a + \mathcal{O}(\omega^{-2})\big]\big)$$

Plugging this into Maxwell's equations,

Order	Evolution eqn.	Constraint eqn.
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PDEs are turned into an infinite hierarchical sequence of ODEs.

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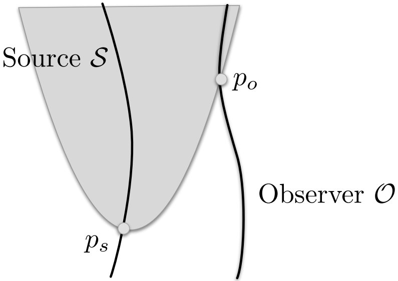
3. Polarization becomes complicated.

Tails: A paradox?

From the theory of PDEs, information propagates in null and timelike directions; there are *tails*.



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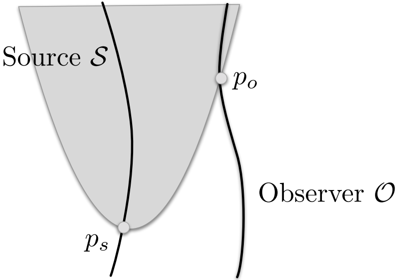

But to all orders, in the high-frequency expansion, evolution is only along null geodesics.

Resolution of the tail paradox

As $\omega \to \infty$, nonzero signals can fall off faster than *any power* of ω .

$$A_a = \mathrm{Re}\left[e^{i\omega\varphi} \left(\sum_{n=0}^{\infty} \omega^{-n} \mathcal{A}_a^n + \mathcal{O}(e^{-k\omega}) \right) \right].$$

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But are they negligible?

Conclusions

1. Geometric optics provides a foundation for gravitational lensing and connects it to the underlying fields.
2. Everything just involves ODEs along null geodesics.
3. Tails are non-perturbative in frequency.
4. Are there interesting situations where corrections are important?