General parametrization of spherically and axially symmetric black holes

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Based on:
R. K., A. Zhidenko Phys.Rev. D101 (2020) no.12, 124004
R. K., T. Pappas, A. Zhidenko Phys.Rev. D101 no.4, 044054 (2020)
R. K., A. Zhidenko Phys.Rev. D100 no.4, 044015 (2019)

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Motivation

Alternative gravitational theories are expected to solve

- Problem of quantization of gravitational field;
- Hierarchy;
- Singularities;
- Cosmological constant (dark energy);
- Dark matter.

Black holes can be used for testing strong gravity regime:

- Observations of gravitational waves by LIGO/VIRGO;
- Shadow in synchrotron plasma radiation to be observed by Event Horizon Telescope;
- Radio pulsars (coupling with the quadruple momentum).

Phenomena around a black hole

Observation

Comparison with experimental data

Simulation

Solution of equations to describe the process

Theory

A separate consideration of each alternatively theory

Observation

Comparison with experimental data

Simulation

Solution of equations to describe the process

Parameterized metric

Estimations of coefficients determining deviations from Kerr spacetime

Parameterized metric

Values of coefficients determining deviations from Kerr spacetime

Alternative theory

Metric of a black hole in each theory

Here we consider a great number of examples of BH metrics and show that a spherically symmetric asymptotically flat BH can be very well approximated by the following line element

$$\begin{aligned} ds^2 &= -N^2(r)dt^2 + B^2(r)N^{-2}(r)dr^2 + r^2 d\Omega^2, \qquad (1) \\ N^2(r) &= 1 - r_0(\epsilon+1)/r + r_0^3(\epsilon+a_1)/r^3 - r_0^4 a_1/r^4, \\ B^2(r) &= (1 + r_0^2 b_1/r^2)^2, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \end{aligned}$$

The approximation (1) can be extended to the small rotation regime as $ds_a^2 = ds^2 - (4Ma\sin^2\theta/r)dtd\phi$, which implies that corrections owing to the modification of gravity must be much larger than those due to rotation, i.e. $a/M \ll a_1, b_1$, but also that the second order corrections given by a_2 and b_2 are negligible. Thus, in the hierarchy of corrections, the above $\sim dtd\phi$-term is between the first- and second-order corrections in the radial direction.

I. Parametrization: spherical symmetry

Rezzolla, Zhidenko PRD **90**, 084009 (2014):

$$ds^2 = -N(r)^2 dt^2 + \frac{B(r)^2}{N(r)^2} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

$$N^2 = xA(x), \qquad A(x) > 0, \quad B(x) > 0, \quad x \equiv 1 - \frac{r_0}{r}.$$

Schwarzschild black hole: $A(x) = B(x) = 1$.

We introduce the parameters: $\epsilon, a_0, a_1, a_2, \ldots, b_0, b_1, b_2, \ldots$

$$\begin{aligned} A(x) &= 1 - \epsilon(1-x) + (a_0 - \epsilon)(1-x)^2 + \tilde{A}(x)(1-x)^3, \\ B(x) &= 1 + b_0(1-x) + \tilde{B}(x)(1-x)^2, \end{aligned}$$

$$\tilde{A}(x) = \frac{a_1}{1 + \frac{a_2 x}{1 + \frac{a_3 x}{1 + \ldots}}}, \qquad \tilde{B}(x) = \frac{b_1}{1 + \frac{b_2 x}{1 + \frac{b_3 x}{1 + \ldots}}}.$$

I. Two sets of parameters determining deviations from Einsteinian geometry

- ϵ, a_0, b_0 are fixed asymptotically, at $x = 1$.
  Comparison with the post-Newtonian expansion:

$$
\begin{aligned}
N^2(r) &= 1 - \frac{2M}{r} + (\beta - \gamma)\frac{2M^2}{r^2} + \mathcal{O}\left(r^{-3}\right), \\
\frac{B^2}{N} &= 1 + \gamma\frac{2M}{r} + \mathcal{O}\left(r^{-2}\right),
\end{aligned}
$$

$$
\epsilon = \frac{2M - r_0}{r_0}, \quad a_0 = \frac{(\beta - \gamma)(1+\epsilon)^2}{2}, \quad b_0 = \frac{(\gamma - 1)(1+\epsilon)}{2}.
$$

- Parameters $a_1, a_2, a_3 \ldots$ and $b_1, b_2, b_3 \ldots$ are found via comparison with the expansion of metric near the horizon ($x = 0$). Convergence of the continuous fraction allows one to approximate the metric of a black hole as well as one wants.

I. Spherical black holes

a) The **Einstein-Weyl (EW)** theory is

$$\mathcal{L} = R - \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \tag{2}$$

b) The **Einstein-scalar-Gauss-Bonnet (EsGB)** theory is described by the Lagrangian

$$\mathcal{L} = R + f(\phi) R^2_{GB} - \frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi \tag{3}$$

c) The **Einstein-scalar-Maxwell (EsM)** theory is given by the Lagrangian

$$\mathcal{L} = R - 2 g^{\mu\nu} \partial_\mu \phi \partial^\mu \phi - f(\phi) F_{\mu\nu} F^{\mu\nu} \tag{4}$$

I. Spherical black holes

theory, $\mathcal{L}$	numerical	analytical
EsGB, $f(\phi) \sim e^{\phi}$	1	5
EsGB, $f(\phi) = \frac{1}{4}\log\phi$	1	6
EsGB, $f(\phi) = \frac{1}{4}\phi^2$	2	6
EsGB, $f(\phi) = \frac{1}{4}\phi^3$	2	6
EsGB, $f(\phi) = \frac{1}{4}\phi^{-1}$	2	6
EsM, $f(\phi) = e^{-\alpha\phi^2}$	3	7
EsM, $f(\phi) = \cosh(\sqrt{-2\alpha}\phi)$	3	7
EsM, $f(\phi) = 1 - \alpha\phi^2$	3	7
EsM, $f(\phi) = \frac{1}{1+\alpha\phi^2}$	3	7
EW	8	4

Table: Analytical approximations obtained for numerical spherically symmetric BH solutions

I. Spherical black holes

1) Kanti, P.,Mavromatos, N. E., Rizos, J., Tamvakis, K., Winstanley, E. **Phys. Rev. D** 54, 5049-5058 (1996)

2) Antoniou, G. and Bakopoulos, A. and Kanti, P. **Phys. Rev. Lett.** 120, 13, 131102 (2018)

3) C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual and J. A. Font, **Phys. Rev. Lett.** 121, no. 10, 101102 (2018)

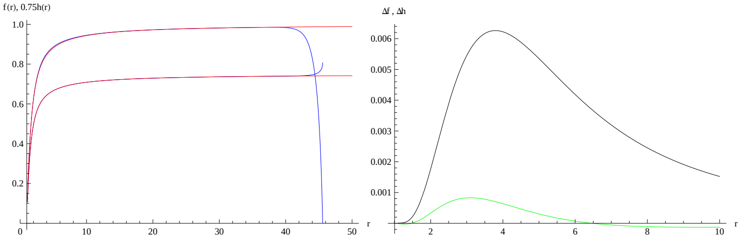
4) K. Kokkotas, R. A. Konoplya and A. Zhidenko, **Phys. Rev. D** 96, no. 6, 064007 (2017)

5) K. D. Kokkotas, R. A. Konoplya and A. Zhidenko, **Phys. Rev. D** 96, no. 6, 064004 (2017)

6) R. Konoplya, T. Pappas, A. Zhidenko, arxiv 1907.10112 (2019)

7) R. Konoplya, A. Zhidenko, **Phys. Rev. D** 100, no. 4, 044015 (2019)

8) H. Lu, A. Perkins, C. Pope, K. Stelle **Phys. Rev. Lett.** 114, no. 17, 171601 (2015)



$$ds^2 = -h(r)dt^2 + dr^2 f^{-1}(r) + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Comparison of the numerical solution and analytical approximation for the EW BH, ($r_0 = 1$, $\alpha = 0.5$). **We observe that for all the above theories the truncation of the continued fraction at the second order is sufficient to reproduce observable effects with the accuracy** $\sim 0.1\%$. **This means that only five parameters** ϵ, a_1, a_2, b_1, b_2 **provide accurate approximation for the BH spacetime.**

II. Axisymmetric black holes

Konoplya, Rezzolla, Zhidenko PRD **94**, 084025 (2016):

$$
\begin{aligned}
ds^2 &= -\frac{N^2(r,\theta) - W^2(r,\theta)\sin^2\theta}{K^2(r,\theta)}dt^2 - 2W(r,\theta)r\sin^2\theta dt d\phi \\
&+K^2(r,\theta)r^2\sin^2\theta d\phi^2 + \Sigma(r,\theta)\left(\frac{B^2(r,\theta)}{N^2(r,\theta)}dr^2 + r^2 d\theta^2\right).
\end{aligned}
$$

where $\Sigma(r,\theta) = 1 + \dfrac{a^2\cos^2\theta}{r^2}, \quad a = J/M.$

Coordinate choice:

$$K^2\left(r,\frac{\pi}{2}\right) - \frac{a}{r}W\left(r,\frac{\pi}{2}\right) = 1 + \frac{a^2}{r^2}.$$

II. Polar expansion

Expansion around the equatorial plane:

$$\begin{aligned}
N^2(r,\theta) &= xA_0(x) + \sum_{i=1}^{\infty} A_i(x)(\cos\theta)^i\,,\\
B(r,\theta) &= 1 + \sum_{i=0}^{\infty} B_i(x)(\cos\theta)^i\,,\\
W(r,\theta) &= \sum_{i=0}^{\infty} \frac{W_i(x)(\cos\theta)^i}{\Sigma}\,,\\
K^2(r,\theta) &= 1 + \frac{AW(r,\theta)}{r} + \sum_{i=0}^{\infty} \frac{K_i(x)(\cos\theta)^i}{\Sigma}\,,
\end{aligned}$$

where $x \equiv 1 - \dfrac{r_0}{r}$, r_0 is the radius of the horizon in the equatorial plane.

II. Radial expansion

$$\begin{aligned}
B_i(x) &= b_{i0}(1-x) + \tilde{B}_i(x)(1-x)^2 \,, \\
W_i(x) &= w_{i0}(1-x)^2 + \tilde{W}_i(x)(1-x)^3 \,, \\
K_i(x) &= k_{i0}(1-x)^2 + \tilde{K}_i(x)(1-x)^3 \,, \\
A_0(x) &= 1 - \epsilon_0(1-x) + (a_{00} - \epsilon_0 + k_{00})(1-x)^2 \\
&\quad + \tilde{A}_0(x)(1-x)^3 \,, \\
A_{i>0}(x) &= K_i(x) + \epsilon_i(1-x)^2 + a_{i0}(1-x)^3 + \\
&\quad + \tilde{A}_i(x)(1-x)^4 \,.
\end{aligned}$$

$$\tilde{A}_i(x) = \cfrac{a_{i1}}{1 + \cfrac{a_{i2}x}{1 + \cfrac{a_{i3}x}{1+\dots}}} \,, \qquad \tilde{B}_i(x) = \cfrac{b_{i1}}{1 + \cfrac{b_{i2}x}{1 + \cfrac{b_{i3}x}{1+\dots}}}$$

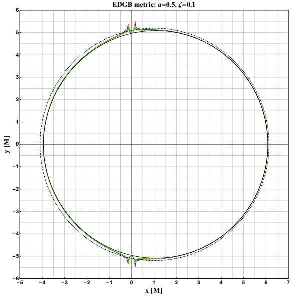
$$\tilde{W}_i(x) = \cfrac{w_{i1}}{1 + \cfrac{w_{i2}x}{1 + \cfrac{w_{i3}x}{1+\dots}}} \,, \qquad \tilde{K}_i(x) = \cfrac{k_{i1}}{1 + \cfrac{k_{i2}x}{1 + \cfrac{k_{i3}x}{1+\dots}}} \,.$$

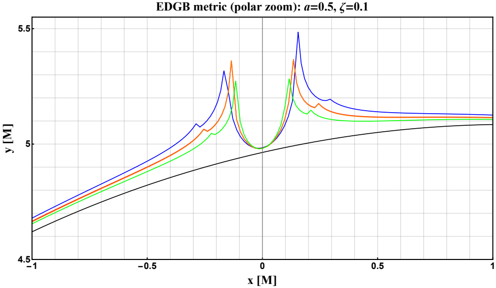
II. Coefficients determining deviations from Kerr spacetime

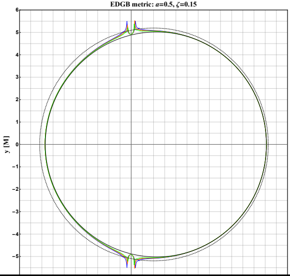
- Asymptotic parameters:
  - $\epsilon_0 = \dfrac{2M - r_0}{r_0}, \qquad k_{00} = \dfrac{a^2}{r_0^2}, \qquad w_{00} = \dfrac{2Ma}{r_0^2},$
  - a_{00} e b_{00} defined by the post-Newtonian parameters,
  - a_{20} related to the quadrupole momentum,
  - $\epsilon_i, \;\; a_{i0}, \;\; b_{i0}, \;\; w_{i0}, \;\; k_{i0}.$

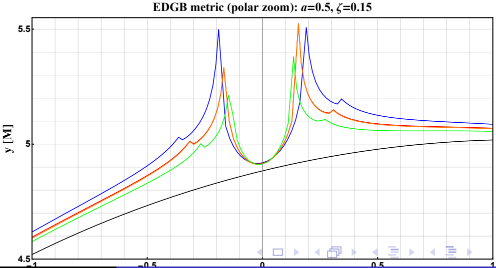
- Deformations determined by the behavior near the horizon:
  - $k_{01} = 0,$ (coordinate choice)
  - $a_{i1}, a_{i2}, a_{i3}, \ldots$ (g_{tt})
  - $w_{i1}, w_{i2}, w_{i3}, \ldots$ $(g_{t\phi})$
  - $b_{i1}, b_{i2}, b_{i3}, \ldots$ (g_{rr})
  - $k_{i1}, k_{i2}, k_{i3}, \ldots$ $(g_{\phi\phi})$

II. Example: Black holes in the Einstein-Gauss-Bonnet-dilaton theory









In this case the problem of computation of x_m is reduced to the solution of the quadric equation, hence the shadow radius (R_s) can be found in a closed but cumbersome form. It depends almost linearly on a_1 and decreases as a_1 grows,

$$\begin{aligned}\frac{r_0^2}{R_s^2} &= \frac{(1-x_0)^2 x_0^2 (2x_0-3)}{5x_0^2-10x_0+2} + (1-x_0)^5 x_0 a_1 \\ &+ \frac{(1-6x_0)^2(1-x_0)^8(5x_0^2-10x_0+2)}{12(5x_0^3-10x_0^2+5x_0-1)} a_1^2 + \mathcal{O}(a_1^3),\end{aligned} \tag{5}$$

where x_0 is the compact coordinate for the photon circular orbit, satisfying the cubic equation, $1-2\epsilon-3(1-4\epsilon)x_0-15\epsilon x_0^2+5\epsilon x_0^3=0$, and monotonously increases with ϵ. For small ϵ we find $x_0=(1/3)+(14/81)\epsilon+(154/729)\epsilon^2+(3122/19683)\epsilon^3+\mathcal{O}(\epsilon^4)$.

Similarly, for the Lyapunov exponent (λ) we obtain

$$\lambda^2 R_s^2 = \frac{3(5x_0^3 - 10x_0^2 + 5x_0 - 1)}{(5x_0^2 - 10x_0 + 2)^2(1 + b_1(1 - x_0)^2)^2} \tag{6}$$
$$-\frac{(1 + x_0)^5(120x_0^4 - 255x_0^3 + 145x_0^2 - 47x_0 + 7)}{2(5x_0^3 - 10x_0^2 + 5x_0 - 1)(1 + b_1(1 - x_0)^2)^3} b_1 a_1$$
$$-\frac{(1 + x_0)^3(5x_0^3 - 15x_0^2 - 3x_0 + 3)}{2(5x_0^3 - 10x_0^2 + 5x_0 - 1)(1 + b_1(1 - x_0)^2)^3} a_1 + \mathcal{O}(a_1^2).$$

Since both quantities depend almost linearly on a_1, one can expect that the error due to the approximation remains one order smaller than the effect as long as the metric stays moderate. If one needs to achieve the approximation in which the error would be two orders less than the effect, then the second order can be used via consideration of non-zero a_2 and b_2.

black hole	R_{sh}	effect	E_1	E_2	λ	effect	E_1	E_2
Æther1	1.666	35.9%	0	0	1.14826	198.3%	0	0
Æther2	2.043	21.4%	0	0	0.67377	75.1%	0	0
KS	2.149	17.3%	1.729%	0.1674%	0.58866	52.9%	5.234%	0.5282%
HE	1.929	25.7%	2.871%	0.3659%	0.82911	115.4%	7.958%	0.4749%
Hayward	3.972	52.9%	4.031%	3.3394%	0.20282	47.3%	2.213%	2.6678%
Bronnikov	3.687	41.9%	0.126%	0.0323%	0.18628	51.6%	0.158%	0.1026%
Bardeen	3.247	25.0%	0.194%	0.1486%	0.23945	37.8%	0.624%	0.5249%
EdM	3.266	25.7%	0.078%	0.0229%	0.24206	37.1%	0.974%	0.1061%
EsM	3.084	18.7%	0.582%	0.3303%	0.27603	28.3%	3.120%	2.1431%
E-Weyl	1.916	26.3%	0.664%	0.5862%	0.72329	87.9%	0.905%	0.7578%
CFM1	2.598	0	0	0	0.54433	41.4%	1.823%	0.1732%
JP1	2.027	22.0%	0	0	0.42855	11.3%	0.518%	0.0064%
EdGB	2.700	3.9%	0.345%	0.2299%	0.36206	5.9%	5.613%	1.3019%
EsGB1	2.699	3.9%	0.386%	0.2206%	0.36245	5.8%	5.494%	1.2332%
EsGB2	2.868	10.4%	1.197%	1.0279%	0.32947	14.4%	12.916%	1.3233%
CFM2	2.598	0	0	0	0.31740	17.5%	56.759%	4.330%
JP2	2.270	12.6%	28.91%	9.4261%	0.43759	13.7%	5.978%	14.963%

Radius of shadow, Lyapunov exponent for a number of BHs, the relative effect compared to the Schwarzschild, and relative errors, E_1 and E_2, due to approximations of the first and second orders, respectively.

black hole Ω_{ISCO}	effect	E_1	E_2	
Æther1	0.030101	77.9%	0	0
Æther2	0.046342	65.9%	0	0
KS	0.117155	13.1%	4.474%	0.4327%
HE	0.158422	16.4%	10.443%	3.7952%
Hayward	0.092482	32.0%	9.583%	7.9520%
Bronnikov	0.120621	11.4%	0.291%	0.0657%
Bardeen	0.121428	10.8%	0.405%	0.2966%
EdM	0.138402	1.7%	0.172%	0.0412%
EsM	0.143746	5.6%	1.694%	0.7214%
E-Weyl	0.026784	80.3%	35.057%	35.3715%
CFM1	0.136083	0	0	0
JP1	0.138963	2.1%	0	0
EdGB	0.131958	3.0%	0.739%	0.6754%
EsGB1	0.132027	3.0%	0.900%	0.5882%
EsGB2	0.127488	6.3%	2.786%	2.3443%
CFM2	0.136083	0	0	0
JP2	0.310425	128.1%	87.588%	13.2753%

Frequency at ISCO for a number of BHs, the relative effect compared to the Schwarzschild, and relative errors, E_1 and E_2, due to approximations of the first and second orders, respectively.

Further reading

- Parameterized BHs in the Einsteinian cubic gravity:
  R. A. Hennigar, M. Poshteh, R. B. Mann Phys.Rev. D97 (2018) no.6, 064041
- Parametrized BHs in quartic gravity:
  H. Khodabakhshi, A. Giaimo, R. B. Mann Phys.Rev. D102 (2020) no.4, 044038
- General spherically symmetric parametrization for D-dimensional black holes:
  R. K., T. D. Pappas, Z. Stuchlík [arXiv:2007.14860], Phys. Rev. D, in press (2020)

Conclusions

The general parametrization of a black-hole spacetime in arbitrary metric theories of gravity includes an infinite set of parameters. It is natural to suppose that essential astrophysically observable quantities, such as quasinormal modes, parameters of shadow, electromagnetic radiation and accreting matter in the vicinity of a black hole, must depend only on a few of these parameters. Starting from the parametrization for spherically symmetric configurations in the form of infinite continued fraction, we suggest a compact representation of the asymptotically flat spherically symmetric and slowly rotating black holes in terms of only three and four parameters respectively. This approximate representation of a black-hole metric should allow one to describe physical observables in the region of strong gravity.