Quark number susceptibilities from resummed perturbative QCD

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In collaboration with Jens O. Andersen (Trondheim), Nan Su and Aleksi Vuorinen (Bielefeld).
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2 Hard-Thermal-Loop perturbation theory

- Basic formalism
- About the leading order result
- Branch cuts technique
- High-T truncation vs weak coupling expansion
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Why finite $T$ and $\mu$ QCD via analytical techniques?
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- What do we intend to do here?
  1. Studying to which extent one can understand recent high precision lattice data, using weakly coupled quasiparticle picture.
Problem of convergence of the series

\[ N_f=3, \text{ weak coupling expansion up to } \alpha_s^2 \log[\alpha_s]. \]
RECALLS
- Imaginary time formalism,
- Dimensional regularization in the momentum space: 
  \[ T \sum P_0 \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}}, \]
- Lagrangian density of QCD:
  \[
  \mathcal{L}_{\text{QCD}} = -\frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) + i \bar{\psi} \gamma^\mu D_\mu \psi \\
  + \mathcal{L}_{gf} + \mathcal{L}_{\text{ghost}} + \Delta \mathcal{L}_{\text{QCD}},
  \]
- Relation between the thermodynamic potential \( \Omega \) and the diagonal quark number susceptibility \( \chi \):
  \[
  \chi (T) = -\frac{\partial^2 \Omega (T, \mu)}{\partial \mu^2} \bigg|_{\mu=0}.
  \]
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BASIC FORMALISM
Reorganizing the perturbative series of thermal QCD:

\[ \mathcal{L} = (\mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{HTL}}) \bigg|_{g \to \sqrt{\delta} g} + \Delta \mathcal{L}_{\text{HTL}}, \]

With the HTL improvement term:

\[ \mathcal{L}_{\text{HTL}} = -\frac{1}{2} (1 - \delta) m_D^2 \text{Tr} \left( F_{\mu\alpha} \left\langle \frac{y^\alpha y^\beta}{(y \cdot D)^2} \right\rangle y^- F_{\mu\beta} \right) \]

\[ + (1 - \delta) i m_q^2 \bar{\psi} \gamma^\mu \left\langle \frac{y_\mu}{y \cdot D} \right\rangle y^+ \psi, \]

\( \delta \): formal expansion parameter, \# of HTL dressed loops; 
\( \Delta \mathcal{L}_{\text{HTL}} \): HTL counterterm(s), \( y^\mu \): lightlike 4-vector; 
\( m_D/m_q \): Debye/quark thermal mass parameters,

Adding \( \mathcal{L}_{\text{HTL}} \) shifts the expansion to an ideal gas of thermal quasiparticles.
ABOUT THE LEADING ORDER RESULT
Prescription for the mass parameters:

1-loop $m_D$ and $m_q$ at finite $T$ and $\mu$, i.e.

$$m_D^2 \equiv \alpha_s \pi \left( \frac{4N_c}{3} T^2 + 2N_f \frac{\mu^2}{\pi^2} \right),$$

$$m_q^2 \equiv \alpha_s \pi \frac{N_c^2 - 1}{4N_c} \left( T^2 + \frac{\mu^2}{\pi^2} \right),$$

Tricky sum-integrals! Integrands with powers of $T_P$:

$$\left( T_P \right)^m \equiv \left( \frac{\Gamma \left( \frac{3}{2} - \epsilon \right)}{\Gamma \left( \frac{3}{2} \right) \Gamma \left( 1 - \epsilon \right)} \right)^m \left( P_0^2 \right)^m \int_0^1 dc_1 \ldots \int_0^1 dc_m \left\{ \frac{(1 - c_1^2)^{-\epsilon}}{(P_0^2 + p^2 c_1^2)} \right\} \ldots \left\{ \frac{(1 - c_m^2)^{-\epsilon}}{(P_0^2 + p^2 c_m^2)} \right\} \equiv \left( 2 \, {}_2F_1 \left( \frac{1}{2}, 1, \frac{3}{2} - \epsilon; -\frac{p^2}{P_0^2} \right) \right)^m,$$
About the leading order result

which look like:

\[ \sum_{\{P\}} \left[ \frac{(P_0)^u (T_P)^m}{(p^2)^w (P^2)^n} \right] \]

- Possible to find analytical expressions in term of hypergeometric functions, for the first powers of $T_P$ (for example in the case of $m = 1$ and $m = 2$),
- Those are needed to compute the High-T truncation, but what about the full result?
- Alternative methods: The branch cuts.
Branch cuts technique
As a first step: from a discrete summation to a contour integral,

'Constraints' given by the branch points, according to the integrand,

Choice of the branch cuts,
Branch cuts

\[ \text{Diagram showing branch cuts with various paths and labels}} \]
Then, renormalization by addition and subtraction of some relevant integrals; those contain the UV divergences without introducing any IR ones,

Must be computable analytically, in order to extract the divergence(s),
\[
\frac{\Omega_{\text{HTL}}^{\text{LO}}}{-d_A \frac{\pi^2 T^4}{45}} \equiv \left\{ \frac{90}{\pi^4} \frac{d_F}{d_A} \left[ \text{Li}_4 \left( -e^{-\beta \mu} \right) + \text{Li}_4 \left( -e^{+\beta \mu} \right) \right] - \frac{1}{2} \right\} \\
+ \left\{ -\frac{15}{4} \right\} \bar{m}_D^2 + \left\{ \frac{45}{4} \log \left( \frac{\bar{\Lambda}}{m_D} \right) \right\} m_D^4 \\
+ \left\{ -\frac{45}{2\pi^4 T^3} \left[ 2 \tilde{I}_{BT} + \tilde{I}_{BL} \right] \right\} \\
+ \frac{d_F}{d_A} \frac{45}{\pi^4 T^3} \left[ \tilde{I}_{F\sim}^+ + \tilde{I}_{F\sim}^- + \tilde{I}_{F\sim}^+ + \tilde{I}_{F\sim}^- \right] \\
+ \frac{45}{2\pi^5 T^4} \left[ \tilde{K}_{BT} \right] - \frac{d_F}{d_A} \frac{45}{\pi^5 T^4} \left[ \tilde{K}_F^+ + \tilde{K}_F^- \right] \right\} \\
+ \left\{ \text{cte}_1 \right\} \bar{m}_D^4 + \left\{ -\frac{d_F}{d_A} \left( \text{cte}_2 \right) \right\} m_q^4, 
\]
High-T truncation vs weak coupling expansion
Scalar sum-integrals expanded in power of $m_D/T$ and $m_q/T$; then truncated at, and including the order $m_D^4/m_q$.

The mass parameters are then treated as $\sim O(\alpha_S^{1/2})$.

Therefore, this truncation is suitable to compare our result with the known weak coupling expansion up to $\alpha_S^2 \log (\alpha_S)$.

Note that the full result includes higher order contributions, to all order beyond this truncation.
High-T truncation vs weak coupling expansion

\[ \frac{X_B}{X_B^0} \text{ vs } T \text{ (MeV)} \]

- **Free case**
- **High-T HTLpt**
- **Weak coupling exp.**
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QNS from resummed perturbative QCD
BASIC FORMALISM
Separation of the pressure into different contributions, according to their typical momentum scales:

\[ p_{QCD} \equiv p_{\text{hard}} + p_{\text{soft}}, \]

- **\( p_{\text{hard}} \):** From the hard modes \((\propto 2\pi T)\), computable via strict perturbative expansion in the 4D theory,
- **\( p_{\text{soft}} \):** From the soft modes \((\propto gT)\), computable via the effective 3D Yang-Mills plus adjoint Higgs theory (EQCD),

Lagrangian density of EQCD:

\[
\mathcal{L}_{\text{EQCD}} = \frac{1}{2} \text{Tr} F_{ij}^2 + \text{Tr} [D_i, A_0]^2 + m_E^2 \text{Tr} A_0^2 + ig^3 \sum_f \mu_f \text{Tr} A_0^3 + \delta \mathcal{L}_E.
\]
ABOUT THE 4-LOOP RESULT
About the 4-loop result

- Already known 4-loop result,
- New resummation scheme, applied to finite $\mu$ case,
- Effect of the strange quark mass.
NEW RESUMMATION SCHEME
New resummation scheme

- Amount to let all of the DR parameters unexpanded,
- Resums to all order a certain class of diagrams,
- Substantially reduce the renormalization scale dependance.
EFFECT OF THE STRANGE QUARK MASS
- rescaling of massive and massless results for the pressure, by appropriate ratio of Stefan-Boltzmann law,

- As the quark contribution to the pressure converge faster than the series as a whole, a rescaling by an appropriate ratio of first order correction to the Stefan-Boltzmann law would not quite modify the story (∼ 5%),

- Hence, good phenomenological way to guess the effect of the presence of a non-vanishing mass for the strange quark.
Effect of the strange quark mass in DR
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Renormalization scale chosen to be an optimal scale; 1-loop corrections to the electric screening mass vanish; In practise, $\Lambda_{N_f=2} \approx 1.29 \times 2\pi T$ and $\Lambda_{N_f=3} \approx 1.44 \times 2\pi T$,

- $\Lambda_{\overline{MS}} = 180 - 200$ MeV for DR, and $\Lambda_{\overline{MS}} \approx 140 - 160$ MeV for HTLpt, respectively for $N_f = 2$ and $N_f = 3$. 

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Computation of $\chi$ in QCD via two different resummation schemes,

The results are consistent with recent lattice data down to $T \approx 250 - 300$ MeV!,

HTLpt results are from the high T expansion of the free quasiparticle gas (1-loop) seem to account for a next to leading order computation; full computation underway

References for previous investigation of the QNS: [A. Vuorinen, 2003],

References for the DR resummation scheme: [M. Laine, Y. Schroder, 2006],

References for the Lattice data: [S. Borsanyi et al., A. Bazavov et al., 2012],

For recent 3-loop HTLpt work: [J. O. Andersen, L. E. Leganger, M. Strickland, and Nan Su, 2011],
\[ \tilde{\mathcal{I}}_{B_{T,L}} \equiv \tilde{\mathcal{I}}_{B_{T,L}}(T, \mu) \equiv \int_0^{\infty} dk \, k^2 \log \left( 1 - e^{-\beta \omega_{T,L}} \right), \]

\[ \tilde{\mathcal{I}}_{F_{\omega \pm}} \equiv \tilde{\mathcal{I}}_{F_{\omega \pm}}(T, \mu) \equiv \int_0^{\infty} dk \, k^2 \log \left( 1 + e^{-\beta \left[ \omega \pm \mu \right]} \right), \]

\[ \tilde{\mathcal{K}}_{B_{TL}} \equiv \tilde{\mathcal{K}}_{B_{TL}}(T, \mu) \equiv \int_0^{\infty} d\omega \, \frac{1}{e^{\beta \omega} - 1} \int_0^{\infty} dk \, k^2 \left( 2\phi_T - \phi_L \right), \]

\[ \tilde{\mathcal{K}}_{F} \equiv \tilde{\mathcal{K}}_{F}(T, \mu) \equiv \int_0^{\infty} d\omega \, \frac{1}{e^{\beta \left[ \omega \pm \mu \right]} + 1} \int_0^{\infty} dk \, k^2 \theta_q, \]