Heavy Mesonic Spectral Functions at Finite Temperature and Finite Momentum

Masayuki Asakawa

Department of Physics, Osaka University
QCD Phase Diagram

- LHC
- RHIC
- 160-190 MeV
- CP (critical point)
- 1st order
- order ?
- CSC (color superconductivity)
- Hadron Phase
  - chiral symmetry breaking
  - confinement
- QGP (quark-gluon plasma)
At finite momentum,

- S and PS channels:
  \[ \rho(\omega, \vec{p}) \]

- V and AV channels:
  \[ \rho_{\mu\nu}(\omega, \vec{p}) = \rho_T(\omega, \vec{p})(P_T)_{\mu\nu} + \rho_L(\omega, \vec{p})(P_L)_{\mu\nu} \]

  \(P_T(P_L)\): transverse (longitudinal) projection operator

Hadrons @ finite temperature and finite momentum,

- potential model ?
- collisions with gluons and quarks (hot wind effect)
  \[ \rightarrow p_T \text{ dependence of } J/\psi \text{ yield etc.} \]
- dispersion relation
- difference between \(\rho_T\) and \(\rho_L\)
- dilepton spectra (\(\rho_T\) and \(\rho_L\)) and photon spectra (\(\rho_T\), on-shell)
What’s measured on the Lattice is
Imaginary Time Correlation Function $D(\tau)$

$$D(\tau) = \int \left\langle O(\tau, \vec{x}) O^{\dagger}(0, \vec{0}) \right\rangle d^3x$$

$D(\tau)$ and $A(\omega) \equiv A(\omega, \vec{0})$ are related by

$$D(\tau) = \int_0^\infty K(\tau, \omega) A(\omega) d\omega$$

$K(\tau, \omega)$: Known Kernel

However,

- Measured in Imaginary Time
- Measured at a Finite Number of discrete points
- Noisy Data

Monte Carlo Method

Direct Inversion: ill-posed !

$\chi^2$-fitting: inconclusive !
Similar Difficulties in Many Areas

- **Lattice**
  - Analytic Continuation to Imaginary Time is measured
  - Measured at a Finite Number of discrete points
  - *Noisy* Data

- **X-ray Diffraction Measurement in Crystallography**
  - Fourier Transformed images are measured
  - Measured at a Finite Number of data points
  - *Noisy* Data

- **Observational Astronomy**
  - Smeared Images due to Finite Resolution are measured
  - Measured by a Finite Number of Pixels
  - *Noisy* Data
MEM

- **Maximum Entropy Method**
  - a method to infer the most statistically probable image (such as $A(\omega)$) given data, instead of solving the (ill-posed) inversion problem

\[
D(\tau) = \int_0^\infty K(\tau, \omega) A(\omega) \, d\omega
\]

- Theoretical Basis: Bayes’ Theorem

\[
P[X|Y] = \frac{P[Y|X]P[X]}{P[Y]}
\]

- Probability of $X$ given $Y$

- In Lattice QCD

$D$: Lattice Data (Average, Variance, Correlation…etc.)

$H$: All definitions and prior knowledge such as $A(\omega) \geq 0$

Bayes' Theorem

\[
P[A|DH] \propto P[D|AH]P[A|H]
\]
Ingredients of MEM

- \( P[D|AH] \)
  \[ = \chi^2\text{-likelihood function} \]
  \[ P[D|AH] = \exp(-L)/Z_L \]

- \( P[A|H] \)
  given by Shannon-Jaynes Entropy

\[
P[A|\alpha m] = \frac{\exp(\alpha S)}{Z_S}
\]

\[
S = \int \left[ A(\omega) - m(\omega) - A(\omega) \log \left( \frac{A(\omega)}{m(\omega)} \right) \right] d\omega
\]

\[
Z_S = \int e^{\alpha S}[dA], \quad \alpha \in \mathbb{R}
\]

Default Model \( m(\omega) \in \mathbb{R} \):
Prior knowledge about \( A(\omega) \)

such as semi-positivity, perturbative asymptotic value, operator renormalization etc.

For further details,
Result of Mock Data Analysis

N(# of data points)-b(noise level) dependence
Result of Mock Data Analysis (cont’d)

N(# of data points)-b(noise level) dependence
Stat. and Syst. Error Analyses in MEM

Generally,

The Larger the Number of Data Points and the Lower the Noise Level

The Closer the Result is to the Original Image

Need to do the following:

• Put Error Bars and Make Sure Observed Structures are Statistically Significant

• Change the Number of Data Points and Make Sure the Result does not Change

in any MEM analysis
Importance of Error Analysis 1

- In MEM analysis, output image itself has little significance
- Only average is of significance

Unfortunately, still many people do MEM analyses without error bars
Importance of Error Analysis 2

From our old paper

\[ T = 1.62T_c \]

\[ T = 1.70T_c \]

V channel \( \langle J/\psi \rangle \)

\[ N_\sigma = 32 \]
Error Analysis in MEM (Statistical)

- MEM is based on Bayesian Probability Theory
  - In MEM, Errors can be and must be assigned
  - This procedure is essential in MEM Analysis

For example, Error Bars can be put to

Average of Spectral Function in $I = [\omega_1, \omega_2]$, 

$$
\langle A_\alpha \rangle_I = \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} A_\alpha(\omega) d\omega
$$

$$
\left\langle \left( \delta A_\alpha \right)^2 \right\rangle_I = \frac{1}{(\omega_2 - \omega_1)^2} \int [dA] \int_{I \times I} d\omega d\omega' \delta A(\omega) \delta A(\omega') P[A | DH \alpha m]
$$

$$
\simeq - \frac{1}{(\omega_2 - \omega_1)^2} \int_{I \times I} d\omega d\omega' \left( \frac{\delta^2 Q(A)}{\delta A(\omega) \delta A(\omega')} \right)_{A = A_\alpha}^{-1}
$$

$$
\delta A(\omega) = A(\omega) - A_\alpha(\omega)
$$

$$
Q(A) = \alpha S - L
$$

$$
[dA] = \prod_{l=1}^{N_\alpha} \frac{dA_l}{\sqrt{A_l}}
$$

For further details, Y. Nakahara, and T. Hatsuda, and M. A., Prog. Part. Nucl. Phys. 46 (2001) 459
Statistical Error Analysis ≠ Jackknife

Lattice fluctuation: included in $\chi^2$, i.e. $P[D|AH]$

On the other hand, MEM calculates $P[A|DH] \propto P[D|AH]P[A|H]$

Statistical error analysis in MEM includes more than Lattice fluctuation

Jackknife Method misses $P[A|H]$ part

Statistical Error with Jackknife Method

Ding et al. (2010)
Stat. and Syst. Error Analyses in MEM

Generally,

The Larger the Number of Data Points and the Lower the Noise Level

The Closer the Result is to the Original Image

Need to do the following:

• Put Error Bars and Make Sure Observed Structures are Statistically Significant

• Change the Number of Data Points and Make Sure the Result does not Change

in any MEM analysis
Importance of Error Analysis 3

- Necessity to use large $\beta (=6/g^2)$ to have many points in $\tau$
- Necessity to change $N_{\text{data}}$

40$^3 \times 30$ lattice
$\beta = 6.47$, 150 confs.
Isotropic lattice ($T<T_c$)
Importance of Error Analysis 4

\[ N_\tau = 46 \ (T = 1.62T_c) \]
V channel \((J/\psi)\)
\[ N_\sigma = 32 \]

\[ N_\tau = 40 \ (T = 1.87T_c) \]
V channel \((J/\psi)\)
\[ N_\sigma = 32 \]
Lattice Parameters

1. Lattice Sizes
   \[64^3 \times N_t\] (previously \[32^3 \times N_t\])

2. \[\beta = 7.0, \quad \xi_0 = 3.5\]
   \[\xi = a_\sigma/a_\tau = 4.0\] (anisotropic)

3. \[a_\tau = 9.75 \times 10^{-3}\] fm
   \[L_\sigma = 2.5\] fm

4. Standard Plaquette Action

5. Wilson Fermion

6. Heatbath: Overrelaxation
   \[= 1 : 4\]
   1000 sweeps between measurements

7. Quenched Approximation

8. Gauge Unfixed

9. Machine: PACS-CS @University of Tsukuba
   BlueGene @KEK
   φ @KMI Nagoya University
Lattice Parameters and # of Conf.

- anisotropic lattice: $\xi = a_\sigma/a_\tau = 4$
- $\beta = 7.0$, $a_\tau = 9.75 \times 10^{-3}$ fm
- large spatial volume: $N_\sigma = 64$ ($L_\sigma = 2.5$ fm)
- $P_{\text{min}} \sim 0.5$ GeV (previously $\sim 1$ GeV)

✓ Large spatial volume: to have small enough $P_{\text{min}}$ to study finite T effects
✓ Large $\beta = $ small $a_\tau$: to have large enough $N_\tau$ for MEM analysis

<table>
<thead>
<tr>
<th>$N_\tau$ ($T/T_c$)</th>
<th>96 (0.78)</th>
<th>54 (1.38)</th>
<th>46 (1.62)</th>
<th>44 (1.70)</th>
<th>42 (1.78)</th>
<th>40 (1.87)</th>
<th>32 (2.33)</th>
</tr>
</thead>
<tbody>
<tr>
<td># of conf.</td>
<td>260</td>
<td>400</td>
<td>400</td>
<td>400</td>
<td>400</td>
<td>400</td>
<td>400</td>
</tr>
</tbody>
</table>
Our Old Result \((N_\sigma=32)\) for PS channel \((\eta_c)\)

\[\eta_c (p = 0)\] disappears between 1.62\(T_c\) and 1.70\(T_c\)

Asakawa and Hatsuda, PRL 2004
PS channel with $N_\sigma=64$

<table>
<thead>
<tr>
<th>$N_\tau$</th>
<th>96 (0.78)</th>
<th>54 (1.38)</th>
<th>46 (1.62)</th>
<th>44 (1.70)</th>
<th>42 (1.78)</th>
<th>40 (1.87)</th>
<th>32 (2.33)</th>
</tr>
</thead>
</table>

PS: $T=0.78T_c$, $p=0$
**PS channel with $N_\sigma=64$**

<table>
<thead>
<tr>
<th>$N_\tau$ ($T/T_c$)</th>
<th>96 (0.78)</th>
<th>54 (1.38)</th>
<th>46 (1.62)</th>
<th>44 (1.70)</th>
<th>42 (1.78)</th>
<th>40 (1.87)</th>
<th>32 (2.33)</th>
</tr>
</thead>
</table>

![Graphs showing $\rho$ vs $\omega$ for different $T/T_c$ values](image)
**PS channel with $N_\sigma=64$**

<table>
<thead>
<tr>
<th>$N_\tau$ ($T/T_c$)</th>
<th>96 (0.78)</th>
<th>54 (1.38)</th>
<th>46 (1.62)</th>
<th>44 (1.70)</th>
<th>42 (1.78)</th>
<th>40 (1.87)</th>
<th>32 (2.33)</th>
</tr>
</thead>
</table>

![Graph showing PS channel with different $N_\tau$ values](image-url)
**PS channel with \( N_\sigma = 64 \)**

<table>
<thead>
<tr>
<th>( N_\tau (T/T_c) )</th>
<th>96 (0.78)</th>
<th>54 (1.38)</th>
<th>46 (1.62)</th>
<th>44 (1.70)</th>
<th>42 (1.78)</th>
<th>40 (1.87)</th>
<th>32 (2.33)</th>
</tr>
</thead>
</table>

\( \eta_c \) melts between \( 1.62T_c \) and \( 1.70T_c \)
**PS channel with \( N_\sigma = 64 \)**

<table>
<thead>
<tr>
<th>( N_\xi ) (( T/T_c ))</th>
<th>96 (0.78)</th>
<th>54 (1.38)</th>
<th>46 (1.62)</th>
<th>44 (1.70)</th>
<th>42 (1.78)</th>
<th>40 (1.87)</th>
<th>32 (2.33)</th>
</tr>
</thead>
</table>

![Graph showing the relationship between \( \omega \) (in GeV) and \( \rho \) for different values of \( N_\xi \) and \( T/T_c \).]
PS channel with $N_\sigma=64$

| $N_\tau$ $(T/T_c)$ | 96 (0.78) | 54 (1.38) | 46 (1.62) | 44 (1.70) | 42 (1.78) | 40 (1.87) | 32 (2.33) |

![Graph of PS channel at different $T/T_c$ values](image)

- **PS: $T=1.78 T_c$**
- **PS: $T=2.33 T_c$**

M. Asakawa (Osaka University)
Data Point Number Dependence

not used in the analysis
note: anisotropy = 4

PS: $T = 1.62 T_c$

$p=0$

$N_t = 39$

$N_t = 38$

$N_t = 36$

$N_t = 34$

$N_t = 32$

$N_t = 30$

$\omega$ (GeV)

$p=0$

$N_t = 39$

$N_t = 38$

$N_t = 36$

$N_t = 34$

$N_t = 32$

$N_t = 30$

$\rho$

max. 39

$N_T$

M. Asakawa (Osaka University)
$N_{\text{data}}$ Dependence of the Peak Shape

1st peak

- Larger $N_{\text{data}}$: sharper signal
- Average value of $\rho$ is almost the same.
- Smaller $N_{\text{data}}$: larger error

It is necessary to have enough number of $N_{\text{data}}$ in MEM analyses
Default Model Dependence

$N_{\text{data}} = 39$

$m_0 = 1.15$

pQCD + Operator Renormalization
(as in Asakawa and Hatsuda (2004))
Error Analysis must also be done

- Peak shape changes as default model changes
- But error size changes as well
- Error size takes minimum at $m_0$ among these four choices of default model

$m_0$ is best choice among these
In the previous calculation, peak shift was not evident due to low statistics and large $p_{\text{min}}$. Now, peak shift is clear, it is upward.
$\eta_c$ at Non-zero Momenta

$P_{\text{min}} \sim 0.5 \text{ GeV}$
η_c at Non-zero Momenta

- η_c exists at non-zero momentum (at least up to ~ 4 GeV)
- Apparently, the peak is lowered at finite momenta
- Default effect (residue decrease) + additional medium effect?

T=1.62T_c
Dispersion Relation below $T_c$

$T=0.78T_c$

$$\omega^2 = m_{p=0}^2 + \hat{p}^2_{\text{lattice}}$$

$$\hat{p}^2_{\text{lattice}} = \frac{2}{a_{\sigma}} \sin(pa_{\sigma}/2)$$

The dispersion relation at $T=0.78T_c$ on the lattice is consistent with that in the vacuum.
Above about $p=3\text{GeV}$, the modification of the dispersion relation becomes evident.

Clue to the structure of the resonance in medium?

\[
\omega^2 = m_{p=0}^2 + \hat{p}_{\text{lattice}}^2 = \frac{2}{a_\sigma} \sin\left(p a_\sigma / 2 \right)
\]
Threshold Enhancement?

Pole mass of charm barely changes as $T$ increases

$E_2$: plasmino

$E_1$: normal mode

$m_p = \text{bare (free) quark mass}$

Karsch and Kitazawa, PRD (2009)

Threshold enhancement cannot explain peak shift of $\eta_c$
Summary

Charmonia at Finite Temperature

- Spectral functions with large spatial volume (finite T effect) with many data points in $\tau$ (needed for MEM analysis)

- At $p=0$
  - $\eta_c$ survives up to $T=1.62T_c$ (confirms previous results)
  - Peak position ("mass") moves up as $T$ increases
  - This cannot be explained by threshold enhancement

- At $p \neq 0$ ($T=0.78T_c$ and $1.62T_c$)
  - $\eta_c$ exists at least up to $p=4$ GeV
  - Peak height decreases as $p$ increases
  - Dispersion relation is modified above $T_c$ (validity of potential model?)

To be Finished Soon

- Separation of the two spectral functions in V channel
- Calculation with smaller quark masses