Characteristic Polynomials of Random Matrices, Disordered Landscapes and Ideal 1/f Noises

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based on:
Y V F, J P Keating under preparation
Y V F, B A Khoruzhenko in progress

closely related to:

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$1/f^{\alpha}$ noises:

Random signals $V(t)$ such that the power $S(f) = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ift} V(t) \, dt \right|^2$ decays like $S(f) \sim f^{-\alpha}$, $0 < \alpha < 3$. Frequently one uses $\alpha = 2H + 1$, with $-1/2 < H < 1$ being the Hurst index for the associated fractional Brownian motion. For $-1/2 < H < 0$ the processes can be stationary, whereas for $0 < H < 1$ they must be with stationary increments, with $H = 1/2$ being the standard Brownian motion (Wiener process). $1/f^{\alpha}$ noises are believed to be ubiquitous in Nature. Engineers frequently use the following code:
**Ideal 1/f noises:**

Random self-similar Gaussian signals with Hurst exponent $H = 0$, i.e. spectral power associated with a given Fourier harmonic is inversely proportional to the frequency. Claimed to be frequently encountered in applications: voltage fluctuations, non-equilibrium phase transitions, spontaneous brain activity, etc.

- **Periodic version:** stationary random Gaussian Fourier series of the form

  $$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[ v_n e^{int} + \overline{v}_n e^{-int} \right], \quad t \in [0, 2\pi)$$

where $v_n, \overline{v}_n$ are complex Gaussian i.i.d. with zero mean and the variance $\mathbb{E}\{v_n \overline{v}_n\} = 1$. It implies

$$\mathbb{E} \left\{ V(t_1) V(t_2) \right\} = 2 \sum_{n=1}^{\infty} \frac{1}{n} \cos n(t_1 - t_2) \equiv -2 \ln |2 \sin \frac{t_1 - t_2}{2}|, \quad t_1 \neq t_2$$

- **stationary-increment version:** a Gaussian process with stationary increments defined via Fourier integral on the half-line $0 < t < \infty$ as

  $$V(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[ v(\omega) e^{i\omega t} - 1 + \overline{v}(\omega) e^{-i\omega t} - 1 \right], \quad \mathbb{E} \left\{ [V(t_1) - V(t_2)]^2 \right\} = \frac{2}{\pi} \ln |t_1 - t_2|$$

with zero-mean complex Gaussian "white noise" Fourier coefficients $v(\omega)$.

The corresponding definitions are formal, as sums/integrals do not converge pointwise, and should be understood as random **generalized functions** (e.g. 1D "projections" of the Gaussian Free Field) or after a proper **regularization**.
Characteristic polynomial of random CUE matrix and periodic 1/f noise:

let $U_N$ be a $N \times N$ unitary matrix, chosen at random from the unitary group $\mathcal{U}(N)$. Introduce its characteristic polynomial $p_N(\theta) = \det (1 - U_N e^{-i\theta})$ and further consider $V_N(\theta) = -2 \log |p_N(\theta)|$. Such an object enjoyed already quite a detailed study by Hughes, Keating & O’Connell 2001 who employed the following representation

$$V_N^{(U)}(\theta) = -2 \log |p_N(\theta)| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} \left[ e^{-in\theta} v_n^{(N)} + \text{comp. conj.} \right],$$

where $v_n^{(N)} = \frac{1}{\sqrt{n}} \text{Tr} \left( U_N^{-n} \right)$.

According to Diaconis & Shahshahani 1994 the coefficients $v_n^{(N)}$ for any fixed $n$ tend in the limit $N \to \infty$ to i.i.d. complex gaussian variables with zero mean and variance $\mathbb{E}\{|\zeta_n|^2\} = 1$.

**Conclusion:** For finite $N$ Log-Mod of the characteristic polynomial of CUE matrices is just a certain regularization of the periodic 1/f noise.
Characteristic polynomial of random GUE matrix and aperiodic 1/f noise:

Similarly, random Hermitian matrix $H$ from GUE can be used to provide a regularized version of the stationary increment 1/f noise (YVF & Khoruzhenko, in progress.) Define parameter $d(N)$ such that $1 \ll d_N \ll N$ for $N \gg 1$, and consider for $\eta > 0$

$$V^{(H)}_{N,x}(t) = -\frac{1}{2\pi} \left\{ \log \det \left[ \left( x + \frac{t}{d(N)} - H \right)^2 + \frac{\eta^2}{d^2(N)} \right] - \log \det \left[ (x - H)^2 + \frac{\eta^2}{d^2(N)} \right]\right\}$$

One can show that

$$V^{(H)}_{N,x}(t) = \frac{1}{\pi} \int_{0}^{\infty} \frac{d\omega}{\sqrt{\omega}} e^{-\eta\omega} \left[ v_{N,x}(\omega) \frac{e^{i\omega t} - 1}{2} + \overline{v}_{N,x}(\omega) \frac{e^{-i\omega t} - 1}{2} \right]$$

where $v_{N,x}(\omega) = \frac{1}{\sqrt{\omega}} e^{ixd(N)\omega} \text{Tr} \left[ e^{-isd(N)H} \right]$.

It can be further verified that for $N \to \infty$ and $x \in (-2, 2)$ the Fourier coefficients $v_{N,x}(\omega)$ tend to the standard Gaussian "white noise":

$$\mathbb{E}\{v_{N,x}(\omega)\} \to 0, \quad \mathbb{E}\{v_{N,x}(\omega_1)v_{N,x}(\omega_2)\} \to \delta(\omega_1 - \omega_2)$$

which implies that $V^{(H)}_{N\gg 1,x}(t)$ is precisely the stationary-increment 1/f noise.
**1/f Noises, Disordered Energy Landscapes, and Burgers Turbulence:**

In the area of *Statistical Mechanics* of *Disordered Systems* 1/f noises have been recently identified as *potential energy landscapes* underlying an intriguing phenomenon of the *freezing transition* which takes place at some finite temperature $T = T_c$ (*Carpentier & Le Doussal* 2001; *Y.V.F & Bouchaud* 2008; *Y.V.F, Le Doussal, & Rosso* 2009). In a related development, it was shown that a freezing transition shows up also in the problem of *decaying Burgers turbulence*, i.e. analysis of the Burgers equation $\partial_t v + (v \nabla)v = \nu \nabla^2 v, \quad \nu > 0$ with random initial conditions given by the gradient of the 1/f noise (*Y.V.F, Le Doussal, & Rosso* 2010 & unpublished).

As a result of those studies we by now have a qualitative, and sometimes, quite precise quantitative (albeit not yet mathematically rigorous) understanding of statistics of *high* and *extreme* values of such random processes. It is natural to try to translate this knowledge into precise conjectures for *high* and *extreme* values of *characteristic polynomials* of random matrices, and then further extend it to associated properties of the *Riemann zeta-function* $\zeta(1/2 + it)$ along the critical line (*Y V F & J.P. Keating*, under preparation).
Characteristic Polynomials as Disordered Energy Landscapes:

To this end, recalling $p_N(\theta) = \det \left( 1 - U_N e^{-i\theta} \right)$ let us define for $\beta > 0$ the following object:

$$Z_N(\beta; L) = \frac{N}{2\pi} \int_0^L |p_N(\theta)|^{2\beta} d\theta \equiv \frac{N}{2\pi} \int_0^L e^{-\beta V_N(\theta)} d\theta , \quad V_N(\theta) = -2 \log |p_N(\theta)|$$

The last form shows that $Z_N(\beta; L)$ can be viewed as a kind of partition function for statistical mechanics of a single classical particle equilibrated at the temperature $T = \beta^{-1}$ in a $1D$ interval $\theta \in [0, L]$, with $V_N(\theta) = -2 \log |p_N(\theta)|$ playing the role of the corresponding energy potential supported in that interval.

Further defining the associated free energy $F_N(\beta) = -\beta^{-1} \log Z_N(\beta)$ we have in the zero-temperature limit:

$$\lim_{\beta \to \infty} F_N(\beta) = \min_{\theta \in (0, L)} V_N(\theta) = -2 \log \left\{ \max_{\theta \in (0, L)} |p_N(\theta)| \right\}$$

Our strategy therefore is to understand the distribution of the free energy as a function of the inverse temperature $\beta$, and relate its zero-temperature limit to statistics of $\max_{\theta \in (0, L)} |p_N(\theta)|$.

In particular, the simplest manifestation of the freezing transition is the following asymptotic:

$$-\mathbb{E} \left\{ F_N \gg 1(\beta) \right\} \approx \left\{ \begin{array}{ll} (\beta + \beta^{-1}) \log N, & \beta < 1 \\ 2 \log N, & \beta \geq 1 \end{array} \right.$$  

which further implies that typically $\max_{\theta \in [0, L)} |p_N(\theta)|_{N \gg 1} \sim N$. Distribution?
Moments of the Partition Function:

**Observation:** one can explicitly evaluate the $N \gg 1$ limit of the positive integer moments

$$\mathbb{E} \left\{ Z_N^n(\beta; L) \right\} = N^n \int_0^L \cdots \int_0^L \mathbb{E} \left\{ |p_N(\theta_1)|^{2\beta} \cdots |p_N(\theta_n)|^{2\beta} \right\} \mathcal{U}(N) \prod_{j=1}^n \frac{d\theta_j}{2\pi},$$

Indeed, the $\mathcal{U}(N)$ group average in the integrand is a certain Toeplitz determinant whose asymptotic $N \gg 1$ behaviour is known due to Fisher-Hartwig and Widom:

$$\mathbb{E} \left\{ |p_N(\theta_1)|^{2\beta} \cdots |p_N(\theta_n)|^{2\beta} \right\} \sim \left[ N\beta^2 \frac{G^2(1+\beta)}{G(1+2\beta)} \right]^n \prod_{r<s} |e^{i\theta_r} - e^{i\theta_s}|^{-2\beta^2}$$

The remaining integration over $\theta_1, \ldots, \theta_n$ can be explicitly performed in two limits: (i) $L = 2\pi$ and (ii) $N^{-1} \ll L \ll 2\pi$ when it is reduced to two versions of the celebrated Selberg integral. In the simplest case $L = 2\pi$ we find:

$$\mathbb{E} \left\{ Z_N^n(\beta; 2\pi) \right\} = Z_e^n \Gamma(1 - n\beta^2), \quad \text{for } 1 < n < 1/\beta^2$$

where $Z_e = N^{1+\beta^2} \frac{G^2(1+\beta)}{G(1+2\beta)\Gamma(1-\beta^2)}$. The ensuing distribution of $Z_{N\gg 1}(\beta < 1; 2\pi)$ was analysed by Y.V.F & Bouchaud 2008. For $\beta > 1$ the integrals diverge, but building on the picture of a "freezing transition" argued to take place at $\beta = 1$ (Carpentier & Le Doussal 2001) one can further conjecture the distribution of the partition function (hence, the associated free energy) for $\beta > 1$. 
Maximum of the modulus of CUE characteristic polynomial:

The maximal value of the modulus of CUE characteristic polynomial $p_N(\theta)$ in an interval $\theta \in [0, L)$, $0 < L \leq 2\pi$ (which contains on average $N_L = N \frac{L}{2\pi}$ eigenvalues of $U_N$) can be written in the limit $N_L \gg 1$ as

$$\max_{\theta \in [0, L)} \log |p_N(\theta)| |N_L| \approx \log N_L - \frac{1}{2} c \log \log N_L - \frac{1}{2} b_{NL} x,$$

where $b_{NL} = 1 + O(1/\log N_L)$, the constant $c$ is conjecturally equal to $\frac{3}{2}$, and the random variable $x$ is distributed with a probability density $\rho(x)$ whose form depends on the chosen $L$ and is understood in two limiting cases. In particular, in the "full-circle" case $L = 2\pi$:

$$\rho(x) = -\frac{d}{dx} \left[ 2 e^{x/2} K_1(2e^{x/2}) \right] = 2e^x K_0 \left( 2e^{x/2} \right)$$

Note that $\rho(x \to -\infty) = -xe^{-x} + \ldots$

In the "mesoscopic" interval $N^{-1} \ll L \ll 2\pi$ one finds $x = y + u \sqrt{-2 \ln L}$, where $u$ is the standard (mean zero, unit variance) gaussian random variable, and $y$ is independent of $u$. The probability density function $\rho(y)$ can be explicitly written as a certain contour integral, whose asymptotic at $y \to -\infty$ is again $\rho(y \to -\infty) = -y e^{-y} + \ldots$. Such a tail shared by two functions is predicted to be universal by a heuristic renormalization group argument (Carpentier & Le Doussal 2001), together with the value $c = 3/2$, and characterizes a new universality class of extreme value statistics different from Gumbel distribution of extremes of short-ranged correlated random variables.
High points of characteristic CUE polynomials:

As asymptotically the highest value \( \max_{\theta \in [0,L]} |p_N(\theta)| N_L \gg 1 \sim N_L = \exp\{\ln N_L\} \) let us call the value of the polynomial to be \( x \)-high if \( |p_N(\theta)| \sim \exp\{x \ln N_L\} \) for some \( 0 < x < 1 \). To quantify the structure associated with \( x \)-high values consider

\[
\mu_N(x; L) = \frac{1}{L} \int_0^L \chi\{\log |p_N(\theta)| - x \log N_L\} d\theta, \quad \chi\{u\} = \begin{cases} 1 & u \geq 0 \\ 0 & u < 0 \end{cases}
\]

which has the meaning of a fraction of the total lengths \( L \) of those intervals in \([0, L]\) where \( |p_N(\theta)| \) exceeds \( x \)-high level. Analysis of \( \mu_N(x; L) \) can be done again in the two limiting cases. Considering the "full-circle" case \( L = 2\pi \) for simplicity, it turns out that the "typical" value \( \mu_e(x) \) of the length \( \mu_N(x; 2\pi) \) is given by

\[
\mu_e(x) = N^{-x^2} \sqrt{\frac{1}{\pi \log N} \frac{2^2(1+x)}{G(1+2x) \Gamma(1-x^2)}}, \quad 0 < x < 1
\]

The nontrivial leading scaling with \( N^{-x^2} \) is a multifractality-type structure of the measure of intervals supporting high values. As long as \( \mu_N(x; 2\pi) \ll 1 \) the probability density for the variable \( \xi = \mu_N(x; 2\pi) / \mu_e(x) \) is predicted to have the following form:

\[
P(\xi) = \frac{1}{x^2} \xi^{-\frac{1}{x^2} - 1} e^{\xi} e^{-\frac{1}{x^2}}, \quad 0 < x < 1
\]
Relation to Riemann $\zeta(1/2 + it)$:

Keating & Snaith 2001 developing the ideas of Montgomery (1973); Odlyzko (1987), etc. convincingly argued that the Riemann zeta-function $\zeta(s)$ along the critical line $s = \frac{1}{2} + it$, $t \in \mathbb{R}$ behaves statistically very much like characteristic polynomials of random matrices of large size $N \sim \log (t/2\pi)$. To this end, define for a fixed real $t$ the function

$$V_t^{(\zeta)}(x) = -2 \log |\zeta \left( \frac{1}{2} + i(t + x) \right)| = -2 \text{Re} \log \zeta \left( \frac{1}{2} + i(t + x) \right)$$

For large $t \to \infty$ the function $V_t^{(\zeta)}(x)$ actually mimics a Gaussian random function of variable $x$ of mean zero and variance $2 \log \log t$ (Selberg, see also Hughes, Keating, O’Connel). Moreover, a simple consideration which uses the Euler product formula for Riemann zeta and the probabilistic properties of primes given by the Prime Number Theorem allows one to find the following small-$x$ behaviour of the covariance high up the critical line:

$$\langle V_t^{(\zeta)}(x_1)V_t^{(\zeta)}(x_2) \rangle \approx \begin{cases} 
-2 \log |x_1 - x_2|, & \text{for } \frac{1}{\log t} \ll |x_1 - x_2| \ll 1 \\
2 \log \log t, & \text{for } |x_1 - x_2| \ll \frac{1}{\log t}
\end{cases}$$

with the averaging going over an interval $[t - h/2, t + h/2]$ such that $\frac{1}{\log t} \ll h \ll t$.

Message: the log-mod of the Riemann zeta-function locally resembles a version of the 1/f noise. One can exploit this fact to make non-trivial predictions for statistics of moments and high values of the Riemann zeta along the critical line using the previously exposed theory.
Our predictions for $\zeta(1/2 + it)$:

We expect a single unitary matrix of size $N = \log (t/2\pi) \gg 1$ to model $\zeta(1/2 + it)$, statistically, over a range of $T \leq t \leq T + 1$. We thus suggest splitting the critical line into ranges of unit length, and making the statistics of $\zeta(1/2 + it)$ over the many ranges. In particular, for the maximum value: $\zeta_{\text{max}}(T) = \max_{T \leq t \leq T+1} |\zeta(1/2 + it)|$ we expect

$$\log \zeta_{\text{max}}(T) \approx \log N - \frac{3}{4} \log \log N - \frac{1}{2} y, \quad N = \log (T/2\pi)$$

and the random $x$ distributed with a probability density behaving as $\rho(y \to -\infty) \approx |y| e^y$.

Moreover, for the measure

$$\mu_T(x) = \text{meas} \left[ T \leq t \leq T + 1 : \log |\zeta(1/2 + it)| \geq x \log N \right]$$

we expect the multifractal scaling for the typical value $\mu_T^{\text{typ}}(x) \sim N^{-x^2}$ and the probability density $\mathcal{P} \left( \xi = \mu_T(x) / \mu_T^{\text{typ}}(x) \right)$ should show the power-law tail $\mathcal{P}(\xi) \sim \xi^{-(1 - \frac{1}{2} x^2)}$ for $\xi \gg 1$.

We also expect the freezing phenomenon to manifest itself in the behaviour for the moments $z(\beta) = N \int_T^{T+1} |\zeta(1/2 + it)|^{2\beta} dt$. In particular asymptotically

$$\beta^{-1} \langle \log z(\beta) \rangle \approx \begin{cases} (\beta + \beta^{-1}) \log N, & \beta < 1 \\ 2 \log N, & \beta \geq 1 \end{cases}$$

Moreover, as long as $0 < \beta < 1$ we expect the probability density $\mathcal{P}(z)$ of $z(\beta)$ to have a power-law right tail $\mathcal{P}(z) \sim z^{-1 - \frac{1}{\beta^2}}$ in an interval $N^{1+\beta^2} \ll z \ll N^2$. In contrast, for $\beta > 1$ we expect the tail exponent to be frozen to $\beta$—independent value: $\mathcal{P}(z) \sim z^{-2}$. 
Summary:

I. log-mod of characteristic polynomials of random matrices are examples of the ideal 1/f noises.

II. Exploiting the methods of statistical mechanics of disordered systems we attempted to understand the statistics of minima/maxima of the CUE characteristic polynomial over various intervals, as well as the related moments and high values.

III. The above picture can be translated into making non-trivial conjectures about statistics of moments and high values of the Riemann zeta along the critical line.

Remark: In a related development, it was found that a freezing phenomenon shows up also in the problem of decaying Burgers turbulence, i.e. analysis of the Burgers equation

$$\partial_t v + (v \nabla) v = \nu \nabla^2 v, \quad \nu > 0$$

with random initial conditions. Namely, for the special initial velocity profile decaying as

$$\mathbb{E}\{v(x, 0)v(x', 0)\} \sim |x - x'|^{-2}$$

the solution $v(x, t > 0)$ abruptly changes statistical properties when viscosity $\nu$ drops below some critical value $\nu_c > 0$.

That set of questions is related to analysing statistics of the position where the maximum is achieved inside a given interval. From the RMT perspective, the story turns out to be intriguingly related to fascinating duality transformations (Desrosiers 2009) for moments of characteristic polynomials of RMT $\beta-$ensembles introduced by Dumitriu & Edelman, 2002

MANY OPEN QUESTIONS REMAIN! (Y V F, P Le Doussal, and A Rosso in progress.)