Finite Rank Perturbations of Non-Hermitian Random Matrices

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Finite rank perturbations

- We consider the limiting eigenvalues as $N \to \infty$ of the matrix

$$X_N + C_N$$

- X_N is an $N \times N$ real random matrix such that

$$\mathbb{E}[X_{ij}] = 0 \quad \mathbb{E}[X_{ij}^2] = 1/N \quad \mathbb{E}[X_{ij}^4] \leq \infty \quad \mathbb{E}[X_{ij}X_{ji}] = \rho/N$$

- $-1 \leq \rho \leq 1$
- Entries are otherwise independent.
- C_N has rank k, not changing with N.
- We study in the large N limit the empirical spectral distribution:

$$\mu_N(x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}(x)$$

Limiting distributions

- If $\rho = 1$ then X_N is a symmetric Wigner matrix and ESD of X_N converges a.s. in distribution to μ_{sc} where

$$\frac{d\mu_{sc}(x)}{dx} = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{[-2,2]},$$

  and the largest (smallest) eigenvalue converges to 2 (-2).
- If all the entries of X_N are i.i.d. then $\rho = 0$ and ESD of X_N converges a.s. in distribution to μ_c where

$$\frac{d\mu_c(z)}{dz} = \frac{1}{2\pi}\mathbf{1}_{|z|\leq 1},$$

  and the magnitude of the largest eigenvalue converges to 1.

Previous work

- The study of finite rank perturbations to Hermitian matrices has received much attention. For example:
  Belinschi, Bercovici, Capitaine, Donati-Martin, Féral;
  Pizzo, R., Soshnikov;
  Benaych-Georges, Guionnet, Nadakuditi, Maïda;
  Knowles, Yin...
- Finite rank perturbations of Non-Hermitian matrices has been less studied. Tao considered the i.i.d. case.

Previous work

- The first step in studying outlier eigenvalues of $X_N + C_N$ is to show that almost surely no eigenvalues of X_N lie outside the support of X_N.
- In the Wigner and i.i.d. case this was done by Bai and Yin.
- They show the spectral radius of X_N converges to the edge of the limiting support.
- They use that $\rho(X^{2k}) \leq \mathrm{tr}(X^{2k})$, and then use the moment method to estimate $\mathrm{tr}(X^{2k})$ for k growing sufficiently fast.

Previous work

- In the Wigner case if λ is an eigenvalue of C_N and $|\lambda| > 1$ then $X_N + C_N$ has an eigenvalue that converges to $\lambda + \frac{1}{\lambda}$.
- In the i.i.d. case if λ is an eigenvalue of C_N and $|\lambda| > 1$ then $X_N + C_N$ has an eigenvalue that converges to $\lambda + \frac{0}{\lambda}$.

Elliptical Law

- The limiting distribution of X_N for general ρ is an ellipse. (Girko; Naumov; O'Rourke, Nguyen)
- Outlier eigenvalues are not controlled by the spectral radius.
- We show that almost surely there are no eigenvalues of X_N by different technique.

No eigenvalues outside Ellipse

- z is not an eigenvalue of X_N if $X_N - z$ is invertible.
- If $|z' - z| \leq \epsilon/2$ then z' is not an eigenvalue of X_N if $X_N - z$ has no eigenvalues less than ϵ.
- Since the smallest singular value of a matrix is less than the magnitude of the smallest eigenvalue it suffices to show there are no small eigenvalues values of $(X_N - z)(X_N - z)^*$.
- Following Haagerup, Thorbjørnsen; Anderson we instead study the self-adjoint linearized problem and consider the matrix

$$\begin{pmatrix} 0 & X - z \\ X^* - \overline{z} & 0 \end{pmatrix}$$

Resolvent

- When studying the spectrum of X_N we use the resolvent

$$G_N(z) = (X_N - z)^{-1}$$

- As well as the hermitianized resolvent

$$R_N(\eta, z) = (H_N - I_N \otimes q)^{-1}$$

  with $(H_N)_{ij} = \begin{pmatrix} 0 & X_{ij} \\ X_{ji} & 0 \end{pmatrix}$, $q = \begin{pmatrix} \eta & z \\ \bar{z} & \eta \end{pmatrix}$. We view these matrices as $N \times N$ matrices with 2×2 matrix entries.
- $m(z) := \lim_{N\to\infty} \mathrm{tr}_N(G_N(z)) = (z - \rho m(z))^{-1}$.
- $\Gamma(\eta, z) := \lim_{N\to\infty} (\mathrm{tr}_N \otimes I_2)(R_N(\eta, z)) = (q - \Sigma(\Gamma(\eta, z)))^{-1}$.
- These matrices are related by $G_N(z) = R_N^{12}(0, z)$.

Eigenvalues of $X_N + C_N$

- z is an eigenvalue of $X_N + A_N B_N$ if

$$\det(X_N + A_N B_N - z) = 0$$

Rearranging and using that

$$\det(I_N + XY) = \det(I_k + YX)$$

we get

$$\det(z + X_N)\det(I_k - B_N G_N(z) A_N) = 0$$

leading to the definitions

$$f(z) = det(I_k - B_N G_N(z) A_N)$$

and

$$g(z) = det(I_k - m(z) B_N A_N)$$

Resolvent limit

- It suffices to show for any unit vectors u_N and v_N

$$u_N(G_N(z) - m(z)I)v_N \to 0$$

- We show that on the event there are no outliers of X_N

$$u_N(G_N(z) - \mathbb{E}[G_N(z)])v_N \to 0$$

  by decomposing into a martingale difference sequence and using standard concentration inequalities.
- Studying the expectation is a bit trickier because $\mathbb{E}[G_{ij}]$ is generally infinite. Instead we study $\mathbb{E}[R_{ij}(\eta, z)]$ for η small.

Resolvent entries

- Schur's Complement gives

$$\mathbb{E}[R_{ii}] = \mathbb{E}[(-q + H_i^{(i)*} R_N^{(i)} H_i^{(i)})^{-1}] \approx \Gamma$$

$$\mathbb{E}[R_{ij}] = \mathbb{E}[R_{ii}(H_{ij} - H_i^{(i)*} R^{(ij)} H_j^{j}) R_{jj}^{(i)}] = O(N^{-3/2})$$

so we conclude almost surely for η small enough

$$\mathbb{E}[u_N^* R^{12}(\eta, z) v_N] = m(z) u_N^* v_N + o(1)$$

Location of Eigenvalues

- $\det(I_k + m(z)A_N B_N) = 0$ if $m(z) = -1/\lambda$. For λ at eigenvalue of C_N.
- Outside the ellipse the range of Stieltjes transform

$$m(z) = \frac{-z + \sqrt{z^2 - 4\rho}}{2\rho}$$

  is the outside of the unit disk. Furthermore, it is 1-1 with inverse

$$m^{\langle -1 \rangle}(z) = -\rho z - \frac{1}{z}$$

- So $m(z) = -1/\lambda$ has solution $z = \lambda + \rho/\lambda$ if $|\lambda| \geq 1$, and no solution otherwise.

Free Probability

- It is helpful to consider the limiting free probability element $\mathcal{E}$

$$\mathcal{E} = \sqrt{\rho}\mathcal{S} + \sqrt{1 - \rho}\mathcal{C}$$

with $\mathcal{S}$ a semicircular element and $\mathcal{C}$ a circular element free from $\mathcal{S}$.

Thank you