Three–dimensional Gaussian fluctuations of noncommutative random surface growth

Jeffrey Kuan

Harvard University

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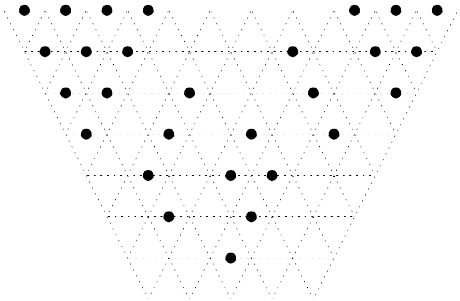
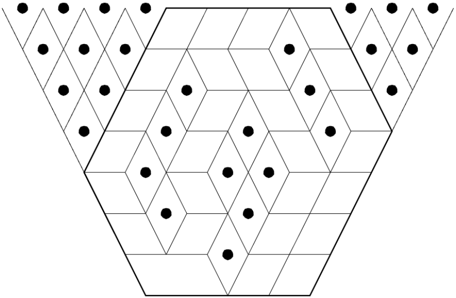
Definition

Push–block dynamics (Borodin–Ferrari ’08)
(Macdonald Process when $q = t$)

The particles live on a lattice in the half plane. There are j particles on the jth level. The particles must satisfy an interlacing property.



Draw lozenges around the particles.

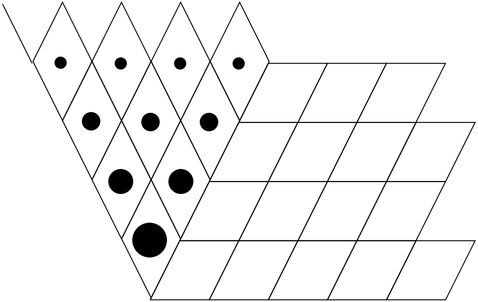


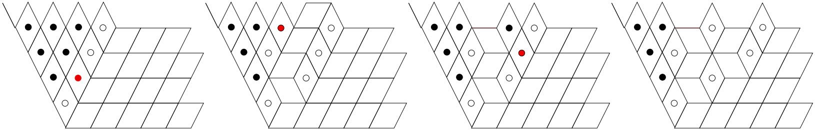
It looks 3D, so we can define a height function

$$H(x,n) = \text{number of particles to the right of } (x,n)$$

Define a Markov Chain as follows:

It starts at $t = 0$ with the densely packed configuration. Imagine that particles have weights that decrease upwards.



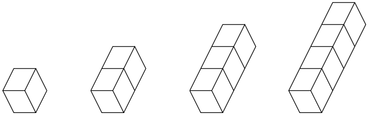


Each particle independently jumps to the right at rate 1. It is blocked by heavier particles and it can push lighter particles.

Connections:

- Can be constructed through representations of unitary groups (Borodin-K 07).
  Different groups lead to different boundary conditions:
  Orthogonal groups (Borodin-K '09) lead to reflecting boundary
  Symplectic groups (Warren-Windridge '08) lead to absorbing boundary
  Also occurs in gauge theory (Majumdar's lectures)
- Processes near these boundaries are universal:
  non–intersecting squared Bessel paths
  (Kuijlaars,Martinez-Finkelshtein, Wielonsky)
  averaged characteristic polynomials of random matrices (Blaizot, Grela, Nowak, Warchol)

Connections:
In terms of the stepped surface in 3d, this can be viewed as adding and removing sticks



This model falls into the Anisotropic Kardar-Parisi-Zhang universality class from mathematical physics.

The Kardar–Parisi–Zhang equation

$$\frac{\partial h}{\partial t} = \underbrace{\nu \bigtriangledown^2 h}_{\text{relaxation}} + \underbrace{\frac{\lambda}{2}(\bigtriangledown h)^2}_{\text{lateral growth}} + \eta,$$

has $t^{1/3}$ fluctuations and the Airy process in the limit.
A variant of KPZ is Anisotropic KPZ (AKPZ).

$$\frac{\partial h(x,y,t)}{\partial t} = \nu_x \nabla_x^2 h + \nu_y \nabla_y^2 h + \frac{\lambda_x}{2}(\nabla_x h)^2 + \frac{\lambda_y}{2}(\nabla_y h)^2 + \eta,$$

with $\nu_x \neq \nu_y$ and $\lambda_x \neq \lambda_y$. Wolf (91) used renormalisation group arguments to predict that when the λ have different sign,

$$h(x,y,t) - \bar{h}(x,y,t) \sim \ln t.$$

Question: Can these fluctuations be proved rigorously? What is the limiting process?

Borodin and Ferrari ('08) confirmed logarithmic fluctuations and showed convergence to the Gaussian free (massless) field along space–like paths

$$\begin{aligned} n_1 \geq \quad \ldots \quad &\geq n_k \\ t_1 \leq \quad \ldots \quad &\leq t_k. \end{aligned}$$

Goal: prove convergence to Gaussian free field without this assumption.

Space–like path condition is necessary because the construction (Diaconis–Fill, Borodin–Ferrari)

$$\begin{array}{ccc} \mathcal{S}_{n+1} & \xrightarrow{P} & \mathcal{S}_{n+1} \\ \Lambda \big\downarrow & & \big\downarrow \Lambda \\ \mathcal{S}_n & \xrightarrow{P} & \mathcal{S}_n \end{array}$$

forces a decrease in the horizontal level with an increase in time.

Given a domain $D \subset \mathbb{R}^d$, let $H_s(D)$ be the space of smooth, real-valued functions that are supported on a compact subset of D. Give $H_s(D)$ the Dirichlet inner product, and let $H(D)$ be its Hilbert space completion.

If $\{e_i\}$ is an orthonormal basis for $H(D)$ and $\{\alpha_i\}$ are i.i.d. $\mathcal{N}(0,1)$, then $h = \alpha_1 e_1 + \alpha_2 e_2 + \dots$ diverges a.s. as an element of $H(D)$. However, for $f \in H(D)$, $\langle h, f \rangle_\nabla$ is a well-defined Gaussian with mean zero and variance $\langle f, f \rangle_\nabla$.

Using integration by parts, it is equivalent to define it as follows.

Definition

A **Gaussian free field** on D is a family of mean zero Gaussian random variables, indexed by $f \in (-\Delta)H(D)$, denoted by (h, f). Their covariance is

$$\mathbb{E}[(h, f)(h, g)] = \int_{D \times D} G(x, y) f(x) g(y) dx dy,$$

where G is the Green's function for the Laplacian on D with Dirichlet boundary conditions.

Example. For $t \in (0,\infty) = D$, let B_t denote $\langle h, \delta_t \rangle$. The Green's function is $G(x,y) = x \wedge y$. Then

1. $B_0 = 0 \;\; a.s.$
2. For $t > s$, $B_t - B_s$ is normal with mean zero and variance $t - s$.
3. $B_t - B_s$ and B_s are independent.

So the Gaussian free field on $(0,\infty)$ is Brownian motion. The Gaussian free field is considered to be a universal object the same way that Brownian motion is universal. It occurs in dimers, Aztec diamond, interacting particles, conformal field theory,...

In higher dimensions, $G(x,x)$ is undefined, so the GFF at a point is undefined. However, for distinct $x_1,\ldots,x_k$ we can formally write

$$\mathbb{E}[\langle h,\delta_{x_1}\rangle\langle h,\delta_{x_2}\rangle] = G(x_1,x_2)$$

Higher moments follow from Wick's (Isserlis') theorem.

Heuristics: In dimension 2, $G(x_1,x_2)\sim\log(x_1-x_2)$ as $|x_1-x_2|\to 0$, so the Gaussian free field has logarithmic fluctuations.

Convergence in moments: (Borodin–Bufetov '13) define the k–th moment of the fluctuations of the random surface to be

$$\int_{-\infty}^{\infty} x^k (H(x,n,t) - \mathbb{E}H(x,n,t))dx \stackrel{IBP}{=} \frac{1}{k+1}(p_{k+1}^{(N)} - \mathbb{E}p_{k+1}^{(N)})$$

where p_k is the k–th power sum polynomial. This converges to the $k--th$ moment of the Gaussian free field. We want the covariances for varying k, N, t without the space–like condition.

This is analog of computing moments $Tr(M^k)$ from random matrix theory.

Basic idea: a probability measure $(\Omega, \mathcal{F}, \mathbb{P})$ defines a commutative von Neumann algebra $L^\infty(\Omega, \mathcal{F})$ and a tracial state $\omega(X) = \mathbb{E}_{\mathbb{P}}[X]$. Conversely, every commutative von Neumann algebra and tracial state comes from a probability measure. Thus, a *noncommutative probability space* is a von Neumann algebra with a tracial state.

$(\Omega, \mathcal{F}, \mathbb{P})$	$\rightsquigarrow$	von Neumann algebra and state $(\mathcal{W}, \omega)$
state space S	$\rightsquigarrow$	C^* algebra $\mathcal{U}$
$X_n : \Omega \to S$	$\rightsquigarrow$	homomorphism $j_n : \mathcal{U} \to \mathcal{W}$
$(\Omega, \mathcal{F}_n)$	$\rightsquigarrow$	subalgebra $\mathcal{W}_n \subset \mathcal{W}$
$\mathbb{E}[\cdot \mid \mathcal{F}_n]$	$\rightsquigarrow$	projection $\mathcal{W} \to \mathcal{W}_n$
transitions $T_t : S \to S$	$\rightsquigarrow$	semigroup of completely positive linear maps $P : \mathcal{U} \to \mathcal{U}$

Definition

The maps j_n are a dilation of the noncommutative Markov chain P if for all $\xi \in \mathcal{W}_n$ and $X \in \mathcal{U}$,

$$\omega(j_{n+1}(X)\xi) = \omega(j_n(PX)\xi)$$

Theorem

There exist a (natural) von Neumann algebra M, state ϕ and a linear $\kappa_t : \mathcal{U}(\mathfrak{gl}_N) \to \mathbb{C}, t \geq 0$ such that the following

$$
\begin{aligned}
\mathcal{U} &= \mathcal{U}(\mathfrak{gl}_N) \\
(\mathcal{W}, \omega) &= M^{\otimes \infty}, \phi^{\otimes \infty} \\
j_n : \mathcal{U} \to \mathcal{W} &\text{ is the } n\text{-fold tensor power representation} \\
\mathcal{W}_n &= M^{\otimes n} \\
M^{\otimes \infty} \to M^{\otimes n} &\text{ is induced from } m_1 \otimes m_2 \mapsto \phi(m_2) m_1 \\
P_t &= (id \otimes \kappa_t) \circ \Delta, \ \text{where } \Delta \text{ is the coproduct} \\
&\quad \text{on } \mathcal{U}(\mathfrak{gl}_N) \text{ as a Hopf algebra}
\end{aligned}
$$

are a noncommutative dilation and Markov chain.

Connections to particle systems:
The Harisch–Chandra isomorphism identifies $Z(\mathcal{U}(\mathfrak{gl}_N))$ and $\{$symmetric polynomials in the variables $\lambda_i - i, 1 \leq i \leq N\}$.

Theorem

- *The restriction to $Z(\mathcal{U}(\mathfrak{gl}_N))$ is still Markov. (In other words, $P_t Z(\mathcal{U}(\mathfrak{gl}_N)) \subset Z(\mathcal{U}(\mathfrak{gl}_N))$).*
- *If Q_t is the markov operator for the particle system on the N–th level, then $P_t f = Q_t f$ for $f \in Z(\mathcal{U}(\mathfrak{gl}_N))$.*

Idea: given (k_1, N_1, t_1) and (k_2, N_2, t_2) with $t_1 > t_2$, replace $p_{k_2}^{N_2} \mapsto P_{t_1 - t_2} p_{k_2}^{N_2}$.

Reduces computation to fixed–time case.

The resulting three–dimensional Gaussian field appears to be different from a three–dimensional Gaussian field resulting from Wigner matrices (Borodin '10)

Why non–commutative probability?

- Without a formula for dynamics along time–like paths, one needs to multiply $X_n, X_m : \Omega \to \mathbb{R}$. Representation theory provides a construction for Ω.
- In representation theory, noncommutative objects are more natural and concrete.
- Under the Harish–Chandra isomorphism, the symmetric polynomial is a sum of noncommutative monomials.

Next steps:

- Relating quantum group $\mathcal{U}_q$ to q–Whittaker processes (Borodin–Corwin '11) ???
- Deform the states by $q^{\mathrm{cr}(\pi)}t^{\mathrm{ne}(\pi)} \to (q,t) - GFF$?
- Orthogonal groups and symplectic groups ??