A q-weighted Robinson-Schensted Algorithm

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Outline

- The Robinson-Schensted algorithm and Pitman's Theorem

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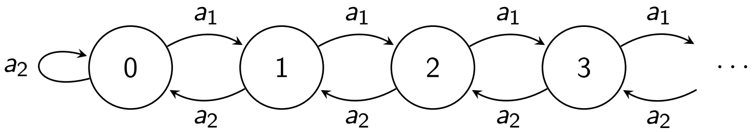
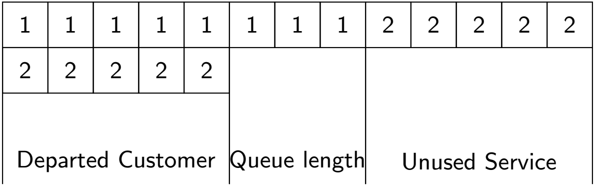
- A q-weighted Robinson-Schensted algorithm and corresponding "Pitman's" Theorem

The M/M/1 queue

The de-Poinssonised M/M/1 queue:
w.p. (with probability) a_1 a customer arrives and increase the queue length by 1.
w.p. a_2 a service happens, such that:
Either a customer departs if the queue length is positive;
Or the service is unused (wasted) if the queue length is 0.



The M/M/1 queue as the RS algorithm



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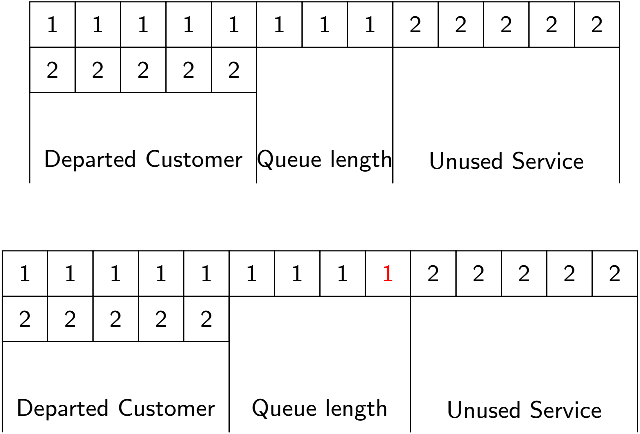

Figure: w.p. a_1 a customer arrives

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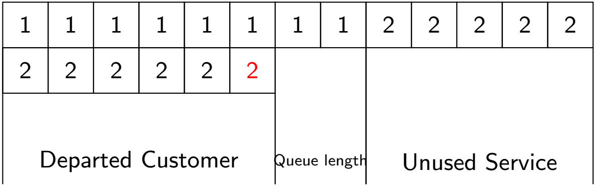
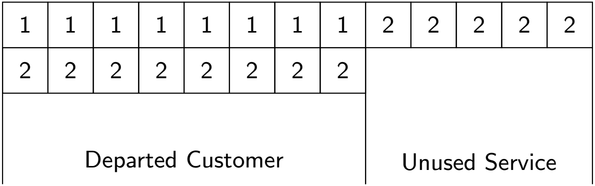


Figure: w.p. a_2 a customer departs if the queue length is greater than 0.

The M/M/1 queue as the RS algorithm

Suppose two more services happen and the queue length is 0 now.



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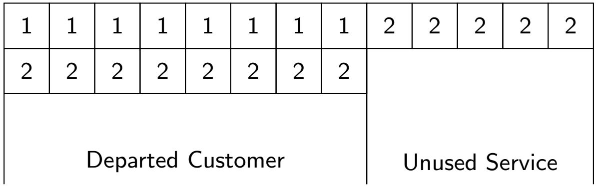

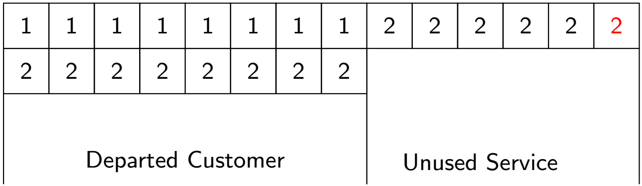

Figure: w.p. a_2 a service is wasted if the queue length is 0.

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When inserting a 2, if the number of 1's in the first row (denoted as λ_1^1) is larger than the number of 2's in the second row (denoted as λ_2^2), then the 2 is inserted to the second row (customer departure), otherwise to the first row (service unused).

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So the M/M/1 queue is interpreted as: w.p. a_i a number i is inserted, $i = 1, 2$.

Evolution of the tableau and the shape

A random word $(w_1, w_2, \dots)$ with $w_i \in \{1, 2\}$ corresponds to a 2-dimensional bernoulli random walk with $S = (S_1, S_2)$:

$$S_i(n) = |\{j \le n : w_j = i\}|.$$

the tableau is a Markov chain. What is more significant is, the shape $(\lambda_1^2, \lambda_2^2)$ is also a Markov chain, with their difference $\lambda_1^2 - \lambda_2^2$ evolving as a discrete 3-dimensional Bessel process with drift $a_1 - a_2$.

The Schur functions

Define the Schur symmetric polynomials s_{λ_1,λ_2} to be

$$s_{\lambda_1^2,\lambda_2^2}(a_1,a_2) = \sum_{\lambda_2^2 \leq \lambda_1^1 \leq \lambda_1^2} a_1^{\lambda_1^1} a_2^{\lambda_1^2+\lambda_2^2-\lambda_1^1} = \frac{a_1^{\lambda_1^2+1} a_2^{\lambda_2^2} - a_2^{\lambda_1^2+1} a_1^{\lambda_2^2}}{a_1 - a_2}.$$

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Then the transition kernel of $(\lambda_1^2, \lambda_2^2)$ is

$$\mathbb{P}(\lambda^2)(n) = \lambda | \lambda^2(n-1) = \mu, \lambda^2(i), i = 1, \ldots, n-2) = \frac{s_\lambda(a)}{s_\mu(a)} \mathbb{I}_{\mu \nearrow \lambda}.$$

where $\mathbb{I}_{\mu\nearrow\lambda}$ is an indicator function with $\mu \nearrow \lambda$ meaning $\lambda = \mu + \mathrm{e}_i$ for some $i \in \{1, 2\}$.

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Again, starting with an empty tableau, the shape of the tableau is an $\ell-$dimensional Markov chain, with transition kernel

$$p(\mu, \lambda) = \frac{s_\lambda(a)}{s_\mu(a)} \mathbb{I}_{\mu \nearrow \lambda}.$$

Connections

When $a_1 > a_2 > \cdots > a_\ell$ the shape evolves as the input random walk conditioned to stay in the non-negative integer Weyl chamber $\{x \in \mathbb{N}^\ell : x_1 \geq x_2 \geq \cdots \geq x_\ell\}$.
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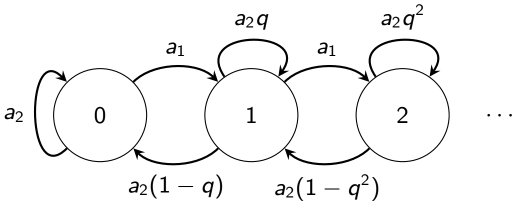
On the other hand, the algorithm is related to the Wigner coefficients of the representations of quantum group $U_q(gl(n))$ when $q \to 0$ by Date-Jimbo-Miwa.

A q-M/M/1 queue

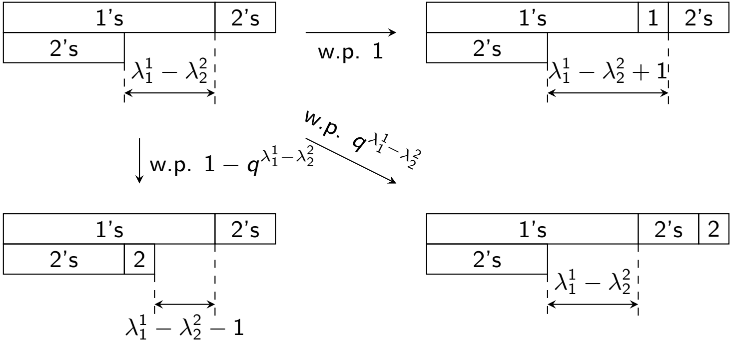
Here we use a different $q \in (0, 1)$.
"Lazy server"



The probability of a service depends on q and the queue length: the more intimidating the queue is, the less lazy the server is...
$q \to 0$ recovers the normal M/M/1 queue.

A rank-1 q-RS algorithm

When inserting a letter 1, it is appended at the end of all the 1's in the tableau, just like in the normal insertion algorithm.
When inserting a letter 2, w.p. $1 - q^{\lambda_1^1 - \lambda_2^2}$ the 2 goes to the second row, otherwise it is inserted to the first row.



The dynamics of the shape

When taking the Bernoulli random walk $(w_1, w_2, \dots)$ parameterised by (a_1, a_2), the shape of the tableau evolves as a Markov chain with kernel:

$$P(\mu, \lambda) = \begin{cases} \frac{\Psi_a(\lambda)}{\Psi_a(\mu)}(1 - q^{\lambda_1 - \lambda_2}), & \text{if } \lambda_1 = \mu_1 + 1; \\ \frac{\Psi_a(\lambda)}{\Psi_a(\mu)}, & \text{if } \lambda_2 = \mu_2 + 1; \\ 0, & \text{otherwise,} \end{cases}$$

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where Ψ_a is a Whittaker function on space of 2-partitions:

$$\Psi_a(\lambda) = \sum_{\lambda_2^2 \le \lambda_1^1 \le \lambda_1^2} a_1^{\lambda_1^1} a_2^{\lambda_1^2 + \lambda_2^2 - \lambda_1^1} \frac{(1-q)^{-1}(1-q^2)^{-1} \dots (1 - q^{\lambda_1^2 - \lambda_1^1})^{-1}}{(1-q)(1-q^2)\dots(1-q^{\lambda_1^1 - \lambda_2^2})}.$$

The difference of the lengths of the two rows $\lambda_1^2 - \lambda_2^2$ is a Markov chain whose transition function is related to the q-Hermite polynomials.

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q-Whittaker functions are eigenfunctions of Ruijsenaars' relativistic Toda difference opearators whose kernels are

$$L(\mu, \lambda) = \begin{cases} 1 - q^{\lambda_i - \lambda_{i+1}}, & \text{if } \lambda_i = \mu_i + 1 \text{ for some } i < \ell; \\ 1 & \text{if } \lambda_1 = \mu_1 + 1; \\ 0 & \text{otherwise .} \end{cases}$$

They are also Macdonald $(q, t)-$symmetric polynomials when $t = 0$.

An example

Suppose we insert a 3 into

1	2	2	2	3	5
2	3	4	4	5	
3					
5					

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1	2	2	2	3	5
2	3	4	4	5	
3					
5					

All possible outputs:

1	2	2	2	3	5	5
2	3	3	4	4		
3						
5						

$$(1-q^2)\frac{1-q}{1-q^3};$$

1	2	2	2	3	4	5
2	3	3	4	5		
3						
5						

$$(1-q^2)\left(1-\frac{1-q}{1-q^3}\right);$$

1	2	2	2	3	3	5
2	3	4	4	5		
3						
5						

$$q^2.$$

Application: q-TASEP

A q-Totally Asymmetric Simple Exclusion Process is an interacting particle system $(X_1, X_2, \ldots, X_\ell) \in W \cap \mathbb{Z}$ with initial condition $X_i(0) = 1 - i$.
Each ith particle independently jumps from X_i to $X_i + 1$ with rate c_i, where

$$c_i = \begin{cases} a_1 & \text{if } i = 1; \\ a_i(1 - q^{X_{i-1} - X_i - 1}) & \text{otherwise.} \end{cases}$$

Then the Poissonised version of $(\mathrm{sh}(P(S))_i^i - i + 1)_{1 \le i \le \ell}$ is a q-TASEP, whose law can be calculated:

$$\mathbb{P}(X_\ell(t) = m - l + 1) = e^{-t} \sum_{\lambda \in C_\ell, \lambda_l = m} \frac{t^{|\lambda|}}{|\lambda|!} (\lambda_\ell)_q^{-1} \Psi_a(\lambda) f^\lambda(q),$$

where $f^\lambda(q)$ is a q-analog of f^λ the number of tableaux of shape λ with distinct entries of $1, 2, \ldots, |\lambda|$.

Other connections

A new combinatorial formula for the Macdonald $(q, t = 0)$-polynomials.

A symmetric property of the algorithm that can be generalised to a family of branching algorithms.

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[SW98, BC13, O'C03, Joh00, GLO10, BP13, DJM90, Gri11, P.13, **?**]