fBm with Hurst index $H = 0$ and statistics of GUE characteristic polynomials

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Randomness in Physics and Mathematics: Bielefeld University
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Outline

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Colours of noise

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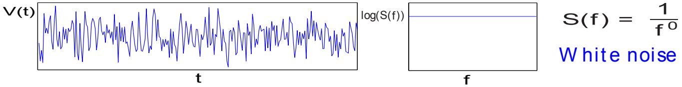

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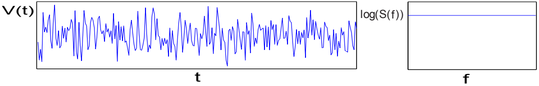

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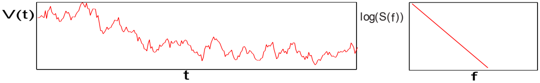
White noise



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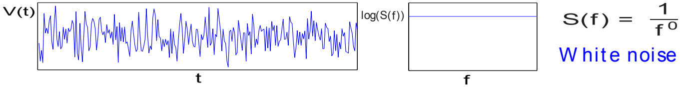

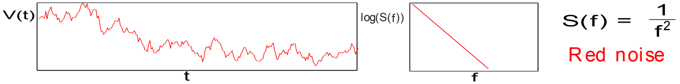

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$$V(t) = \sum_{k=1}^{\infty} X_k \frac{\sqrt{2}\sin(k\pi t)}{k\pi}, \qquad X_k \text{ i.i.d. standard Gaussians.}$$

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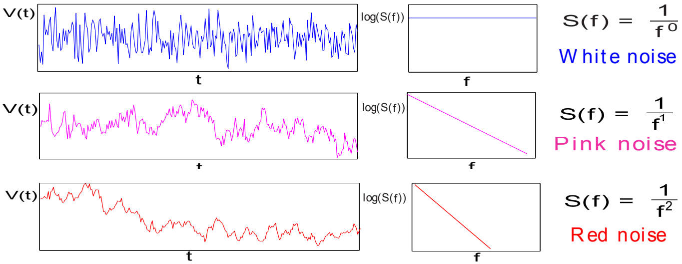

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- Number theory - linear statistics of the Riemann ζ-function relate to 1/f noise (e.g. Bourgade and Kuan '12)
- **Random Matrix Theory**. We shall see how 1/f noises arise in RMT.

In this talk I will focus on random Hermitian matrices.

Fractional Brownian motion

Fractional Brownian motion

Let $B(ds)$ be the Gaussian white noise measure with $\mathbb{E}(B(ds)) = 0$ and $\mathbb{E}(B(ds_1)B(ds_2)) = \delta(s_1 - s_2)ds_1 ds_2$.

For $0 < H < 1$, we define fBm as (see Taqqu et al. '03)

$$B_H(t) = \frac{1}{C}\int_0^\infty [e^{-ift} - 1]\frac{1}{f^{H+1/2}}\, d\tilde{B}(f) + \text{c.c.}$$

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Covariance structure:

$$\mathbb{E}((B_0^{(\eta)}(t_1) - B_0^{(\eta)}(t_2))^2) = \log\left(\frac{(t_1 - t_2)^2}{\eta^2} + 1\right)$$

Thus, our process $B_0^{(\eta)}(t)$ is *logarithmically correlated*.

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The central quantity of interest in this talk is:

$$V_N(z) = -\log \left| \det(z - \mathcal{H}_N) \right|, \quad z \in \mathbb{C}$$

The limit $N \to \infty$ turns out to depend sensitively on how z varies with N.

Scaling regimes in Random Matrix Theory

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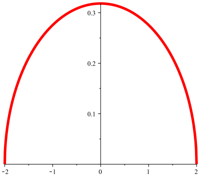
Random matrix statistics can be analyzed at three different scales:



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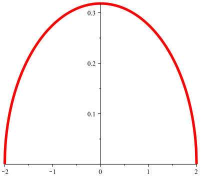

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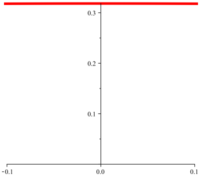

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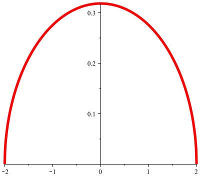

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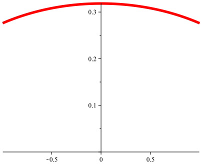

Mesoscopic regime
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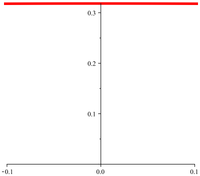

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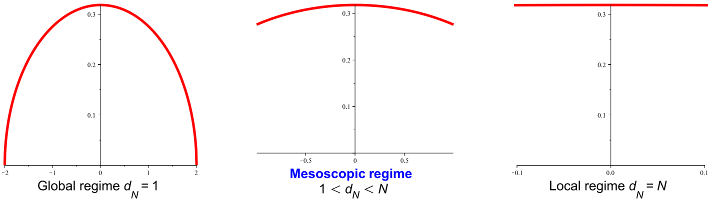

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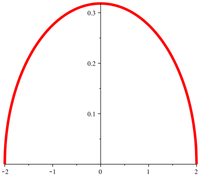

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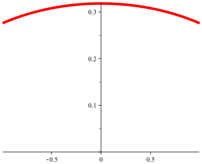

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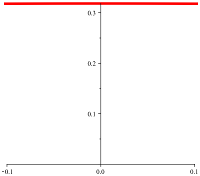

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We will be interested in the *mesoscopic statistics* of the eigenvalues. For $\lambda_0 \in (-2, 2)$, let

$$z(t) := \lambda_0 + \frac{t + i\eta}{d_N}, \qquad \eta > 0, \qquad t \in \mathbb{R}$$

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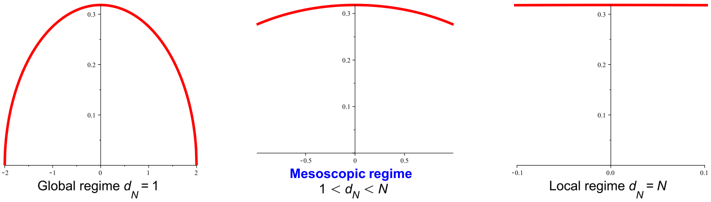

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Our main contribution: Suppose $d_N \to \infty$ with $d_N/N \to 0$. Then

$$\boxed{V_N(z(t)) - V_N(z(0)) \to B_0(t)}$$

as $N \to \infty$.

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Set $z(t) = \lambda_0 + \frac{t+i\eta}{d_N}$ *and introduce*

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Consider the **mesoscopic regime** $d_N \to \infty$ *but* $d_N/N \to 0$. *Then for any* $(t_1, \ldots, t_m) \in \mathbb{R}^m$ *and* $\eta > 0$, *we have*

$$\boxed{(\tilde{W}_N(t_1), \ldots, \tilde{W}_N(t_m)) \overset{d}{\Rightarrow} (B_0^{(\eta)}(t_1), \ldots, B_0^{(\eta)}(t_m)), \quad N \to \infty,}$$

where $B_0^{(\eta)}(t)$ *is our regularization of fBm with* $H = 0$:

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$$\tilde{W}_N(t) = W_N(t) - \mathbb{E}(W_N(t))$$

Consider the **mesoscopic regime** $d_N \to \infty$ *but* $d_N/N \to 0$. *Then for any* $(t_1, \ldots, t_m) \in \mathbb{R}^m$ *and* $\eta > 0$, *we have*

$$\boxed{(\tilde{W}_N(t_1), \ldots, \tilde{W}_N(t_m)) \stackrel{d}{\Rightarrow} (B_0^{(\eta)}(t_1), \ldots, B_0^{(\eta)}(t_m)), \quad N \to \infty,}$$

where $B_0^{(\eta)}(t)$ *is our regularization of fBm with* $H = 0$*:*

$$B_0^{(\eta)}(t) = \int_0^\infty \frac{e^{-\eta f}}{\sqrt{f}}[e^{-itf} - 1]d\tilde{B}(f) + \text{c.c.}$$

The limiting object $B_0^{(\eta)}(t)$ **is a logarithmically correlated Gaussian field with stationary increments.**

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The characteristic function of $(W_N(t_1), \ldots, W_N(t_m))$ is

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Standard RMT techniques cast ϕ_N as the Hankel determinant

$$\phi_N = \det\left\{\int_{\mathbb{R}} x^{k+j} w(x)\, dx\right\}_{k,j=0}^{N-1}, \quad w(x) = e^{-Nx^2/2} \prod_{k=1}^{m+1} |x - z_k|^{\alpha_k},$$

where $\alpha_{m+1} := -\sum_{k=1}^{m} \alpha_k$ and $t_{m+1} \equiv 0$.

Singularities $z_k = \lambda_0 + (\tau_k + i\eta)d_N^{-1}$ **merge together** *in the limit* $N \to \infty$.

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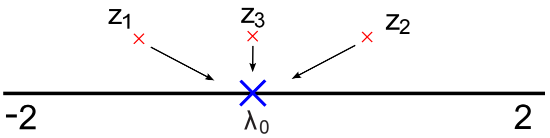

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We solve this via a mapping to the Riemann-Hilbert problem for the OPs. The non-linear steepest descent must be performed in the *mesoscopic* regime.

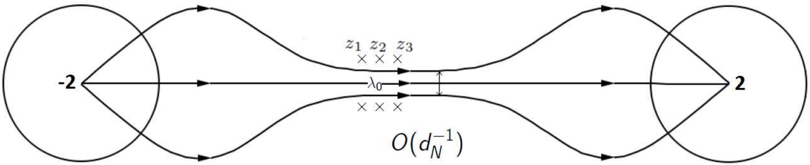

Figure : Steepest descent contours for the Riemann-Hilbert problem

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We proved that in the mesoscopic regime of RMT, the log-mod of characteristic polynomials of random Hermitian matrices converges to $B_0(t)$ - fractional Brownian motion with $H = 0$.

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Thank you for listening!

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For $a > 0$, introduce the space

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Theorem (Fyodorov, Khoruzhenko and Simm '13)

Let $X_1, X_2, \ldots$ be i.i.d. standard Gaussians. Then as $N \to \infty$

$$\boxed{-\log|\det(x - \mathcal{H})| - \mathbb{E}[\ldots] \Rightarrow \sum_{k=1}^{\infty} \frac{T_k(x/2) X_k}{\sqrt{k}}}$$

weakly in $V^{(a)}$ with $a < -1/2$.

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Tightness then follows from Chebyshev's inequality. **QED.**