Distribution of the Smallest Eigenvalue in the Correlated Wishart Model

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Outline

Time Series Analysis

*In **complex systems**, time series are measured which yield rich information about the **dynamics** but also about the **correlations**.*

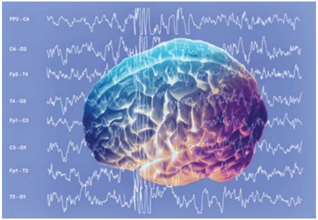
Time Series

A time series X, with n time steps, is a vector

$$X = [X(1) \dots X(n)] \tag{1}$$

examples are :

- amplitudes of brain waves



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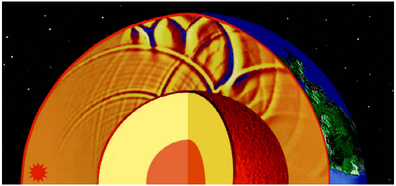

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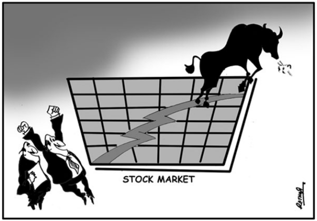
- stock prices



Examples of time Series

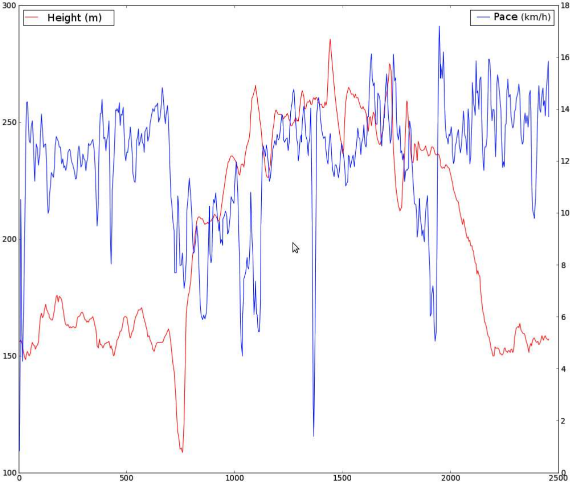


Figure: Myself running 8.1 Km.

Correlation of Time Series

Suppose we have simultaneously measured p time series X_i, $i = 1, \ldots, p$ each of n time steps.

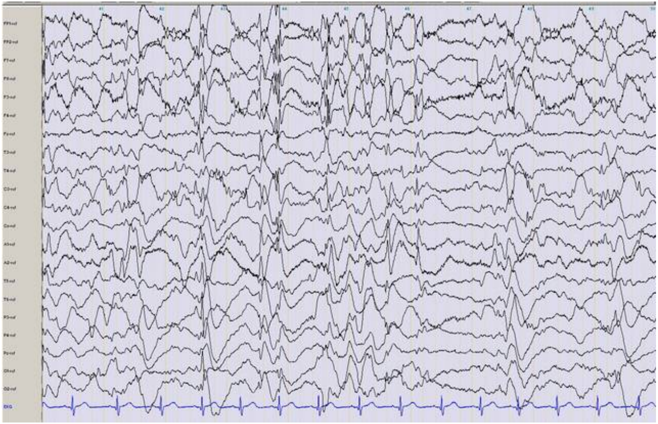


Figure: Brain waves of a sleeping 15 year old boy.

Correlation Matrix

The time series X_i are **normalized to zero mean and unit variance.** We order them into a rectangular $p \times n$–matrix M and define the correlation matrix C, such that

$$M = \begin{bmatrix} M_1(1) & \dots & M_1(n) \\ \vdots & \ddots & \vdots \\ M_p(1) & \dots & M_p(n) \end{bmatrix} \quad \text{and} \quad C = \frac{1}{n} M M^\dagger \; .$$

C is **positive definite** and, due to $n \geq p$, of **full rank.**

Why studying (eigenvalue) statistics of correlation matrices?

Construct an ensemble of
sample correlation matrices
that models the statistics of C.

Statistics of Correlation Matrices

Let W be a rectangular $p \times n$ matrix ($p \leq n$), where $W_{ij} \in \mathbb{R}, \mathbb{C}$ for $\beta = 1, 2$, then

$$WW^{\dagger}$$

has the same **global symmetries** as C.

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Wishart Correlation Matrix

In agreement with empirical studies, we choose the W_{ij} to be **Gaussian distributed**, with mean zero and variance $(C^{-1})_{ji}$, *i.e.* it is **distributed** w.r.t.

$$P(W|C) \sim \exp\left(-\frac{\beta}{2}\mathrm{tr}\, WW^\dagger C^{-1}\right) .$$

$WW^\dagger/n$ drawn from $P(W|C)$ is a (*Gaussian*) **Wishart correlation matrices.**

Properties of the ensemble :

- $WW^\dagger/n$ is **upon average** C, *i.e.*

$$\left\langle \frac{1}{n} WW^\dagger \right\rangle = C$$

- **Invariant observables**

$$f(VWW^\dagger V^\dagger) = f(WW^\dagger) \ ,$$

where $V \in \mathrm{G}_p = \mathrm{O}(p), \mathrm{U}(p)$ for $\beta = 1, 2$, depend **upon average** on the eigenvalues of C, ordered in Λ, **only**.

Eigenvalue Statistics of Correlation Matrices

Joint Eigenvalue Distribution

$$P_\beta(X|\Lambda) = K_{p\times n}\,|\Delta_p(X)|^\beta \det^\gamma X \Phi_\beta(X,\Lambda^{-1}),$$

where $K_{p\times n}$ is a normalization constant, $\gamma = n - p + 1 - 2/\beta$ and

$$\Phi_\beta(X,\Lambda^{-1}) = \int_{G_p} d\mu(V) \exp\left(-\frac{\beta}{2}\mathrm{tr}\, VXV^\dagger\Lambda^{-1}\right) ,$$

is a highly **non-trivial** group integral.

Why the smallest eigenvalue ?

The Smallest Eigenvalue

$E_p^{(\beta)}(t)$ is the probability to find **all eigenvalues** in $[t, \infty)$, *i.e.*

$$E_p^{(\beta)}(t) = K_{p\times n} \int_t^{\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_p P(X|\Lambda)$$

$E_p^{(\beta)}(t)$ is the probability to find **all eigenvalues** in $[t,\infty)$, *i.e.*

$$E_p^{(\beta)}(t) = K_{p\times n} \exp\left(-\mathrm{tr}\frac{\beta t}{2\Lambda}\right) \int_0^\infty d[X]\, |\Delta_p(X)|^\beta \times \det{}^\gamma\left(X + t\mathbb{1}_p\right) \Phi_\beta(X,\Lambda^{-1}) \,.$$

where $\gamma = \beta(n - p + 1 - 2/\beta)/2$. Here we consider $\gamma \in \mathbb{N}$ only.

The Smallest Eigenvalue

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$$\times \det{}^\gamma \left(X + t\mathbb{1}_p\right) \Phi_\beta(X, \Lambda^{-1}) \, .$$

It is related to the distribution of the smallest eigenvalue $\mathcal{P}_{\min}^{(\beta)}(t)$ by

$$\mathcal{P}_{\min}^{(\beta)}(t) = -\frac{d}{dt} E_p^{(\beta)}(t) \, .$$

The small-σ model ($\beta = 1, 2$ only)

$$E_p^{(\beta)}(t) = K_{p\times\bar{n}} \exp\left(-\mathrm{tr}\frac{\beta t}{2\Lambda}\right) \int d[\sigma] \exp(-\mathrm{tr}\sigma) \times f_{\beta,\bar{n}}(\sigma) \prod_{k=0}^{p} \det{}^{\beta/2}\left(\frac{\beta t}{2}\mathbb{1}_{2\gamma/\beta} - \Lambda_k \sigma\right) ,$$

where σ is a $2\gamma/\beta \times 2\gamma/\beta$–dimensional Hermitian matrix, if $\beta = 1$, it is self–dual, $\bar{n} = p + 2/\beta - 1$ and

$$f_{\beta,\bar{n}}(\sigma) = \int d[\varrho] \det{}^{\beta\bar{n}/2}\varrho \exp(-\imath \mathrm{tr}\varrho\sigma)$$

The small-σ model ($\beta = 1, 2$ only)

$$E_p^{(\beta)}(t) = \frac{\exp\left(-\mathrm{tr}\frac{\beta t}{2\Lambda}\right)}{\det^{\gamma}\Lambda} \det^{\beta/2}\left[Q_{ij}^{(\beta,p)}(t)\right]_{i,j=1,\ldots,2\gamma/\beta} ,$$

where

$$Q_{ij}^{(\beta,p)}(t) = q_{ij}\,\Theta(\alpha_{p,\beta}) \sum_{k=0}^{\min(p,\alpha_{p,\beta})} \frac{e_k(\Lambda)\ t^{p-k}}{(\alpha_{p,\beta}-k)!} .$$

$q_{ij} = (j-i)(-1)^{j+i+1}$, $q_{ij} = (-1)^i$ for $\beta = 1, 2$ and
$\alpha_{p,\beta} = p + 2(\gamma+1)/\beta - i - j$

$$e_k(\Lambda) = \sum_{1 \le i_1 < \cdots < i_k \le p} \Lambda_{i_1} \cdots \Lambda_{i_k}$$

The Distribution of the Smallest Eigenvalue ($\beta = 1, 2$ only)

The Distribution of the Smallest Eigenvalue

$$\mathcal{P}^{(\beta)}_{\min}(t) = \mathrm{tr}\frac{\beta}{2\Lambda} E^{(\beta)}_p(t) - \frac{\beta}{2} \frac{\exp\left(-\mathrm{tr}\frac{\beta t}{2\Lambda}\right)}{\det^{\gamma}\Lambda} \times \sum_{l=1}^{\gamma} \frac{\det\left[G^{(l)}_{ij}(t)\right]_{i,j=1,\dots,2\gamma/\beta}}{\det^{1-\beta/2}\left[Q^{(\beta,p)}_{ij}(t)\right]_{i,j=1,\dots,2\gamma/\beta}} ,$$

where

$$G^{(l)}_{ij}(t) = \begin{cases} Q^{(\beta,p)}_{ij}(t) & , l \neq i \\ \frac{\mathrm{d}}{\mathrm{d}t} Q^{(\beta,p)}_{ij}(t) & , l = i \end{cases} .$$

What happens for large correlated Wishart matrices?

Careful analysis shows

$$\langle t\rangle \sim \mathrm{O}(p),$$

if $n-p \sim \mathrm{O}(1)$. If

$$\eta = \frac{1}{p}\sum_{k=1}^{p}\frac{1}{\Lambda_k} = \frac{1}{p}\mathrm{tr}\Lambda^{-1}\ ,$$

converges to a non-vanishing constant, we can study the microscopic limit, *i.e.*, $p \to \infty$ and $n-p \sim O(1)$.

Microscopic Limit

After rescaling of the eigenvalues by $1/4p\eta$, we study the microscopic limits

$$\mathcal{E}^{(\beta)}(u) = \lim_{p\to\infty} E_p^{(\beta)}\left(\frac{u}{4p\eta}\right)$$

and

$$\wp_{\min}^{(\beta)}(u) = \lim_{p\to\infty} \frac{1}{4p\eta}\mathcal{P}_{\min}^{(\beta)}\left(\frac{u}{4p\eta}\right) .$$

The Gap Probability

$$\mathcal{E}^{(\beta)}(u) = \exp\left(-\frac{\beta u}{8}\right) \det{}^{\beta/2}\left[\tilde{q}_{ij}\, L^{(0)}(u)\right]_{i,j=1,\ldots,2\gamma/\beta},$$

where

$$L_{ij}^{(l)}(u) = \sqrt{\frac{u}{4}}^{\,i+j-\kappa'} \mathrm{I}_{\kappa'+\delta_{i-l,0}-i-j}\left(\sqrt{u}\right)$$

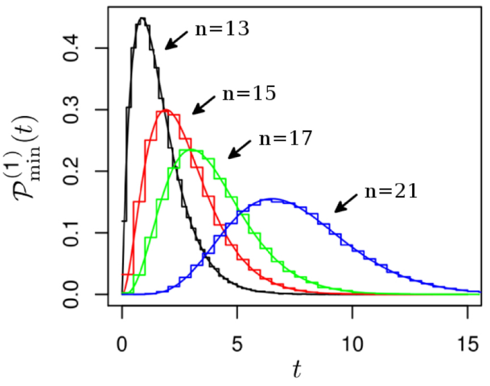
and I_ν is the modified Bessel function of order ν. We defined $\kappa' = 2(\gamma+1)/\beta$ and $\tilde{q}_{ij} = (j-i)$ for $\beta = 1$, $\tilde{q}_{ij} = (-1)^{i+1}$ for $\beta = 2$.

Distribution of the Smallest Eigenvalue

$$\wp_{\min}^{(\beta)}(u) = \frac{\beta}{8}\mathcal{E}^{(\beta)}(u) - \frac{\beta}{8\sqrt{u}} \exp\left(-\frac{\beta u}{8}\right)$$

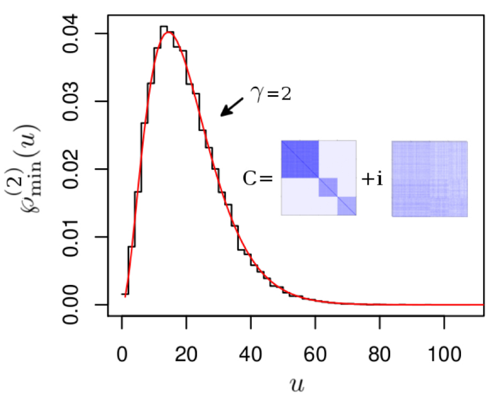
$$\times \frac{\sum_{l=1}^{2\gamma/\beta} \det\left[\tilde{q}_{ij}\; L_{ij}^{(l)}(u)\right]_{i,j=1,\ldots,2\gamma/\beta}}{\det^{1-\beta/2}\left[\tilde{q}_{ij}\; L_{ij}^{(0)}(u)\right]_{i,j=1,\ldots,2\gamma/\beta}} .$$

Exact results ($\beta = 1$)



$\Lambda_k = 0.6, 1.2, 6.7, 9.3, 10.5, 15.5, 17.2, 20.25, 30.1, 35.4.$

Microscopic Limit ($\beta = 2$)



Conclusion

- The exact expression of $E_p^{(p)}(t)$ and $\mathcal{P}_{\text{min}}^{(\beta)}(t)$ show **determinant and Pfaffian structures.**
- Our formulas **easy–to–use** compare to known results.
- We were able to determine a class of empirical correlation matrices yielding the distribution of **uncorrelated Wishart** model in the microscopic limit on a **local scale.**

This work was recently accepted by **PRL**! A preprint is on ArXiv:

math-ph/1306.4790

Thank you for your attention!

A New Correlated Wishart Model ($\beta = 2$ only)

The small-$\overline{W}$ model

$$E_p^{(\beta)}(t) = K_{p\times\bar{n}} \exp\left(-\mathrm{tr}\frac{\beta t}{2\Lambda}\right) \int d[\overline{W}] \det{}^{\gamma}(\overline{W}\,\overline{W}^{\dagger} + t\mathbb{1}_p) \times \exp\left(-\frac{\beta}{2}\mathrm{tr}\overline{W}\,\overline{W}^{\dagger}\Lambda^{-1}\right) ,$$

where $\overline{W}$ is a $p \times \bar{n}$ dimensional matrix, $\bar{n} = p + 2 - \beta$ and $\gamma = \frac{\beta}{2}(n - p + 1) - 1$.

For $\beta = 1$, γ can be half–integer!

Express $\det^{\gamma}(\overline{W}\,\overline{W}^{\dagger} + t\mathbb{1}_p)$ as Gaussian integral over **anticommuting variables**

$$\Downarrow$$

Perform the $\overline{W}$–integral

$$\Downarrow$$

Replace the Grassmannians by a matrix of a particular **symmetry class** using a **delta function**

$$\Downarrow$$

Perform the **Gaussian** integral over the **anticommuting variables**

Mutual Dualities ($\beta = 1, 2$)

$$E_p^{(\beta)}(t) \begin{array}{l} \nearrow \begin{array}{c}\textbf{large-}W\textbf{ model}\\ p \times n\text{-dim.}\end{array} \Leftrightarrow \begin{array}{c}\textbf{large-}\sigma\textbf{ model}\\ (2\gamma\beta|2\gamma\beta) \times (2\gamma\beta|2\gamma\beta)\text{-dim.}\end{array} \\ \searrow \begin{array}{c}\textbf{small-}\overline{W}\textbf{ model}\\ p \times \bar{n}\text{-dim.}\end{array} \Leftrightarrow \begin{array}{c}\textbf{small-}\sigma\textbf{ model}\\ 2\gamma/\beta \times 2\gamma/\beta\text{-dim.}\end{array} \end{array}$$

if $\gamma \in \mathbb{N}$ and

$$E_p^{(\beta)}(t) \begin{array}{l} \nearrow \begin{array}{c}\textbf{large-}W\textbf{ model}\\ p \times n\text{-dim.}\end{array} \Leftrightarrow \begin{array}{c}\textbf{large-}\sigma\textbf{ model}\\ (2\alpha+2|2\alpha) \times (2\alpha+2|2\alpha)\text{-dim.}\end{array} \\ \searrow \begin{array}{c}\textbf{small-}\overline{W}\textbf{ model}\\ p \times \bar{n}\text{-dim.}\end{array} \Leftrightarrow \begin{array}{c}\textbf{small-}\sigma\textbf{ model}\\ (1|2\alpha) \times (1|2\alpha)\text{-dim.}\end{array} \end{array}$$

if $\beta = 1$ and $\gamma \in \frac{1}{2}\mathbb{N}$ with $\gamma = (2\alpha + 1)/2$.