Polarized Ensembles of Random Quantum States

Summer School - Randomness in Physics and Mathematics

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Intro

Some preliminaries

A **physical system** is specified by the observables $A, B, C, \dots \in \mathcal{A}$ that we can measures. The observables form an algebra $\mathcal{A}$.

In classical mechanics $\mathcal{A}$ is commutative. In quantum mechanics $\mathcal{A}$ is not.

A **state** of a system is an assignement of a number $\varphi(\mathcal{A})$ for each element $\mathcal{A}$ of the algebra of observables. Such a number $\varphi(\mathcal{A})$ is the outcome of a measurement of the observable $\mathcal{A}$.

$$\varphi : A \to \mathbb{C} \quad \varphi \text{ linear, } \varphi \geq 0, \ \varphi(\mathbf{1}) = 1.$$

Remark

The set of states $\mathcal{S}$ is convex.

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The set of states $\mathcal{S}$ is convex.

What happens if the state φ of the system is random?
What about its typical properties?

Intro

Quantum mechanics

In a quantum mechanical setting:

- $\mathcal{A} = \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ Hilbert space;
- $\varphi = \mathrm{tr}(\rho\,\cdot)$, where $\rho = \rho^\dagger$, $\rho \geq 0$, $\mathrm{tr}\rho = 1$.

Examples $\mathcal{H} = \mathbb{C}^N$

$\rho = \frac{\mathbf{1}}{N}$ maximally mixed state

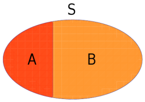
$\rho = |\psi\rangle\langle\psi|$ pure state (a rank-1 projection operator).

Remark

- The pure states $|\psi\rangle$ are normalized vectors $\|\psi\|^2 = 1$.
- The set of quantum states $\mathcal{S}(\mathcal{H})$ is convex, the pure states being its extremal points.

Composite Quantum Systems

$$S = A + B$$
$$\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$$



$$\dim \mathcal{H}_A = N$$
$$\dim \mathcal{H}_B = M$$

"...the quantum theory subscribes to the view that *the whole is greater than the sum of its parts.*" (H.Weyl)

Given a state φ_{AB} of the whole system, the state of subsystem A is given by the **reduced state** φ_A:

$$\varphi_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \xrightarrow{\mathrm{Tr}_B} \varphi_A \in \mathcal{S}(\mathcal{H}_A)$$

Definition (Entanglement)

A pure state $\varphi_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable if $\varphi_A \in \mathcal{S}(\mathcal{H}_A)$ is pure.

A non separable state is said *entangled*.

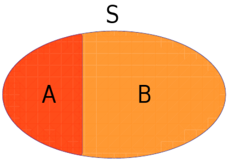
The problem

Typicality in Hilbert Spaces

Let us consider a bipartite quantum system:

$$S = A + B \ .$$

Let us choose randomly a state $|\psi\rangle_{AB}$ of system S:
What about the typical properties of the subsystem A?



Rephrasing:

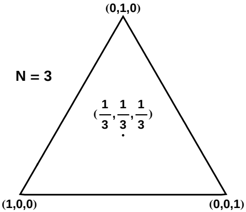
- Fix an ensemble of random pure quantum states $|\psi\rangle_{AB}$ of a bipartite space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$.
- Consider the consequent ensemble of reduced density matrices $\rho_A \in \mathcal{S}(\mathcal{H}_A)$.
- What are the typical properties of ρ_A?

Entanglement characterization

Thm (A consequence of SVD)

The pure state $|\psi\rangle_{AB}$ is separable iff $\mathrm{rank}\rho_A = 1$.

Let $(\lambda_1, \ldots, \lambda_{d_A}) = (1, 0, \ldots, 0)$ be the eigenvalues of ρ_A.
If $(\lambda_1, \ldots, \lambda_{d_A}) = (1, 0, \ldots, 0)$ then $|\psi\rangle$ is separable.
If at least two eigenvalues are $\neq 0$ the state $|\psi\rangle$ is entangled.



How to quantify the non-separability of $|\psi\rangle$?

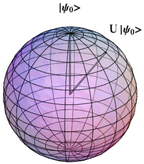
Local Purity: $$\boxed{\pi_{AB}(\psi) = \sum_i \lambda_i^2} \qquad 1/d_A \leq \pi_{AB} \leq 1$$

The lower the local purity the more entangled is $|\psi\rangle$.

Random Pure States

Unbiased ensemble

There exists a "uniform" measure μ_{Haar} on pure states (from topological reasons)



maximal simmetry in the Hilbert space $\mathcal{H}$

$\Updownarrow$

minimal prior knowledge of system S.

This ensemble has been intensively investigated (lot of RMT technology).

Induced measure on density matrices

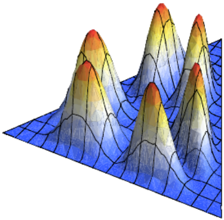
The unitarily invariant measure on pure states μ_{Haar} induces a measure ν_A on the reduced density matrices (the *push-forward measure* of μ_{Haar} w.r.t. the map Tr_B):

$$\boxed{\nu_A = (\mathrm{Tr}_B)_* \mu_{\mathrm{Haar}} = f \times \mu}$$

Joint pdf of the eigenvalues

The eigenvalues are distributed according to the following density:

$$f_{N,M}(\lambda_1, \ldots, \lambda_N) = C_{N,M} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_l \lambda_l^{M-N}$$



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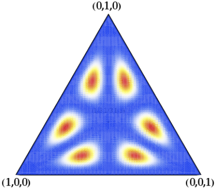
$C_{N,M}$: normalization constant

$\prod_{i<j} (\lambda_i - \lambda_j)^2$: levels repulsion

$\prod_l \lambda_l^{M-N}$: subsystem A is strongly correlated with B



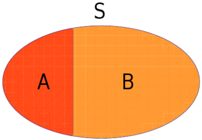
The pdf is supported in the domain defined by

$$\lambda_i \geq 0 \; , \quad \sum \lambda_k = 1 \; .$$

Want to study different physically motivated measures on the space of pure states.

The setup

A bipartite quantum system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with $d_A = \dim \mathcal{H}_A \leq \dim \mathcal{H}_B = d_B$.



Suppose that the state of the quantum system has the form (up to normalization)

$$|\psi\rangle = \alpha |\psi_1\rangle + \beta |\psi_2\rangle \,, \tag{1}$$

where $|\psi_1\rangle \,, |\psi_2\rangle \in \mathcal{H}$ are random states.

Once the probability distributions of $|\psi_1\rangle$ and $|\psi_2\rangle$ are specified,
the random state $|\psi\rangle$ is characterized by a well defined distribution.

$|\psi\rangle$ = superposition of a fixed pure state and an unbiased random one.

We get the following one-parameter ensemble

$$\boxed{|\psi\rangle = \left[\epsilon \mathbb{1}_{AB} + \sqrt{1-\epsilon^2} U_{AB}\right] |\phi_0\rangle} \tag{2}$$

where:

- the normalized state $|\phi_0\rangle \in \mathcal{H}$ is fixed;
- $\epsilon \in [0, 1]$ is a tunable parameter;
- $\mathbb{1}_{AB}$ is the identity operator;
- $U_{AB} \in \mathcal{U}(\mathcal{H})$ is a random unitary (Haar distributed).

$$|\psi\rangle = \epsilon |\phi_0\rangle + \sqrt{1-\epsilon^2} |\phi\rangle \tag{3}$$

Typical local purity

We study the local purity of one of its party

$$\pi_{AB}(\psi) = \mathrm{Tr}\rho_A^2 \in [1/d_A, 1] \ . \tag{4}$$

The lower the local purity the more entangled is $|\psi\rangle$.

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The typical purity of the random state (2) is the average value of (4):

$$\boxed{\mathbb{E}[\pi_{AB}(\psi)] = \epsilon^4 \pi_0 + \left(1 - \epsilon^4\right) \pi_{\mathrm{unb}} + O\left(\frac{1}{d_A d_B}\right)} \tag{5}$$

where $\pi_{\mathrm{unb}} = \frac{d_A + d_B}{d_A d_B}$, $\pi_0 = \pi_{AB}(|\phi_0\rangle)$ is the local purity of the bias $|\phi_0\rangle$.

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The average value (5) is typical (concentration of measure phenomenon):

$$\mathbf{Pr}\Big\{\Big|\pi_{AB}(\psi) - \mathbb{E}[\pi_{AB}(\psi)]\Big| > \alpha\Big\} \leq 2\exp\left(-\frac{d_A d_B \alpha^2}{32(1-\epsilon^2)}\right) \tag{6}$$

Isopurity manifolds

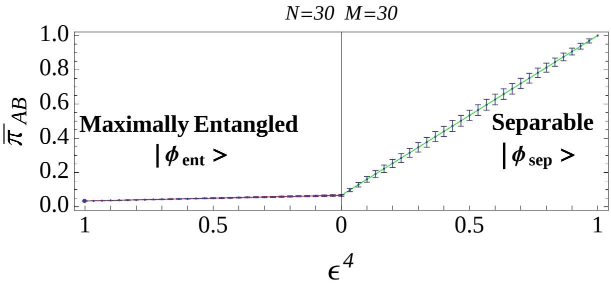
$$|\psi\rangle = \left[\epsilon U_A \otimes U_B + \sqrt{1-\epsilon^2} U_{AB}\right] |\phi_0\rangle$$
$$\pi_{AB} \simeq \epsilon^4 \pi_0 + \left(1-\epsilon^4\right) \pi_{\rm unb} \tag{7}$$

Separable bias:

$$|\phi_0\rangle = |\phi_{\rm sep}\rangle \quad \Rightarrow \quad \pi_0 = 1 \tag{8}$$

Maximally entangled bias:

$$|\phi_0\rangle = |\phi_{\rm ent}\rangle_{AB} \quad \Rightarrow \quad \pi_0 = 1/d_A \tag{9}$$



Generation of random pure states with fixed purity

An inexpensive strategy for generating random pure states with fixed value of π_{AB}:

1. Choose $\epsilon \in [0,1]$ such that

$$\pi_{AB} = \epsilon^4 \pi_0 + \left(1 - \epsilon^4\right) \pi_{\mathrm{unb}}, \tag{10}$$

   where $\pi_0 = 1$ or $\pi_0 = 1/N$ if the desired value of π_{AB} is, respectively, larger or smaller than the unbiased typical value π_{Haar}.

2. Generate a pure state $|\psi\rangle$ by superposition

$$|\psi\rangle = \epsilon \,|\phi_0\rangle + \sqrt{1 - \epsilon^2}\,|\phi\rangle \ \ , \tag{11}$$

   where $|\phi\rangle \sim \mu_{\mathrm{Haar}}$ and $|\phi_0\rangle$ is a random separable or maximally entangled pure state sampled, according to the value of π_0 chosen in (10).

Proposals

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- Typicality in multipartite systems??
- What is the role of the entanglement of random states in Statistical Mechanics?

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Robustness of separability

Application: stability of separability of quantum states with respect to random additive perturbations. If the state of a bipartite system $|\xi_0\rangle_A \otimes |\chi_0\rangle_B$ is separable, how much noise η is necessary to make the its reduced state ρ_A distinguishable from a pure state?

$$|\psi\rangle = \sqrt{1-\eta^2}\,|\xi_0\rangle_A \otimes |\chi_0\rangle_B + \eta\,|\phi\rangle \ , \quad 0 \leq \eta \leq 1, \tag{12}$$

where

- $|\phi\rangle \sim \mu_{NM}$ is an unbiased random perturbation;
- η measures the strength of the noise.

In the limit of large system sizes, $d_A, d_B \to \infty$, the threshold critical value becomes

$$\eta_\star^2 = \left(1 - \frac{1}{\sqrt{2}}\right) + O\left(\frac{1}{d_A}\right), \tag{13}$$

As long as the state $|\psi\rangle$ of the large quantum system has the form (12) with

$$\eta < \sqrt{1 - \frac{1}{\sqrt{2}}} \simeq 0.54, \tag{14}$$

the effective dimension is $d^{\mathrm{eff}}(\rho_A) < 2$, and separability will be (approximately) preserved.