Variational Formula for First Passage Percolation

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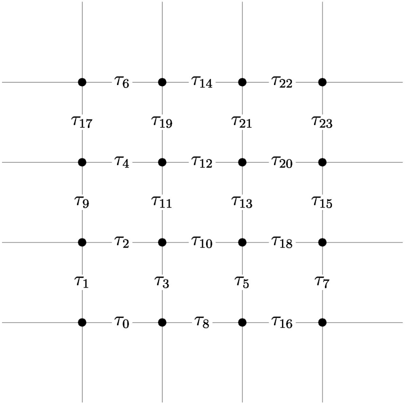
First Passage Percolation on the Lattice



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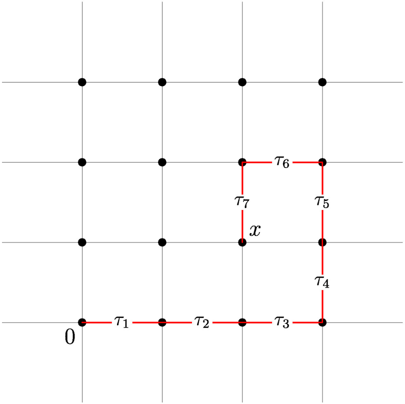

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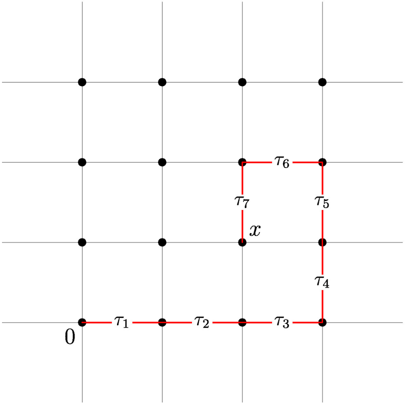

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- Path $\gamma(0,x)$ has total weight $W(\gamma(0,x))$ = sum of edge-weights
- First-Passage Time:

$$T(0,x) = \inf_{\gamma} W(\gamma(0,x))$$

We want to compute the time-constant

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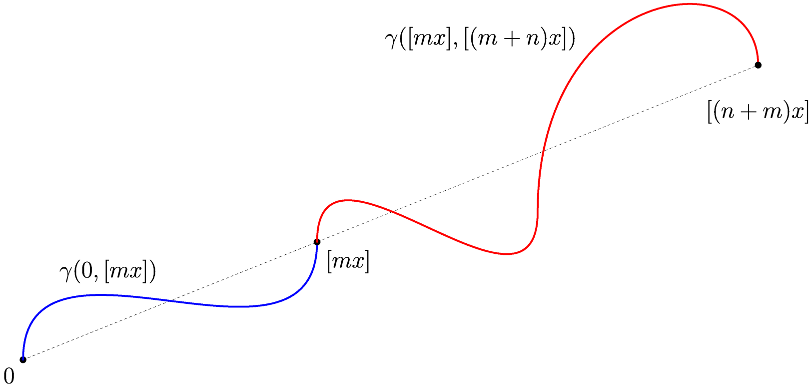

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Kingman's Theorem

- Subadditive Ergodic Theorem(Kingman, 1968): the following limit exists almost surely

$$\lim_{n\to\infty} \frac{T(0,[nx])}{n} =: \mu(x).$$

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- If the measure is ergodic, $\mu(x)$ is deterministic; hence the name time-constant.

Motivation: the limit-shape

Why is the time constant interesting?

Consider sites occupied by time t:

$$A_t := \{x \in \mathbb{R}^d \,|\, T([x]) \leq t\},$$

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Theorem: (Cox and Durrett, 1981)

$$\lim_{t \to \infty} A_t / t = \{x \mid \mu(x) \leq 1\} =: B_0$$

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- Variance is supposed to go as $n^{2/3}$, and exponents are thought to satisfy the KPZ scaling relation (Chatterjee (2011)).
- For directed FPP with special edge-weights, fluctuations follow the Tracy-Widom GUE (Johansson, 2000). Using asymptotics of orthogonal polynomials.

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- For some f, consider $u_n(x,t) = E_x[f(S_n(t))]$, and $u(x,t) = E_x[f(B(t))]$.
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- Want to show

$$\partial_t u + \partial_{xx} u = 0$$

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- If time permits, will do a heuristic proof sketch

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- $\tau(x, \cdot, \omega)$ is stationary with respect to translation on $\mathbb{Z}^d$
- Probability space Ω with measure ergodic under translation
- S set of functions $f : \mathbb{Z}^d \to \mathbb{R}$ with discrete derivatives $f(x + \alpha, \omega) - f(x, \omega)$ that are stationary and mean zero.

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$$\hat{H}(Df + p, x, \omega) = \sup_{\alpha \in \mathcal{A}} \left\{ -\frac{(f(x+\alpha,\omega) - f(x,\omega)) + p \cdot \alpha}{\tau(x,\alpha,\omega)} \right\}$$

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Solving the equation $H(D\mu(x)) = 1$

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- Hence μ is a norm.
- By a Hopf-Lax formula, H is the dual-norm of μ!

$$H(p) = \sup_{\mu(x)=1} x \cdot p$$

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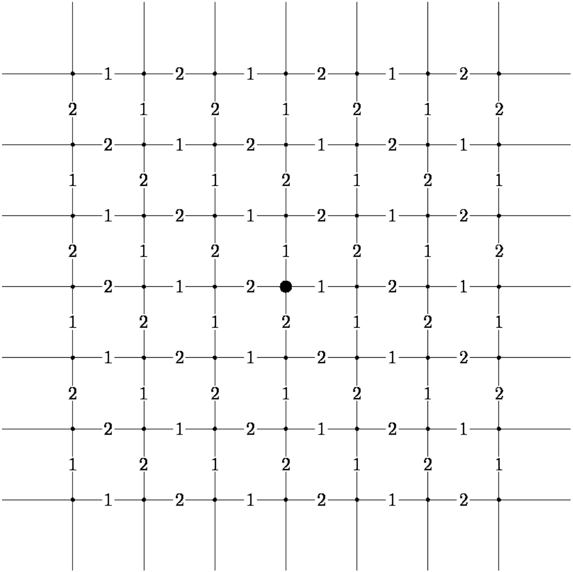
- We will see the level sets $H(p) = 1$.
- The "bigger" the Hamiltonian level-set, the slower the percolation. It's a speed-time duality.

Example: Periodic Edge-Weights

$\tau_e \in \{1, 2\}$



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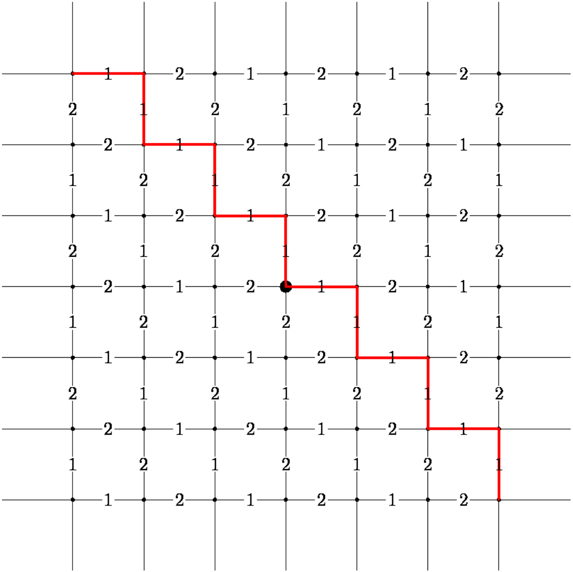
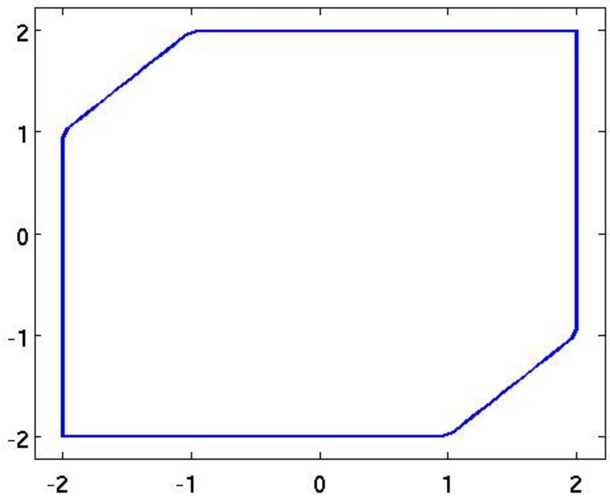
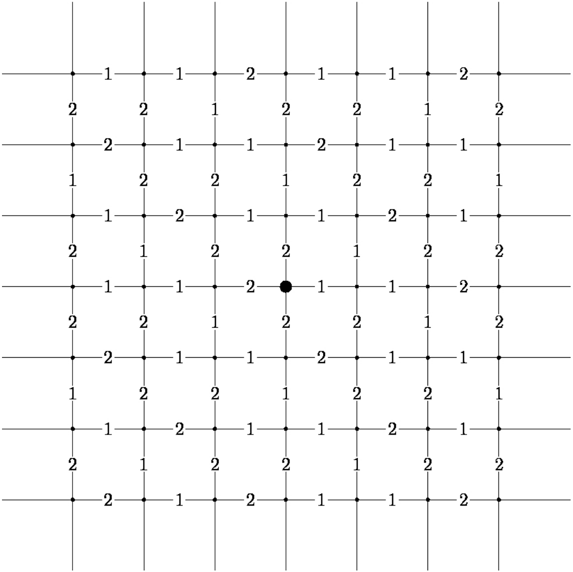


Limit Shape: Periodic Medium

$\tau \in \{1, 2\}$, Plot of $H(p) = 1$



Random Medium



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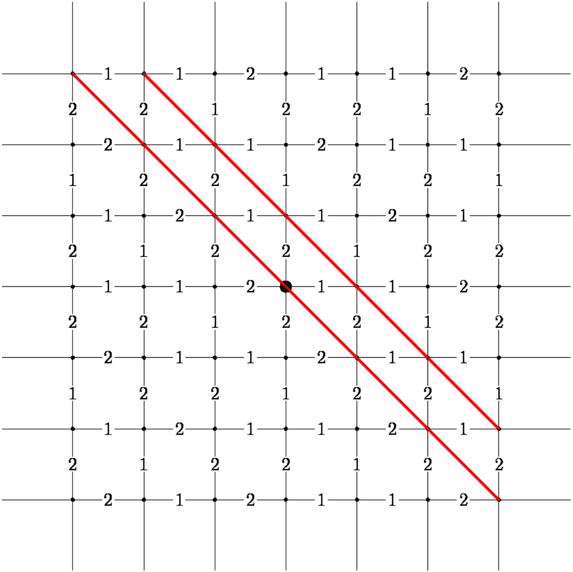

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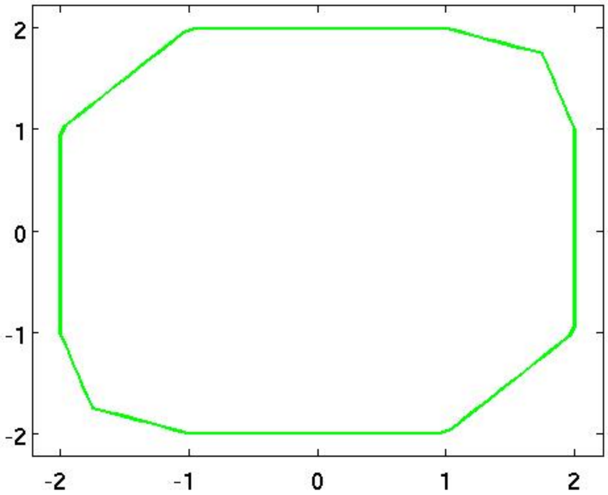
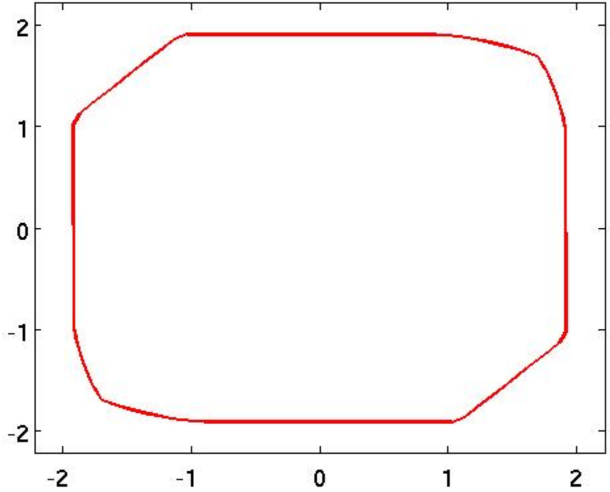
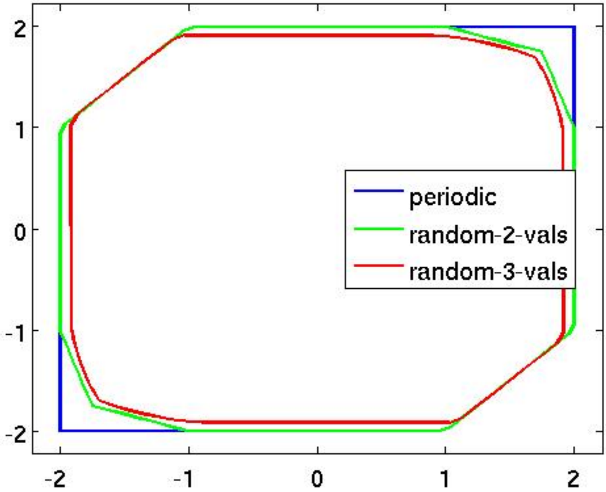


Limit Shape: Random Medium

$\tau \in \{1, 1.5, 2\}$, Plot of $H(p) = 1$



Limit Shape: Comparison of Periodic and Random limit shapes



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- Implication: if edge-weights takes n values, we need to solve a deterministic convex minimization problem in n^2 variables.

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- Can complex analysis applied to the problem? Minimization is over discrete analytic functions.

Acknowledgements

- S.R.S Varadhan
- Sourav Chatterjee
- Matan Harel, Behzad Mehrdad, Jim Portegies

Preprint coming soon on arxiv!