Universality in Quantum Chaos - Exercises

Summer School on Randomness in Physics and Mathematics, Bielefeld 2013

Classical chaos

1. *Ergodicity.*

   (a) Consider a trajectory with speed v in an ergodic billiard. We follow this trajectory for a time T. We denote by $N(T)$ the number reflections occurring during this time at a given boundary segment of length Δs with $p = \sin\beta$ (where β is the reflection angle) in the interval $[p_0, p_0 + \Delta p)$. Determine $\lim_{T\to\infty} \frac{N(T)}{T}$.

   Hint: In which part of the energy shell can the trajectory be found in the time interval τ before a reflection? Approximate the overall time spent in this part using ergodicity.

   (b) Determine $\lim_{T\to\infty} \frac{N_{\text{total}}(T)}{T}$ where $N_{\text{total}}(T)$ is the overall number of times the trajectory hits the boundary at any position and with any angle. Use your result to determine the length of mean free path in a billiard.

2. *Circle billiard (example for an integrable system).*

   Prove that all periodic orbits in a circular billiard have tr $M = 2$.

   To show this it helps to first calculate the stability matrix of a part of the orbit consisting of one half of the straight-line segment preceding a reflection and one half of the straight-line segment following this reflection. Then you can consider powers of this matrix, and compute their trace.

Semiclassics

3. *Propagators.*

   (a) Verify the formula for the free propagator given in the lecture.

   (b) Determine the propagator for a particle in one dimension with positions in $(0, \infty)$, no potential, and Dirichlet boundary condition at $q = 0$: $K(0, q_0, t) = 0$. Do the same for periodic boundary conditions where the points 0 and 1 are identified.

   Hint: The results will be sums over contributions of classical trajectories existing in the system. All summands can be written in terms of free propagators with suitably changed arguments q_0.

4. *Bessel function.*

   Use stationary phase approximations to give an approximation for the Bessel function $J_\nu(y)$ in the limit $y \to \infty$. Here ν is a natural number and the Bessel function is defined by

$$J_\nu(y) \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-i\nu x + iy \sin x) dx$$

5. *Stationary phase approximation in* $\mathbb{R}^2$.

   (a) Let A be a real symmetric positive definite 2×2 matrix. Show that

   $$\int_{\mathbb{R}^2} e^{-\boldsymbol{x}\cdot A\boldsymbol{x}} d^2x = \frac{\pi}{\sqrt{\det(A)}} .$$

   First consider the case that A is a diagonal matrix. To generalise your result write A in the form $\mathcal{O}D\mathcal{O}^{-1}$ where D is diagonal and $\mathcal{O}$ is orthogonal and choose $\boldsymbol{y} = \mathcal{O}^{-1}\boldsymbol{x}$ as a new integration variable.

   (b) Now evaluate the integral

   $$\int_{\mathbb{R}^2} e^{i\boldsymbol{x}\cdot A\boldsymbol{x}} d^2x$$

   using similar ideas.

   (c) Determine the stationary phase approximation of the two-dimensional integral

   $$I = \int_{\mathbb{R}^2} A(\boldsymbol{x}) e^{iN\phi(\boldsymbol{x})} d^2x$$

   in the limit $N \to \infty$. (You can assume that the stationary points of $\phi(\boldsymbol{x})$ are isolated and that $\phi(\boldsymbol{x})$ and $A(\boldsymbol{x})$ do not have any singularities.)

6. *Trace formula in one dimension.*

   (a) Using the formula for the free propagator and a stationary-phase approximation show that the Green function for a particle of mass M in one dimension without boundaries or potential can be approximated (for small $\hbar$, real E, and $q \neq q_0$) by

   $$G_f(q, q_0, E) \approx -\frac{iM}{\hbar^2 k} e^{ik|q-q_0|}$$

   where $k = \frac{\sqrt{2ME}}{\hbar}$.

   (b) Using the results of 6(a) and 3(b) determine the oscillatory part of the level density of a one dimensional system with periodic boundary conditions identifying 0 and 1. (An alternative way to get this result would be Poisson summation.)

Semiclassical approach to spectral statistics

7. τ^3 *contribution to the spectral form factor.*

   Derive the τ^3 contribution to the spectral form factor using the diagrams and rules given in the lecture.

   Note that the formula for contributions to the spectral form factor given in the lecture refers to a "structure" where one of the links, or alternatively one of the encounter stretches, is singled out as the first. Hence for each of the five diagrams shown you also have to determine the number of associated structures, i.e., you have to consider different ways of singling out a link as the first, and check whether they are topologically equivalent.

   Finally, you can check whether the list of diagrams given in the lecture is exhaustive, by trying to construct other diagrams that would also contribute to the order τ^3. If you find any candidates, why are they not relevant for the small-τ form factor?

8. *3-encounters.*

   Consider an encounter of three almost parallel stretches inside a periodic orbit. Let s_1, u_1 denote the stable and unstable deviation between the second and the first encounter stretch, and s_2, u_2 the stable and unstable deviation between the third and the first encounter stretch. How can the connections inside the encounter be changed? Consider the two ways of changing connections that affect all three encounter stretches, and derive the action difference for each of them.

   Hint: The reconnections can be performed in steps.

9. *Hannay-Ozorio de Almeida sum rule.*

   Using the following information formulate a condition under which the Hannay-Ozorio de Almeida sum rule is true. It turns out that this condition is satisfied for hyperbolic systems.

   The Frobenius-Perron operator P_t describes the classical evolution of densities in phase space over the time t. It is defined by $(P_t\rho)(\boldsymbol{x}) = \int d^n x_0\, \delta(\boldsymbol{x} - \Phi_t(\boldsymbol{x}_0))\rho(\boldsymbol{x}_0)$ where $\Phi_t(\boldsymbol{x}_0)$ is the phase space point reached after time t starting from the point $\boldsymbol{x}_0$ and $\rho(\boldsymbol{x})$ is a density in phase space. P_t has eigenvalues of the form $e^{-\gamma_j t}$, $\mathrm{Re}\,\gamma_j \geq 0$ and its trace can be expressed as a sum of periodic orbits $\mathrm{tr}\, P_t = \sum_p \frac{T_p^{\mathrm{prim}}}{|\det(M_p - 1)|}\delta(t - T_p)$.

10. *Diagrams arising in mesocopic quantum transport.*

    We consider mesoscopic cavities whose boundary has an irregular shape such that the classical motion inside becomes chaotic. If the cavity is opened up and one applies a voltage difference between the openings a current can flow. The conductance and related properties of the cavity can be computed using random matrix theory as well as semiclassical methods. The semiclassical approach now involves open trajectories connecting the openings.

    (a) The *conductance* is determined by pairs of trajectories α, β. α and β must begin and end close to each other, and they must have similar classical action. They may differ in encounters. Find the semiclassical diagrams in this setting that have the smallest $L - V$.

    (b) The semiclassical approach to *shot noise* (related to fluctuations of the current due to the discreteness of the electric charge) involves quadruplets of trajectories α, β, γ, δ. The cumulative action of the "partner trajectories" β and δ has to be close to the cumulative action of the "original trajectories" α and γ. β connects the beginning of γ to the end of α, and δ connects the beginning of α to the end of γ. Again find the semiclassical diagrams with the smallest $L - V$.

Resummation

11. *Generating function.*

    Complete the derivation of the generating function for systems without time-reversal invariance given in the lecture. To do so, you can use that within the diagonal approximation each orbit belonging to one of the sets A or C must also belong to one of the sets B or D. Hence you can sum over the intersections $A \cap B$, $A \cap D$, $C \cap B$, $C \cap D$. Moreover you can use the so-called dynamical zeta function

$$\zeta(s) = \sum_\Gamma |F_\Gamma|^2 (-1)^{n_\Gamma} e^{-sT_\Gamma} = \left(\sum_\Gamma |F_\Gamma|^2 e^{-sT_\Gamma} \right)^{-1}$$

where the sum goes over sets of orbits Γ and n_Γ is the number of orbits included; for small s the dynamical zeta function can be approximated by its argument. You can also leave out the restrictions $T_C, T_D < T_H/2$ and Taylor-expand the actions for small $\alpha, \beta, \gamma, \delta$.