Products of Rectangular Gaussian Matrices

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Why Products of Random Matrices?

- Transport in disordered materials
- Wireless telecommunications
- Quantum entanglement
- Finance
- Etc.

A Single Matrix

What is a Ginibre Matrix?

Non-Hermitian Gaussian Matrix (Ginibre Matrix)

- $\mathbb{X}_1$: Rectangular $N_1 \times N_0$ matrix
- Entries $[\mathbb{X}_1]_{ab}$: Independent Complex Gaussians
- Notation: $\nu_1 \equiv N_1 - N_0 \geq 0$

$$d\mu(\mathbb{X}_1) = e^{-\operatorname{Tr}\mathbb{X}_1 \mathbb{X}_1^\dagger} d\mathbb{X}_1$$

Ginibre (1965), etc.

What is an Induced Ginibre Matrix?

Induced Ginibre Matrix

$$\mathbb{X}_1 = U_1 \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$$

- $\mathbb{X}_1$: $N_1 \times N_0$ Ginibre matrix
- $U_1 \in U(N_1)/[U(N_0) \times U(\nu_1)]$
- X_1: $N_0 \times N_0$ Induced Ginibre matrix

$$d\mu_{\mathrm{Ind}}(X_1) = \det{}^{\nu_1}[X_1 {X_1}^\dagger]\, e^{-\mathrm{Tr}\, X_1 {X_1}^\dagger} dX_1$$

Fischmann, Bruzda, Khoruzhenko, Sommers & Życzkowski (2012)

Eigenvalues

Schur Decomposition

$$X_1 = U(Z + T)U^\dagger$$

- X_1: (Induced) Ginibre matrix
- $U \in U(N_0)/U(1)^{N_0}$
- $Z = \text{diag}(z_1, \ldots, z_{N_0})$: Eigenvalues
- T: Strictly upper triangular matrix

Eigenvalues

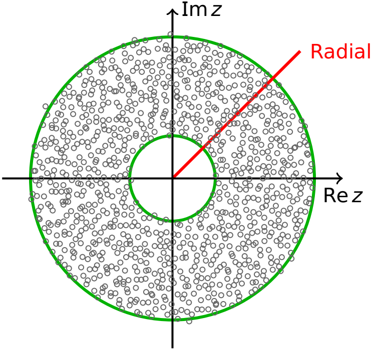
Joint Probability Density Function

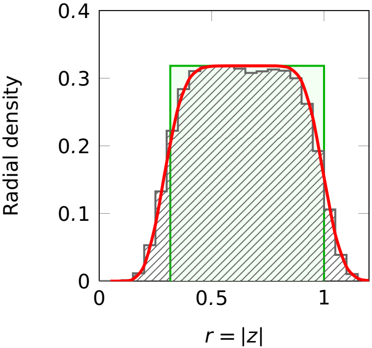
$$P_{\mathrm{jpdf}}(Z) \propto \left(\prod_{\ell=1}^{N_0} |z_\ell|^{2\nu_1} e^{-|z_\ell|^2} \right) \det_{i,j} \left[(z_i)^{j-1} \right] \det_{a,b} \left[(z_a^*)^{b-1} \right]$$

Correlation Functions

$$R_k^\nu(z_1, \ldots, z_k) = \frac{N_0!}{(N_0 - k)!} \prod_{\ell=k+1}^{N_0} \int_{\mathbb{C}} d^2 z_\ell \, P_{\mathrm{jpdf}}(z_1, \ldots, z_{N_0})$$

Eigenvalue Distribution





- Numerical
- Radial cut
- Edges

  $r_{\text{in}} = \sqrt{\mathcal{V}_1} \quad r_{\text{out}} = \sqrt{N_1}$

- Numerical
- Analytical
- Macroscopic

Singular Values

Singular Value Decomposition

$$\mathbb{X}_1 = U\Sigma V^\dagger$$

- $\mathbb{X}_1$: Ginibre matrix
- U, V: Unitary matrices
- $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_{N_0})$: Singular values

Singular Values

Joint Probability Density Function

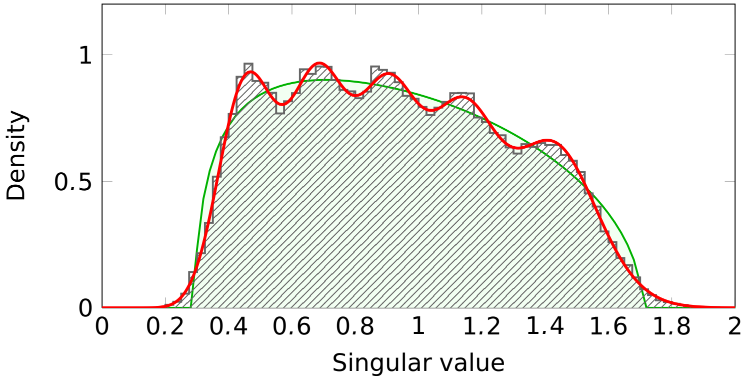
$$P_{\mathrm{jpdf}}(\Sigma) \propto \left(\prod_{\ell=1}^{N_0} \sigma_\ell \right) \det_{i,j} \left[\sigma_i^{2(j-1)} \right] \det_{a,b} \left[\sigma_a^{2(\nu_1+b-1)} e^{-\sigma_a^2} \right]$$

Correlation Functions

$$R_k^\nu(\sigma_1, \ldots, \sigma_k) = \frac{N_0!}{(N_0-k)!} \prod_{\ell=k+1}^{N_0} \int_0^\infty d\sigma_\ell \, P_{\mathrm{jpdf}}(\sigma_1, \ldots, \sigma_{N_0})$$

Singular Values

- Numerical
- Analytical
- Macroscopic (Marčenko–Pastur)



Products of Matrices

Products Ginibre Matrices

The Product Matrix

$$\mathbb{Y}_M = \mathbb{X}_M \cdots \mathbb{X}_2 \mathbb{X}_1$$

- $\mathbb{Y}_M$: $N_M \times N_0$ matrix
- $\mathbb{X}_i$: $N_m \times N_{m-1}$ Ginibre Matrix
- Notation: $\nu_m \equiv N_m - N_0 \geq 0$
  (N_0 is smallest matrix size)

Finding Eigen- & Singular Values

Two Main Points from the Derivation

1. Rectangular matrices $\leftrightsquigarrow$ Induced matrices
2. Integration over Meijer G-functions

Parameterization

The Product Matrix

$$\mathbb{Y}_M = \mathbb{X}_M \cdots \mathbb{X}_2 \mathbb{X}_1$$

Parameterization

$$\mathbb{X}_1 = \mathcal{U}_1 \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$$

- X_1: $N_0 \times N_0$ matrix
- $\mathcal{U}_1 \in \mathrm{U}(N_1) \big/ [\mathrm{U}(N_0) \times \mathrm{U}(N_1 - N_0)]$

Parameterization

The Product Matrix

$$\mathbb{Y}_M = \mathbb{X}_M \cdots \mathbb{X}_3 \mathbb{X}_2 \mathcal{U}_1 \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$$

Parameterization

$$\mathbb{X}_2 \mathcal{U}_1 = \mathcal{U}_2 \begin{pmatrix} X_2 & A_2 \\ 0 & B_2 \end{pmatrix}$$

- X_2: $N_0 \times N_0$ matrix
- A_2: $N_0 \times (N_1 - N_0)$ matrix
- B_2: $(N_2 - N_0) \times (N_1 - N_0)$ matrix
- $\mathcal{U}_2 \in \mathrm{U}(N_2) \big/ [\mathrm{U}(N_0) \times \mathrm{U}(N_2 - N_0)]$

Parameterization

The Product Matrix

$$\mathbb{Y}_M = \mathbb{X}_M \cdots \mathbb{X}_4 \mathbb{X}_3 \mathcal{U}_2 \begin{pmatrix} X_2 & A_2 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$$

Parameterization

$$\mathbb{X}_3 \mathcal{U}_2 = \mathcal{U}_3 \begin{pmatrix} X_3 & A_3 \\ 0 & B_3 \end{pmatrix}$$

- X_3: $N_0 \times N_0$ matrix
- A_3: $N_0 \times (N_2 - N_0)$ matrix
- B_3: $(N_3 - N_0) \times (N_2 - N_0)$ matrix
- $\mathcal{U}_3 \in \mathrm{U}(N_3) \Big/ [\mathrm{U}(N_0) \times \mathrm{U}(N_3 - N_0)]$

Parameterization

Final Parameterization

$$\mathbb{Y}_M = \mathcal{U}_M \begin{pmatrix} X_M & A_M \\ 0 & B_M \end{pmatrix} \cdots \begin{pmatrix} X_2 & A_2 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} X_1 \\ 0 \end{pmatrix} = \mathcal{U}_M \begin{pmatrix} Y_M \\ 0 \end{pmatrix}$$

- $Y_M = X_M X_{M-1} \cdots X_1$: $N_0 \times N_0$ matrix
- X_m: Induced Ginibre matrices

Finding Eigen- & Singular values

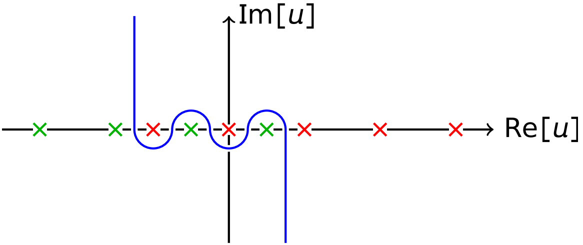
Two Main Points from the Derivation

1. Rectangular matrices $\rightsquigarrow$ Induced matrices
2. Integration over Meijer G-functions

Meijer G-function

Definition

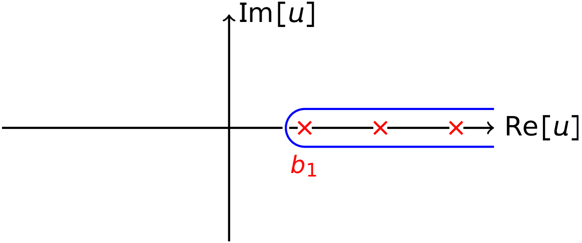
$$G^{m,n}_{p,q}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle| z\right) = \frac{1}{2\pi i}\int_{Path} du\, z^u \frac{\prod_{i=1}^{m}\Gamma[b_i-u]\prod_{i=1}^{n}\Gamma[1-a_i+u]}{\prod_{i=n+1}^{p}\Gamma[a_i-u]\prod_{i=m+1}^{q}\Gamma[1-b_i+u]}$$



Meijer G-function

A Simple Example

$$G^{1,0}_{0,1}\left(\begin{matrix} - \\ b_1 \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_{path} du\, z^u\, \Gamma[b_1 - u] = \sum_{k=0}^{\infty} R_k = z^{b_1} e^{-z}$$



Joint Probability Density Functions

JPDF for Eigenvalues

$$P_{\text{jpdf}}^{\text{EV}}(Z) \propto \left(\prod_{\ell=1}^{N_0} |z_\ell|^{2\nu_1} e^{-|z_\ell|^2} \right) \det_{i,j} \left[(z_i)^{j-1} \right] \det_{a,b} \left[(z_a^*)^{b-1} \right]$$

JPDF for Singular Values

$$P_{\text{jpdf}}^{\text{SV}}(\Sigma) \propto \left(\prod_{\ell=1}^{N_0} \sigma_\ell \right) \det_{i,j} \left[\sigma_i^{2(j-1)} \right] \det_{a,b} \left[\sigma_a^{2(\nu_1+b-1)} e^{-\sigma_a^2} \right]$$

Joint Probability Density Functions

JPDF for Eigenvalues

$$P_{\text{jpdf}}^{\text{EV}}(Z) \propto \left(\prod_{\ell=1}^{N_0} G_{0,1}^{1,0}\left({- \atop \nu_1} \,\middle|\, |z_\ell|^2 \right) \right) \det_{i,j}\left[(z_i)^{j-1} \right] \det_{a,b}\left[(z_a^*)^{b-1} \right]$$

JPDF for Singular Values

$$P_{\text{jpdf}}^{\text{SV}}(\Sigma) \propto \left(\prod_{\ell=1}^{N_0} \sigma_\ell \right) \det_{i,j}\left[\sigma_i^{2(j-1)} \right] \det_{a,b}\left[G_{0,1}^{1,0}\left({- \atop \nu_1+b-1} \,\middle|\, \sigma_a^2 \right) \right]$$

Meijer G-function

Integration formula

$$\int_0^\infty dt\, e^{-t} t^{b_0-1} G^{m,n}_{p,q}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, \frac{s}{t} \right) = G^{m+1,n}_{p,q+1}\left(\begin{matrix} a_1, \dots, a_p \\ b_0, b_1, \dots, b_q \end{matrix} \,\middle|\, s \right)$$

Joint Probability Density Functions

JPDF for Eigenvalues

$$P_{\text{jpdf}}^{\text{EV}}(Z) \propto \left(\prod_{\ell=1}^{N_0} G_{0,M}^{M,0}\left({- \atop \nu_M, \ldots, \nu_1} \middle| |z_\ell|^2 \right) \right) \times \det_{i,j}\left[(z_i)^{j-1} \right] \det_{a,b}\left[(z_a^*)^{b-1} \right]$$

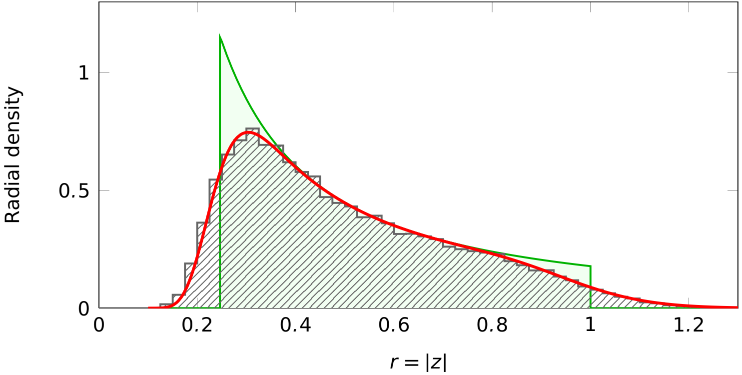
JPDF for Singular Values

$$P_{\text{jpdf}}^{\text{SV}}(\Sigma) \propto \left(\prod_{\ell=1}^{N_0} \sigma_\ell \right) \det_{i,j}\left[\sigma_i^{2(j-1)} \right] \times \det_{a,b}\left[G_{0,M}^{M,0}\left({- \atop \nu_M, \ldots, \nu_2, \nu_1+b-1} \middle| \sigma_a^2 \right) \right]$$

Eigenvalue Distribution

- Numerical
- Analytical
- Macroscopic

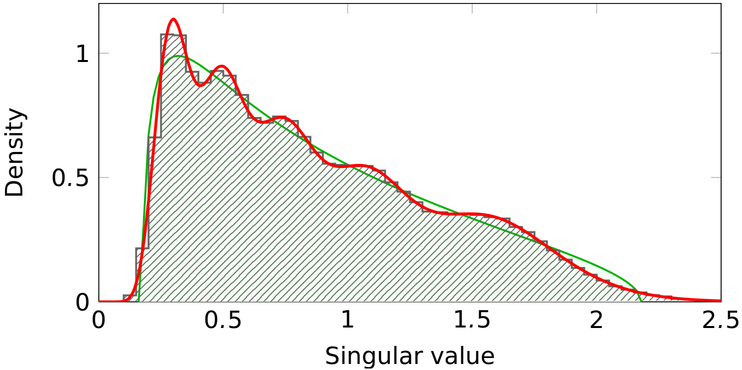
  $r_{\text{in}} = \sqrt{\nu_1 \cdots \nu_M}, \quad r_{\text{out}} = \sqrt{N_1 \cdots N_M}$



Singular Value Distribution

- Numerical
- Analytical
- Macroscopic

Burda, Jarosz, Livan, Nowak & Swiech (2010)



Correlation Functions

Correlations for Eigenvalues

$$R_k^{\mathrm{EV}}(z_1,\ldots,z_k) = \prod_{\ell=1}^{k} G^{M,0}_{0,M}\left({- \atop \nu_M,\ldots,\nu_1} \,\middle|\, |z_\ell|^2 \right)$$

$$\times \det_{1\le a,b\le k}\left[\sum_{n=0}^{N_0-1} \frac{z_a z_b^*}{\prod_{m=1}^{M}(n+\nu_m)!} \right]$$

Correlations for Singular Values

$$R_k^{\mathrm{SV}}(s_1,\ldots,s_k) = \det_{1\le a,b\le k}\left[\sum_{n=0}^{N_0-1} G^{1,0}_{1,M+1}\left({n+1 \atop 0,-\nu_M,\ldots,-\nu_1} \,\middle|\, s_a \right) \right.$$

$$\left. \times\, G^{M,1}_{1,M+1}\left({-n \atop \nu_M,\ldots,\nu_1,0} \,\middle|\, s_b \right) \right]$$