

Distribution of Bipartite Entanglement of a Random Pure State

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[J. Stat. Phys. 142, 403 \(2011\)](#) (long version)

Plan

Plan:

- Bipartite entanglement of a random pure state

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- Reduced density matrix \rightarrow random matrix theory

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- Distribution of the Renyi entropy

Results

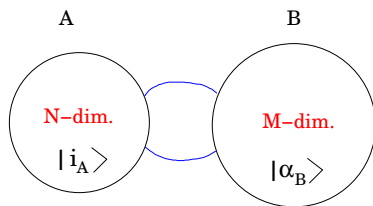
Coulomb gas technique for large systems

Phase transitions

Plan:

- Bipartite entanglement of a random pure state
- Reduced density matrix \rightarrow random matrix theory
- Distribution of the Renyi entropy
 - Results
 - Coulomb gas technique for large systems
 - Phase transitions
- Summary and Conclusions

Coupled Bipartite System



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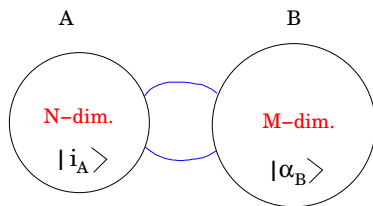
$$N \ll M$$

Bipartite quantum system $A \times B$: Hilbert space $H_A \otimes H_B$

subsystem A: dimension N (the system to study)

subsystem B: dimension $M \geq N$ ("environment")

Coupled Bipartite System



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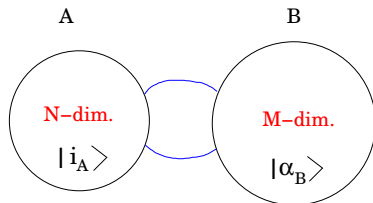
Bipartite quantum system $A \times B$: Hilbert space $H_A \otimes H_B$

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Large system: limit $N \gg 1$ and $M \gg 1$ with $M \approx N$.

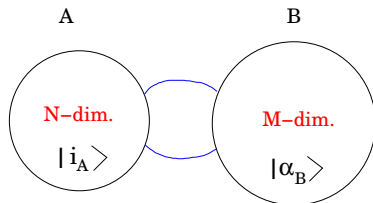
Pure State and Reduced Density Matrix



Coupled Bipartite System

$$N \ll M$$

Pure State and Reduced Density Matrix



Coupled Bipartite System

$$N \ll M$$

Any **pure** state of the full system:

$$|\psi\rangle = \sum_{i,\alpha} x_{i,\alpha} |i_A\rangle \otimes |\alpha_B\rangle$$

$X = [x_{i,\alpha}] \rightarrow (N \times M)$ rectangular **Coupling** matrix

Pure State of Bipartite System



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Pure State of Bipartite System

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$$|\psi\rangle = \sum_{i,\alpha} x_{i,\alpha} |i_A\rangle \otimes |\alpha_B\rangle$$

- If $x_{i,\alpha} = a_i b_\alpha$ then

$$|\psi\rangle = \sum_i a_i |i_A\rangle \otimes \sum_\alpha b_\alpha |\alpha_B\rangle = |\Phi_A\rangle \otimes |\Phi_B\rangle$$

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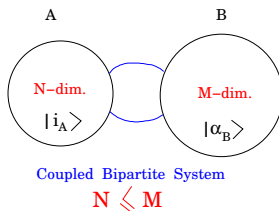
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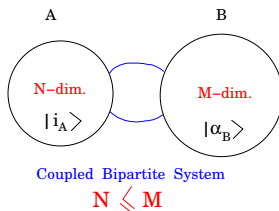
$$\hat{\rho} = |\psi\rangle\langle\psi| \text{ with } \text{Tr}[\hat{\rho}] = 1$$

- Pure state: $\hat{\rho} \neq \sum_k p_k |\psi_k\rangle\langle\psi_k|$ → not a Mixed state

Reduced Density Matrix of subsystem A:

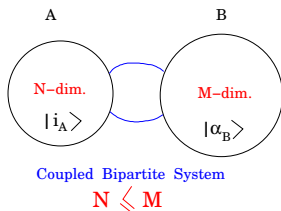


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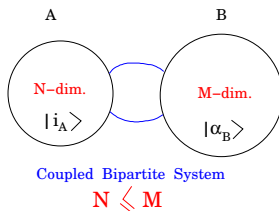
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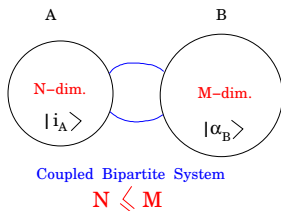


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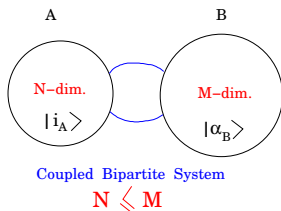


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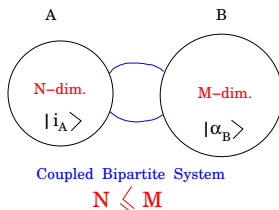


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$$= \sum_{i,j=1}^N W_{i,j} |i_A\rangle \langle j_A|$$

where the $N \times N$ matrix: $W = XX^\dagger$

Eigenvalues of W :

- In the diagonal representation

$$\hat{\rho}_A = \sum_{i,j=1}^N W_{i,j} |i_A\rangle\langle j_A| \rightarrow \sum_{i=1}^N \lambda_i |\lambda_i^A\rangle\langle \lambda_i^A|$$

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- Schmidt decomposition:

$$|\psi\rangle = \sum_{i=1}^N \sqrt{\lambda_i} |\lambda_i^A\rangle \otimes |\lambda_i^B\rangle$$

Entanglement

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and

$$\sum_{i=1}^N \lambda_i = 1$$

(i) **Unentangled**

$$\lambda_{i_0} = 1, \lambda_j = 0 \quad \forall j \neq i_0$$

$$|\psi\rangle = |\lambda_{i_0}^A\rangle \otimes |\lambda_{i_0}^B\rangle$$

is **separable**

$$\hat{\rho}_A = |\lambda_{i_0}^A\rangle\langle\lambda_{i_0}^A| \text{ is pure}$$

(ii) **Maximally entangled**

$$\lambda_j = 1/N \text{ for all } j$$

(all eigenvalues equal)

$|\psi\rangle$ is a superposition of all product states

$$\hat{\rho}_A = \frac{1}{N} \sum_{i=1}^N |\lambda_i^A\rangle\langle\lambda_i^A| = \frac{1}{N} \mathbf{1}_A \text{ is completely mixed}$$

Entanglement entropy

Subsystem A described by $\hat{\rho}_A = \sum_{i=1}^N \lambda_i |\lambda_i^A\rangle \langle \lambda_i^A|$

- **Von Neuman entropy:** $S_{\text{VN}} = -\text{Tr} [\hat{\rho}_A \ln \hat{\rho}_A] = -\sum_{i=1}^N \lambda_i \ln \lambda_i$

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- $S_{q \rightarrow 1} = S_{\text{VN}}$ and $S_{q \rightarrow \infty} = -\ln(\lambda_{\max})$
- $\Sigma_2 = \exp[-S_{q=2}] \rightarrow$ **Purity**

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$$\lambda_{i_0} = 1, \lambda_j = 0 \quad \forall j \neq i_0$$

$$\hat{\rho}_A = |\lambda_{i_0}^A\rangle\langle\lambda_{i_0}^A| \text{ is pure}$$

$S_{\text{VN}} = 0 = S_q$ is **minimal**

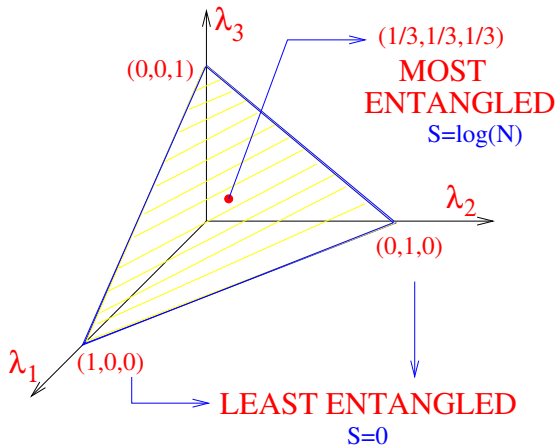
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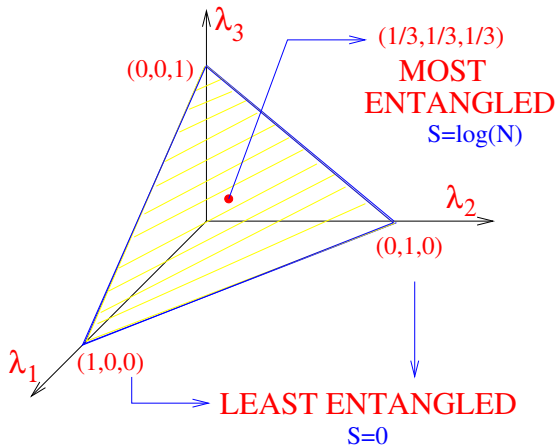
$$\hat{\rho}_A = \frac{1}{N} \sum_{i=1}^N |\lambda_i^A\rangle\langle\lambda_i^A| \text{ mixed}$$

$S_{\text{VN}} = \ln N = S_q$ is **maximal**

A Simple Diagram for $N=3$



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Random Pure State: Haar Measure

$$|\psi\rangle = \sum_{i,\alpha} x_{i,\alpha} |i_A\rangle \otimes |\alpha_B\rangle = \sum_{i=1}^N \sqrt{\lambda_i} |\lambda_i^A\rangle \otimes |\lambda_i^B\rangle$$

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- **random pure state** \rightarrow where $X = [x_{i,\alpha}]$ are **uniformly distributed** among the sets of $\{x_{i,\alpha}\}$ satisfying $\sum_{i,\alpha} |x_{i,\alpha}|^2 = \text{Tr}\rho = 1$

\rightarrow **Haar measure**

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- Equivalently, for the $N \times M$ matrix

$$P(X) \propto \delta(\text{Tr}(XX^\dagger) - 1)$$

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- Equivalently, for the $N \times M$ matrix

$$P(X) \propto \delta(\text{Tr}(XX^\dagger) - 1)$$

- In the basis $|i^A\rangle$ of H_A : $\hat{\rho}_A = W = XX^\dagger$

\rightarrow Distribution of the eigenvalues λ_i of $\hat{\rho}_A$?

\rightarrow **Distribution of the entanglement entropies** S_{VN}, S_q ?

Joint PDF of Eigenvalues

- the joint pdf $P(\lambda_1, \lambda_2, \dots, \lambda_N)$
where $\{\lambda_1, \lambda_2, \dots, \lambda_N\} \rightarrow$ eigenvalues of the Wishart matrix

$$W = \mathbf{X}\mathbf{X}^\dagger \quad (\mathbf{X} \rightarrow N \times M \text{ Gaussian random matrix})$$

with an additional constraint

$$\sum_{i=1}^N \lambda_i = 1$$

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- Joint distribution of Wishart eigenvalues (James '64):

$$P(\{\lambda_i\}) \propto \exp \left[-\frac{\beta}{2} \sum_{i=1}^N \lambda_i \right] \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j < k} |\lambda_j - \lambda_k|^\beta$$

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(Lloyd & Pagels '88, Zyczkowski & Sommers '2001)

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- Joint distribution of Wishart eigenvalues (James '64):

$$P(\{\lambda_i\}) \propto \exp \left[-\frac{\beta}{2} \sum_{i=1}^N \lambda_i \right] \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j < k} |\lambda_j - \lambda_k|^\beta \times \delta \left(\sum_{i=1}^N \lambda_i - 1 \right)$$

(Lloyd & Pagels '88, Zyczkowski & Sommers '2001)

- Given this pdf, what is the distribution of the Renyi entropy:

$$S_q = \frac{1}{1-q} \ln \left[\sum_{i=1}^N \lambda_i^q \right]$$

Random Pure State: Well Studied

- Average von Neumann entropy: $\langle S_{VN} \rangle \rightarrow \ln N - \frac{N}{2M}$ for $M \sim N \gg 1$
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- Laplace transform of the purity ($q = 2$) distribution for large N [Facchi et. al. (2008, 2010)]

Results: Entropy Distribution

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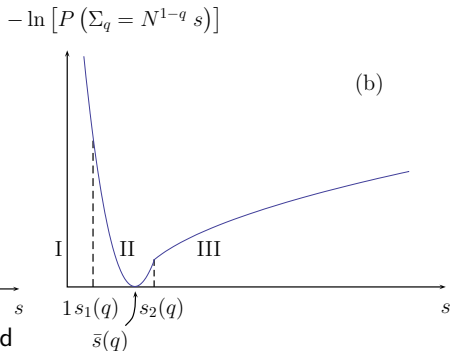
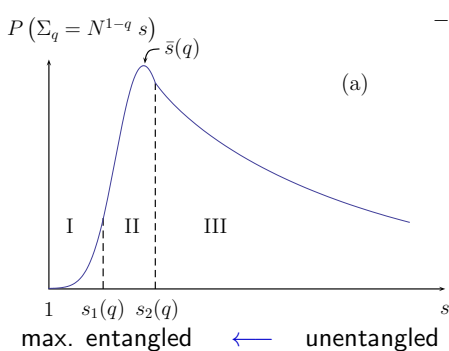
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For $q = 2$, $\langle \Sigma_2 \rangle \approx \frac{2}{N}$ (purity); For $q \rightarrow 1$, $\langle S_{VN} \rangle \approx \ln N - \frac{1}{2}$

Results: pdf of $\Sigma_q = \sum_i \lambda_i^q$

$$P(\Sigma_q = N^{1-q} s) \approx \begin{cases} \exp\{-\beta N^2 \Phi_I(s)\} & \text{for } 1 \leq s < s_1(q) \\ \exp\{-\beta N^2 \Phi_{II}(s)\} & \text{for } s_1(q) < s < s_2(q) \\ \exp\{-\beta N^{1+\frac{1}{q}} \Psi_{III}(s)\} & \text{for } s > s_2(q) \end{cases}$$



Computation of the pdf of Σ_q : Coulomb gas method

- PDF of $\Sigma_q = \sum_i \lambda_i^q$

$$P(\Sigma_q, N) = \int P(\lambda_1, \dots, \lambda_N) \delta\left(\sum_i \lambda_i^q - \Sigma_q\right) \left(\prod_i d\lambda_i\right).$$

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- The joint pdf of the eigenvalues can be interpreted as a **Boltzmann weight** at inverse temperature β :

$$\begin{aligned} P(\lambda_1, \dots, \lambda_N) &\propto \delta\left(\sum_i \lambda_i - 1\right) \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}(M-N+1)-1} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \\ &\propto \exp\{-\beta E[\{\lambda_i\}]\} \end{aligned}$$

where

$$E[\{\lambda_i\}] = -\gamma \sum_{i=1}^N \ln \lambda_i - \sum_{i<j} \ln |\lambda_i - \lambda_j| \quad (\text{with } \sum_i \lambda_i = 1)$$

→ effective energy of a **2D** Coulomb gas of charges

Charge Density and Effective Energy

$$E\{\lambda_i\} = -\gamma \sum_{i=1}^N \ln \lambda_i - \sum_{i < j} \ln |\lambda_i - \lambda_j| \quad \text{with} \quad \sum_i \lambda_i = 1$$

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Continuous **charge density** in the large N limit (regimes **I**, **II**):

$$\{\lambda_i\} \longrightarrow \rho(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i N)$$

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with **effective energy**

$$\begin{aligned} E_s[\rho] = & -\frac{1}{2} \int_0^\infty \int_0^\infty dx dx' \rho(x) \rho(x') \ln |x - x'| \\ & + \mu_0 \left(\int_0^\infty dx \rho(x) - 1 \right) \\ & + \mu_1 \left(\int_0^\infty dx x \rho(x) - 1 \right) + \mu_2 \left(\int_0^\infty dx x^q \rho(x) - s \right) \end{aligned}$$

where μ_0 , μ_1 and μ_2 are Lagrange multipliers

Saddle Point Solution

$$E_s[\rho] = -\frac{1}{2} \int_0^\infty \int_0^\infty dx dx' \rho(x) \rho(x') \ln|x-x'| + \mu_0 \left(\int_0^\infty dx \rho(x) - 1 \right) \\ + \mu_1 \left(\int_0^\infty dx x \rho(x) - 1 \right) + \mu_2 \left(\int_0^\infty dx x^q \rho(x) - s \right)$$

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$V(x) \rightarrow$ **effective potential** for the charges.

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- Taking derivative with respect to x gives

$$\mathcal{P} \int_0^\infty dx' \frac{\rho_c(x')}{x-x'} = \mu_1 + q \mu_2 x^{q-1} = V'(x)$$

Steps for Computing the PDF of Σ_q

(i) find the **solution** $\rho_c(x)$ of the integral equation

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Tricomi's solution: density $\rho_c(x)$ with finite support $[L_1, L_2]$:

$$\rho_c(x) = \frac{1}{\pi \sqrt{x-L_1} \sqrt{L_2-x}} \left[C - \mathcal{P} \int_{L_1}^{L_2} \frac{dy}{\pi} \frac{\sqrt{y-L_1} \sqrt{L_2-y}}{x-y} V'(y) \right],$$

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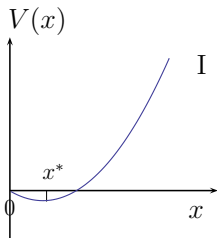
(iii) evaluate the **saddle point energy** $E_s[\rho_c]$

Phase Transitions

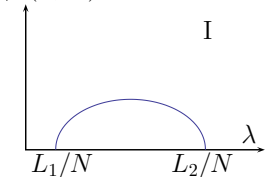
Regime I:

$$1 < s < s_1$$

$$(s_1(q=2) = 5/4)$$



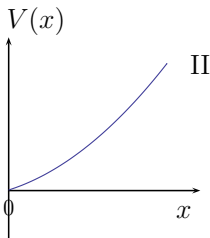
(a)
 $\rho_c(\lambda, N)$



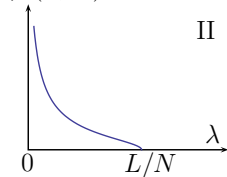
Regime II:

$$s_1 < s < s_2$$

$$(s_2(q=2) = 2 + \frac{2^{4/3}}{N^{1/3}})$$



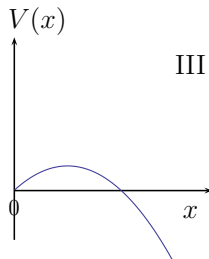
(b)
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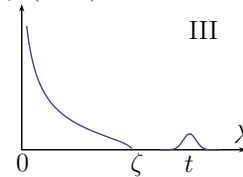
Regime III:

$$s > s_2$$

(weakly entangled)



(c)
 $\rho_c(\lambda, N)$



$q = 2$: Purity

- **Regime I:** $1 < s < 5/4$

$$\rho_c(x) = \frac{\sqrt{L_2 - x} \sqrt{x - L_1}}{2\pi(s - 1)},$$

$$P\left(\Sigma_2 = \frac{s}{N}, N\right) \propto e^{-\beta N^2 \Phi_I(s)}, \quad \Phi_I(s) = -\frac{1}{4} \ln(s - 1) - \frac{1}{8}$$

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- **Regime II:** $5/4 < s < 2 + \frac{2^{4/3}}{N^{1/3}}$

$$\rho_c(x) = \frac{1}{\pi} \sqrt{\frac{L - x}{x}} (A + Bx) \quad \text{with } L = 2 \left(3 - \sqrt{9 - 4s}\right)$$

$$P\left(\Sigma_2 = \frac{s}{N}, N\right) \propto e^{-\beta N^2 \Phi_{II}(s)}, \quad \Phi_{II}(s) = -\frac{1}{2} \ln\left(\frac{L}{4}\right) + \frac{6}{L^2} - \frac{5}{L} + \frac{7}{8}$$

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- **3-rd order phase transition at** $s = 5/4$

$$\left. \frac{d^3 \Phi_I(s)}{ds^3} \right|_{s=5/4} = -32 \quad \text{while} \quad \left. \frac{d^3 \Phi_{II}(s)}{ds^3} \right|_{s=5/4} = -16$$

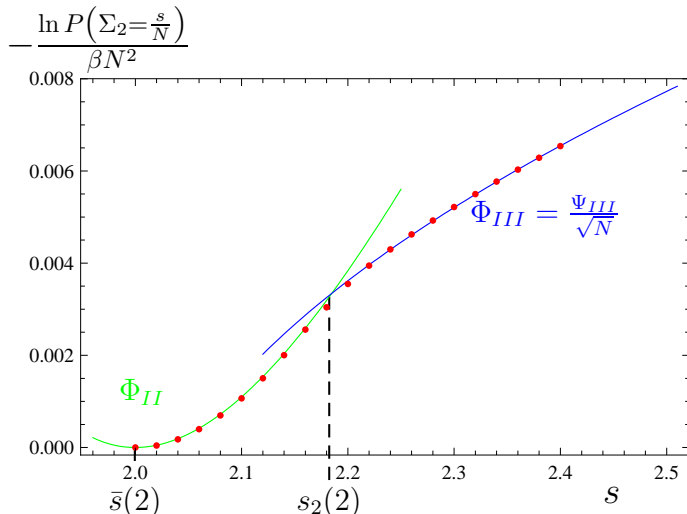
Regime III

- **Regime III:** $s > 2 + \frac{2^{4/3}}{N^{1/3}}$ Continuous density with support $[0, \zeta]$ with $\zeta \approx \frac{4}{N}$ and separated $\lambda_{\max} \gg \zeta$: $\lambda_{\max} = t \approx \frac{\sqrt{s-2}}{\sqrt{N}}$

$$P\left(\Sigma_2 = \frac{s}{N}, N\right) \approx e^{-\beta N^{\frac{3}{2}} \Psi_{III}(s)}, \quad \Psi_{III}(s) = \frac{\sqrt{s-2}}{2}$$

Numerical simulations

Monte Carlo Simulations (non-standard Metropolis algorithm):
pdf of Σ_2 (purity) for $N = 1000$ (second phase transition)

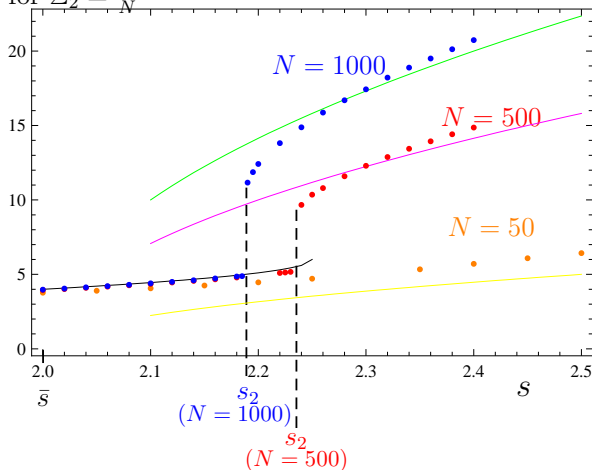


Numerical Simulations (2)

Jump of the maximal eigenvalue (rightmost charge) at $s = s_2$ for $q = 2$ and different values of N

$$N t = N \lambda_{\max}$$

for $\Sigma_2 = \frac{s}{N}$



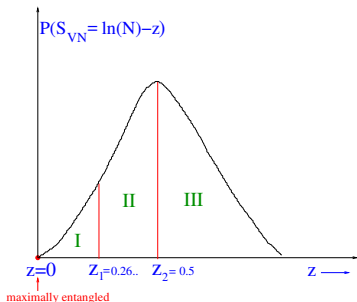
The limit $q \rightarrow 1$ — von Neumann Entropy

$$S_{VN} = S_{q \rightarrow 1} = - \sum_{i=1}^N \lambda_i \ln(\lambda_i) = \ln(N) - z$$

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$$P(S_{VN} = \ln(N) - z) \approx \begin{cases} \exp \{ -\beta N^2 \phi_I(z) \} & \text{for } 0 \leq z < z_1 = \frac{2}{3} - \ln\left(\frac{3}{2}\right) = 0.26 \\ \exp \{ -\beta N^2 \phi_{II}(z) \} & \text{for } 0.26.. < z < z_2 = 1/2 \\ \exp \left\{ -\beta \frac{N^2}{\ln(N)} \phi_{III}(z) \right\} & \text{for } z > z_2 = 1/2 \end{cases}$$



The limit $q \rightarrow \infty$ — Maximal eigenvalue

- $S_{q \rightarrow \infty} = -\ln(\lambda_{\max})$

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- Around its mean, $\langle \lambda_{\max} \rangle = 4/N$, the typical fluctuations ($\sim O(N^{-5/3})$) are described by the Tracy-Widom distribution

$$\lambda_{\max} = \frac{4}{N} + 2^{4/3} N^{-5/3} Y_\beta$$

$Y_\beta \rightarrow$ random variable distributed via Tracy-Widom law

The limit $q \rightarrow \infty$ — Maximal eigenvalue

- $S_{q \rightarrow \infty} = -\ln(\lambda_{\max})$

$$P\left(\lambda_{\max} = \frac{t}{N}\right) \approx \begin{cases} \exp\{-\beta N^2 \chi_I(t)\} & \text{for } 1 \leq t < t_1 = \frac{4}{3} \\ \exp\{-\beta N^2 \chi_{II}(t)\} & \text{for } \frac{4}{3} < t < t_2 = 4 \\ \exp\{-\beta N \chi_{III}(t)\} & \text{for } t > t_2 = 4 \end{cases}$$

- Around its mean, $\langle \lambda_{\max} \rangle = 4/N$, the typical fluctuations ($\sim O(N^{-5/3})$) are described by the Tracy-Widom distribution

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- For finite N and M , see [P. Vivo \(2011\)](#)

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Work in progress (C. Nadal, S.M with C. Pineda and T. Seligman)

Distribution of entropy for N finite but $M \gg N$ (large environment)?

On the large N distribution of the Renyi entropy:

- C. Nadal, S.M. and M. Vergassola, *Phys. Rev. Lett.* 104, 110501 (2010)
- long version: *J. Stat. Phys.* 142, 403 (2011)

On the distribution of the minimum Schmidt number λ_{\min} :

- S.M., O. Bohigas and A. Lakshminarayan, *J. Stat. Phys.* 131, 33 (2008)
- see also S.M. in “Handbook of Random Matrix Theory” (ed. by G. Akemann, J. Baik and P. Di Francesco and forwarded by F.J. Dyson) (Oxford Univ. Press, 2011) (arXiv:1005.4515)