The Spectral Density of Scale-free Networks

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References:

[1] T. Nagao and G.J. Rodgers, J. Phys. A: Math. Theor. 41 (2008) 265002.

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[3] T. Nagao, J. Phys. A: Math. Theor. 46 (2013) 065003.

Introduction

The theory of complex networks has dramatically been developed since the end of the last century. It is based on the observation that there are universal features in real biological and social networks.

One of such features is the scale-free property, meaning that the degree (the number of edges directly connected to each node) distribution function $P(\Delta)$ obeys a power law $P(\Delta) \propto \Delta^{-\lambda}$ for large Δ.

Barabási and Albert argued that the scale-free property is caused by the process of "preferential attachment". For example, if the Frankfurt airport has many flight connections to other airports, then more airports like to have connections to the Frankfurt airport.

As a result, the Frankfurt airport tends to have a huge number of flight connections. In such a situation, the degree distribution of the airline network tends to obey a power law.

An airport like the Frankfurt airport is called a “hub” in the airline network. In network theory, such special nodes connected to singularly many edges are called hubs. The existence of hubs makes the network structure inhomogeneous. Such an inhomogeneous structure is a typical feature of a scale-free network.

In this talk, the spectral densities of the adjacency and Laplacian matrices describing scale-free networks are analyzed by using a standard technique of random matrix theory.

Outline

1. Spectral density of networks

2. GKK model of scale-free networks

3. Replica method

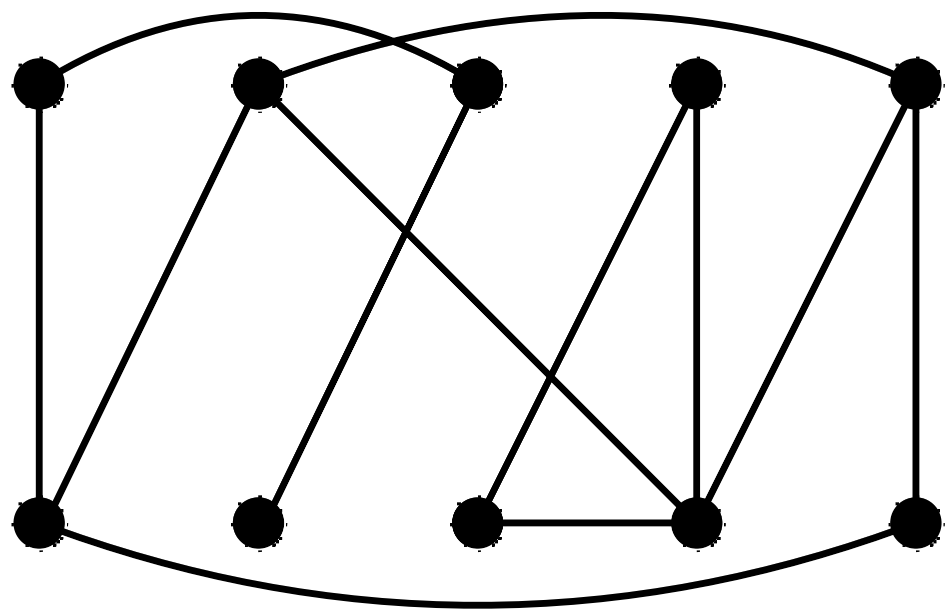
4. Bipartite scale-free networks

5. Effective medium approximation

6. Summary

1. Spectral density of networks

Let us consider a network (graph) with N nodes and examine the asymptotic behavior in the limit $N \to \infty$.



Example: a network with N = 10

The connection pattern of the network is described by the adjacency matrix A, which is an $N \times N$ symmetric matrix with

$$A_{jl} = \begin{cases} 1, & \text{if } j\text{−th and } l\text{−th nodes are directly connected,} \\ 0, & \text{otherwise.} \end{cases}$$

The number of edges attached to the j-th node

$$k_j = \sum_{l=1}^{N} A_{jl}$$

is called the degree.

The degree distribution function $P(\Delta)$ is defined as

$$P(\Delta) = \left\langle \frac{1}{N} \sum_{j=1}^{N} \delta(\Delta - k_j) \right\rangle ,$$

where the brackets stand for the average over the probability density function of the adjacency matrices.

If the degree distribution function $P(\Delta)$ obeys a power law

$$P(\Delta) \propto \Delta^{-\lambda}, \quad \Delta \to \infty,$$

then the network is said to be <u>scale-free</u>.

In spectral theory of networks, the Laplacian matrix is of another interest. The Laplacian matrix L is an $N \times N$ symmetric matrix defined as

$$L_{jl} = \begin{cases} k_j, & j = l, \\ -A_{jl}, & j \neq l. \end{cases}$$

The Laplacian matrix has non-negative eigenvalues and the smallest eigenvalue is always zero.

The Laplacian matrix is important in physical applications. For example, it appears in the equation

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = D \sum_{l=1}^{N} L_{jl} u_l,$$

which describes a diffusive transportation on the network. Here D denotes the mobility parameter.

The spectral density of the adjacency (or Laplacian) matrix characterizing the network is defined as

$$\rho(\mu) = \left\langle \frac{1}{N} \sum_{j=1}^{N} \delta(\mu - \mu_j) \right\rangle,$$

where $\mu_1, \mu_2, \cdots, \mu_N$ are the eigenvalues of the adjacency matrix A (or the Laplacian matrix L).

For scale-free complex networks, it is expected that the spectral density of adjacency matrices also obeys a power law

$$\rho(\mu) \propto \mu^{-\gamma}, \quad \mu \to \infty.$$

In order to evaluate the asymptotic behavior of the above quantities in the limit $N \to \infty$,

the <u>generating function</u>

$$G(\{t_{jl}\}) = \ln \left\langle \exp\left(-i \sum_{j<l}^{N} A_{jl} t_{jl} \right) \right\rangle$$

is useful. Here t_{jl} are parameters independent of N.

2. GKK model of scale-free networks

Goh, Kahng and Kim introduced a simple model (GKK model) of scale-free networks. Suppose that there are N nodes and that the j-th node is assigned a probability

$$P_j = \frac{j^{-\alpha}}{\sum_{j=1}^{N} j^{-\alpha}} \sim (1-\alpha) N^{\alpha-1} j^{-\alpha}, \quad 0 < \alpha < 1.$$

In each step two nodes are chosen with the assigned probabilities and connected unless they are already connected. Such a step is repeated $pN/2$ times.

Then the j-th and l-th nodes are connected with a probability

$$f_{jl} = 1 - (1 - 2P_jP_l)^{pN/2} \sim 1 - \mathrm{e}^{-pNP_jP_l},$$

so that the adjacency matrix A of the network is distributed according to a probability density function

$$\mathcal{P}_{jl}(A_{jl}) = (1 - f_{jl})\delta(A_{jl}) + f_{jl}\delta(A_{jl} - 1), \quad j < l.$$

For the generating function $G(\{t_{jl}\})$ of the GKK model, Kim, Rodgers, Kahng and Kim proved a useful asymptotic formula in the limit $N \to \infty$:

$$G(\{t_{jl}\}) \sim pN \sum_{j<l}^{N} P_jP_l(e^{-it_{jl}} - 1).$$

As a special case, we find an estimate

$$F_j(t) \equiv \ln \left\langle e^{-i \sum_{l=1}^{N} A_{jl} t} \right\rangle \sim pNP_j(e^{-it} - 1).$$

Now let us calculate the degree distribution. One obtains

$$\langle k_j \rangle = \sum_{l=1}^{N} \langle A_{jl} \rangle = i \frac{\partial}{\partial t} F_j(t) \Big|_{t=0} \sim pNP_j,$$

so that the mean degree is

$$\frac{1}{N} \sum_{j=1}^{N} \langle k_j \rangle \sim p.$$

Moreover it follows that the degree distribution function is

$$\begin{aligned} P(\Delta) &= \frac{1}{2\pi N}\sum_{j=1}^{N}\int dt\ e^{i\Delta t+F_j(t)} \\ &\sim \frac{1}{2\pi}\int dt\int_0^1 dx\ \mathrm{e}^{i\Delta t+p(1-\alpha)x^{-\alpha}(e^{-it}-1)}. \end{aligned}$$

Then, in the limit $\Delta \to \infty$, we find

$$\begin{aligned} P(\Delta) &\sim \int_0^1 dx\ \delta\left\{\Delta - p(1-\alpha)x^{-\alpha}\right\} \\ &= \frac{\{p(1-\alpha)\}^{1/\alpha}}{\alpha}\frac{1}{\Delta^{1+(1/\alpha)}}. \end{aligned}$$

Therefore the exponent λ of the GKK model is equal to $1+(1/\alpha)$.

3. Replica method

In order to calculate the spectral densities of the adjacency and Laplacian matrices of the GKK model, let us apply the replica method in statistical physics.

To begin with, we introduce the partition function

$$Z(\mu) = \int \prod_{j=1}^{N} d\phi_j \ \exp\left(\frac{i}{2}\mu \sum_{j=1}^{N} \phi_j^2 - \frac{i}{2} \sum_{jl}^{N} J_{jl}\phi_j\phi_l \right).$$

In terms of the partition function, we write the spectral density $\rho(\mu)$ in the form

$$\rho(\mu) = \frac{2}{N\pi}\text{Im}\frac{\partial}{\partial\mu}\langle \ln Z(\mu + i\epsilon)\rangle, \quad \epsilon \downarrow 0.$$

Here J is the adjacency matrix A or the Laplacian matrix L. Since there is a relation

$$\langle \ln Z\rangle = \lim_{n\to 0} \frac{\ln\langle Z^n\rangle}{n},$$

we wish to evaluate the average $\langle Z^n\rangle$.

Let us first evaluate the spectral density of adjacency matrices A. Using the vector of replica variables

$$\vec{\phi}_j = (\phi_j^{(1)}, \phi_j^{(2)}, \cdots, \phi_j^{(n)})$$

and the measure

$$d\vec{\phi}_j = d\phi_j^{(1)} d\phi_j^{(2)} \cdots d\phi_j^{(n)},$$

we find

$$\begin{aligned} \langle Z^n \rangle &= \int \prod_{j=1}^{N} d\vec{\phi}_j \; \mathrm{e}^{\frac{i}{2}\mu \sum_{j=1}^{N} \vec{\phi}_j^2} \left\langle \exp\left(-\frac{i}{2} \sum_{jl} A_{jl} \vec{\phi}_j \cdot \vec{\phi}_l \right) \right\rangle \\ &= \int \prod_{j=1}^{N} d\vec{\phi}_j \; \exp\left\{ \frac{i}{2}\mu \sum_{j=1}^{N} \vec{\phi}_j^2 + G\left(\frac{\vec{\phi}_j \cdot \vec{\phi}_l}{2} \right) \right\} \end{aligned}$$

in terms of the generating function G.

Using the asymptotic relation for the generating function G, we obtain

$$\langle Z^n \rangle \sim \int \prod_{j=1}^{N} \mathcal{D}\xi_j(\vec{\phi})\ \mathrm{e}^{S_0+S_1+S_2}.$$

The functional integrations are taken over the auxiliary functions $\xi_j(\vec{\phi})$ satisfying

$$\int \mathrm{d}\vec{\phi}\ \xi_j(\vec{\phi}) = 1.$$

The functionals S_0, S_1 and S_2 are defined as

$$S_0 = -\sum_{j=1}^{N} \int \mathrm{d}\vec{\phi}\ \xi_j(\vec{\phi}) \ln \xi_j(\vec{\phi}),$$

$$S_1 = \frac{i}{2}\mu \sum_{j=1}^{N} \int \mathrm{d}\vec{\phi}\ \xi_j(\vec{\phi})\vec{\phi}^2$$

and

$$S_2 = \frac{pN}{2} \sum_{j,k=1}^{N} P_j P_k \int \mathrm{d}\vec{\phi} \int \mathrm{d}\vec{\psi}\ \xi_j(\vec{\phi})\ \xi_k(\vec{\psi}) \left\{ \mathrm{e}^{-i\vec{\phi}\cdot\vec{\psi}} - 1 \right\}.$$

The functional integrations over $\xi_j(\vec{\phi})$ are dominated by the stationary point satisfying

$$\delta \left\{ S_0 + S_1 + S_2 + \sum_{j=1}^{N} \theta_j \left(\int \mathrm{d}\vec{\phi}\ \xi_j(\vec{\phi}) - 1 \right) \right\} = 0,$$

where θ_j are the Lagrange multipliers.

Then we can derive the variational equations

$$\xi_j(\vec{\phi}) = \Theta_j \ \exp\left\{\frac{i}{2}\mu\vec{\phi}^2 + pNP_j \sum_{k=1}^{N} P_k \int \mathrm{d}\vec{\psi}\xi_k(\vec{\psi}) \left(\mathrm{e}^{-i\vec{\phi}\cdot\vec{\psi}} - 1\right)\right\},$$

where Θ_j are normalization constants.

In the limit of large mean degree $p \to \infty$, the variational equations are satisfied by the Gaussian ansatz

$$\xi_j(\vec{\phi}) = \frac{1}{(2\pi i\sigma_j)^{n/2}} \ \exp\left(-\frac{\vec{\phi}^2}{2i\sigma_j}\right).$$

This property simplifies the problem and enables us to evaluate the asymptotic spectral density[2].

Putting the Gaussian ansatz into the variational equations, we find

$$\mu - \frac{1}{\sigma_j} - pNP_j \sum_{k=1}^{N} P_k \sigma_k = 0,$$

which lead to the integral equation

$$Ex^\alpha - \frac{x^\alpha}{s(x)} - (1-\alpha)^2 \int_0^1 y^{-\alpha} s(y) \mathrm{d}y = 0,$$

where $E = \mu/\sqrt{p}$, $s(x) = \sqrt{p}\sigma_j$ $(x = j/N)$.

Rodgers, Austin, Kahng and Kim solved the integral equation and succeeded in deriving

$$\rho(\mu) \sim \frac{2}{\sqrt{p}} \frac{(1-\alpha)^{1/\alpha}}{\alpha} \frac{1}{E^{1+(2/\alpha)}}$$

for large E. Therefore, in the limit $p \to \infty$, the exponent γ is $1 + (2/\alpha)$, so that the relation with $\lambda = 1 + (1/\alpha)$ is $\gamma = 2\lambda - 1$.

Let us moreover consider the spectral density of Laplacian matrices. As shown by Kim and Kahng, it again follows from the asymptotic relation of the generating function G that

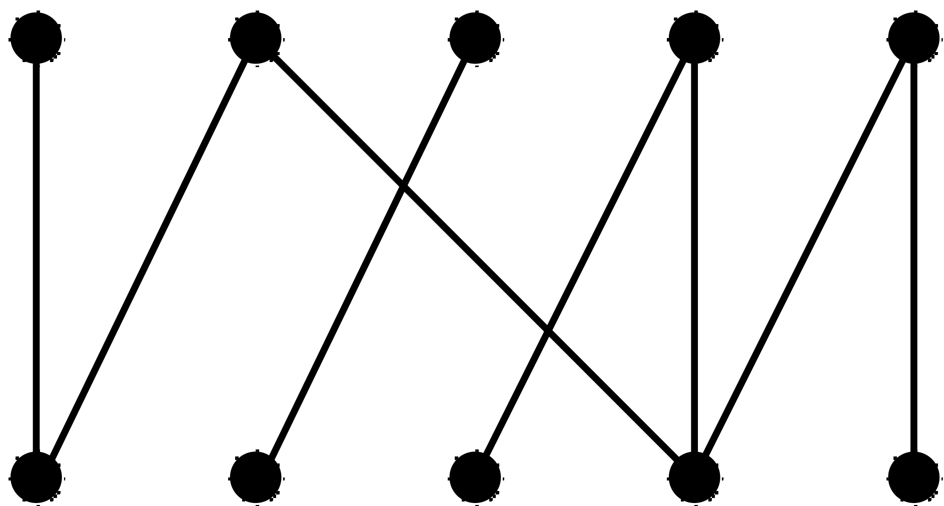
$$\rho(\mu) = \frac{\{p(1-\alpha)\}^{1/\alpha}}{\alpha} \frac{1}{\mu^{(1/\alpha)+1}} H\{\mu - p(1-\alpha)\}$$

in the region $\mu = O(p)$ with $p \to \infty$. Here

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

4. Bipartite scale-free networks

Let us next consider a bipartite network with N nodes of type A and M nodes of type B ($N \geq M$). Connections are made only between the nodes of different types.



Example: a bipartite network with M = N = 5

Bipartite networks are applied to the analysis of human sexual contacts, and of the connection patterns between collaborators and collaboration acts, such as actors and movies, scientists and papers.

We are interested in the asymptotic behavior of the network in the limit

$$N \to \infty \text{ and } M \to \infty \text{ with } c = \frac{N}{M} \text{ fixed.}$$

Let us suppose that the nodes of type A and B are connected according to the following procedure[3].

In each step we choose a node j of type A and a node k of type B with probabilities P_j and Q_k, respectively.

Then the nodes j and k are connected, unless they are already connected.

After repeating such a step pN times, a node j of type A and a node k of type B is connected with a probability

$$f_{jk} = 1 - (1 - P_j Q_k)^{pN} \sim 1 - \mathrm{e}^{-pNP_jQ_k}.$$

Let us consider an $N \times M$ random matrix C ($N \geq M$), where $C_{jk} = 1$ if the node j of type A is directly connected to the node k of type B, and $C_{jk} = 0$ otherwise.

Eacn matrix element C_{jk} is independently distributed with the probability density function

$$\mathcal{P}_{jk}(C_{jk}) = (1 - f_{jk})\delta(C_{jk}) + f_{jk}\delta(1 - C_{jk}).$$

We assume that P_j and Q_k are given by

$$P_j = \frac{j^{-\alpha}}{\sum_{l=1}^{N} l^{-\alpha}} \sim (1-\alpha)N^{\alpha-1}j^{-\alpha}, \quad 0 < \alpha < 1$$

and

$$Q_k = \frac{k^{-\beta}}{\sum_{l=1}^{M} l^{-\beta}} \sim (1-\beta)M^{\beta-1}k^{-\beta}, \quad 0 < \beta < 1.$$

There are thus two parameters α and β controlling the probability distribution function of the matrix C.

We define the degree d_j of the type-A node j as the number of directly connected type-B nodes:

$$d_j = \sum_{k=1}^{M} C_{jk}.$$

Then the type-A node degree distribution function is given by

$$P^{(A)}(\Delta) = \left\langle \frac{1}{N} \sum_{j=1}^{N} \delta(\Delta - d_j) \right\rangle,$$

where the brackets denote the average over the probability density function of the matrix C.

We can similarly introduce the degree e_k of the type-B node k:

$$e_k = \sum_{j=1}^{N} C_{jk}$$

and the type-B node degree distribution function

$$P^{(B)}(\Delta) = \left\langle \frac{1}{M} \sum_{k=1}^{M} \delta(\Delta - e_k) \right\rangle .$$

The generating function for this scale-free network is defined as

$$\mathcal{G}(\{t_{jk}\}) = \ln \left\langle \exp \left(-i \sum_{j=1}^{N} \sum_{k=1}^{M} C_{jk} t_{jk} \right) \right\rangle .$$

A useful asymptotic relation[3]

$$\mathcal{G}(\{t_{jk}\}) \sim pN \sum_{j=1}^{N} \sum_{k=1}^{M} P_j Q_k (\mathrm{e}^{-it_{jk}} - 1)$$

can be derived in the limit $N, M \to \infty$. Here t_{jk} depends on neither N nor M.

As special cases, we can derive asymptotic relations for

$$F_j^{(A)}(t) \equiv \ln \left\langle \mathrm{e}^{-id_j t} \right\rangle, \quad F_k^{(B)}(t) \equiv \ln \left\langle \mathrm{e}^{-ie_k t} \right\rangle$$

as

$$F_j^{(A)}(t) \sim pNP_j(\mathrm{e}^{-it} - 1), \quad F_k^{(B)}(t) \sim pNQ_k(\mathrm{e}^{-it} - 1).$$

Then we can see that

$$\begin{aligned} \langle d_j \rangle &= i \frac{\partial}{\partial t} F_j^{(A)}(t) \Big|_{t=0} \sim pNP_j, \\ \langle e_k \rangle &= i \frac{\partial}{\partial t} F_k^{(B)}(t) \Big|_{t=0} \sim pNQ_k, \end{aligned}$$

so that the mean degree $m^{(A)}$ of the type-A node is

$$m^{(A)} = \frac{1}{N} \sum_{j=1}^{N} \langle d_j \rangle \sim p,$$

while the mean degree $m^{(B)}$ of the type-B node is

$$m^{(B)} = \frac{1}{M} \sum_{k=1}^{M} \langle e_k \rangle \sim pc.$$

It can be seen that the type-A node degree distribution function is written as

$$P^{(A)}(\Delta) = \frac{1}{2\pi N}\sum_{j=1}^{N}\int \mathrm{d}t\ \mathrm{e}^{i\Delta t+F_j^{(A)}(t)}$$
$$\sim \frac{1}{2\pi}\int \mathrm{d}t \int_0^1 \mathrm{d}x\ \mathrm{e}^{i\Delta t+p(1-\alpha)x^{-\alpha}(\mathrm{e}^{-it}-1)}.$$

Then in the limit $\Delta \to \infty$ we find

$$P^{(A)}(\Delta) \sim \frac{\{p(1-\alpha)\}^{1/\alpha}}{\alpha}\frac{1}{\Delta^{(1/\alpha)+1}}$$

and similarly obtain

$$P^{(B)}(\Delta) \sim \frac{\{pc(1-\beta)\}^{1/\beta}}{\beta}\frac{1}{\Delta^{(1/\beta)+1}}.$$

Thus we have seen that the network has the scale-free property. The exponents of the power laws defined as

$$P^{(A)}(\Delta) \propto \Delta^{-\lambda_A}, \quad P^{(B)}(\Delta) \propto \Delta^{-\lambda_B}, \quad \Delta \to \infty$$

are found to be $\lambda_A = (1/\alpha) + 1$ and $\lambda_B = (1/\beta) + 1$.

We can also study the adjacency and Laplacian matrices of this scale-free network. The adjacency matrix $\mathcal{A}$ of this network is defined as

$$\mathcal{A} = \begin{pmatrix} O_N & C \\ C^{\mathsf{T}} & O_M \end{pmatrix},$$

where C^{T} is a transpose of C and O_n is an $n \times n$ matrix with zero elements.

The Laplacian matrix $\mathcal{L}$ is an $(N+M)\times(N+M)$ symmetric matrix with

$$\mathcal{L}_{jl} = \begin{cases} d_j, & j=l \text{ and } 1 \le j \le N, \\ e_{j-N}, & j=l \text{ and } N+1 \le j \le N+M, \\ -\mathcal{A}_{jl}, & j \ne l. \end{cases}$$

Let us suppose that J is the adjacency matrix $\mathcal{A}$ or the Laplacian matrix $\mathcal{L}$. The spectral density of J is defined as

$$\rho(\mu) = \left\langle \frac{1}{N+M} \sum_{j=1}^{N+M} \delta(\mu - \mu_j) \right\rangle,$$

where μ_j, $j = 1, 2, \cdots, N+M$ are the eigenvalues of J.

In order to calculate $\rho(\mu)$, we introduce the partition function

$$Z(\mu) = \int \prod_{j=1}^{N+M} \mathrm{d}\Phi_j \, \exp\left(\frac{i}{2}\mu \sum_{j=1}^{N+M} \Phi_j^2 - \frac{i}{2} \sum_{j=1}^{N+M} \sum_{l=1}^{N+M} J_{jl}\Phi_j\Phi_l \right).$$

Using the partition function Z, we can write the spectral density as

$$\begin{aligned} \rho(\mu) &= \frac{2}{(N+M)\pi} \mathrm{Im} \frac{\partial}{\partial \mu} \langle \ln Z(\mu + i\epsilon) \rangle \\ &= \lim_{n \to 0} \frac{2}{(N+M)n\pi} \mathrm{Im} \frac{\partial}{\partial \mu} \ln \langle \{Z(\mu + i\epsilon)\}^n \rangle, \quad \epsilon \downarrow 0. \end{aligned}$$

Therefore we again need to evaluate the average $\langle Z^n \rangle$.

Let us first consider the spectral density of the adjacency matrices $\mathcal{A}$. In terms of the replica variables

$$\vec{\phi}_j = (\phi_j^{(1)}, \phi_j^{(2)}, \cdots, \phi_j^{(n)}), \quad \vec{\psi}_k = (\psi_k^{(1)}, \psi_k^{(2)}, \cdots, \psi_k^{(n)})$$

and

$$\mathrm{d}\vec{\phi}_j = \mathrm{d}\phi_j^{(1)} \mathrm{d}\phi_j^{(2)} \cdots \mathrm{d}\phi_j^{(n)}, \quad \mathrm{d}\vec{\psi}_k = \mathrm{d}\psi_k^{(1)} \mathrm{d}\psi_k^{(2)} \cdots \mathrm{d}\psi_k^{(n)},$$

we obtain

$$\begin{aligned} \langle Z^n \rangle = & \int \prod_{j=1}^{N} \mathrm{d}\vec{\phi}_j \int \prod_{k=1}^{M} \mathrm{d}\vec{\psi}_k \\ & \times \exp\left\{ \frac{i}{2}\mu \sum_{j=1}^{N} \vec{\phi}_j^2 + \frac{i}{2}\mu \sum_{k=1}^{M} \vec{\psi}_k^2 + \mathcal{G}\left(\frac{\vec{\phi}_j \cdot \vec{\psi}_k}{2} \right) \right\}. \end{aligned}$$

It follows in the limit $N, M \to \infty$ that

$$\langle Z^n \rangle \sim \int \prod_{j=1}^{N} \mathcal{D}\xi_j(\vec{\phi}) \prod_{k=1}^{M} \mathcal{D}\eta_k(\vec{\psi}) \ \mathrm{e}^{S_0+S_1+S_2}.$$

The functional integrations are taken over the auxiliary functions $\xi_j(\vec{\phi})$ and $\eta_k(\vec{\psi})$ satisfying

$$\int \mathrm{d}\vec{\phi} \ \xi_j(\vec{\phi}) = \int \mathrm{d}\vec{\psi} \ \eta_k(\vec{\psi}) = 1.$$

The functionals S_0, S_1 and S_2 are defined as

$$S_0 = -\sum_{j=1}^{N} \int \mathrm{d}\vec{\phi}\ \xi_j(\vec{\phi}) \ln \xi_j(\vec{\phi}) - \sum_{k=1}^{M} \int \mathrm{d}\vec{\psi}\ \eta_k(\vec{\psi}) \ln \eta_k(\vec{\psi}),$$

$$S_1 = \frac{i}{2}\mu \sum_{j=1}^{N} \int \mathrm{d}\vec{\phi}\ \xi_j(\vec{\phi})\vec{\phi}^2 + \frac{i}{2}\mu \sum_{k=1}^{M} \int \mathrm{d}\vec{\psi}\ \eta_k(\vec{\psi})\vec{\psi}^2$$

and

$$S_2 = pN \sum_{j=1}^{N} \sum_{k=1}^{M} P_j Q_k \int \mathrm{d}\vec{\phi} \int \mathrm{d}\vec{\psi}\ \xi_j(\vec{\phi})\ \eta_k(\vec{\psi}) \left\{ \mathrm{e}^{-i\vec{\phi}\cdot\vec{\psi}} - 1 \right\}.$$

The functional integrations over $\xi_j(\vec{\phi})$ and $\eta_k(\vec{\psi})$ are dominated by the stationary point satisfying

$$\delta \left\{ S_0 + S_1 + S_2 + \sum_{j=1}^{N} \theta_j \left(\int \mathrm{d}\vec{\phi}\ \xi_j(\vec{\phi}) - 1 \right) \right. \\ \left. + \sum_{k=1}^{M} \omega_k \left(\int \mathrm{d}\vec{\psi}\ \eta_k(\vec{\psi}) - 1 \right) \right\} = 0,$$

where θ_j and ω_k are the Lagrange multipliers.

Then we can derive the variational equations

$$\begin{aligned}
\xi_j(\vec{\phi}) &= \Theta_j \exp\left\{\frac{i}{2}\mu\vec{\phi}^2 + pNP_j \sum_{k=1}^{M} Q_k \int \mathrm{d}\vec{\psi}\eta_k(\vec{\psi}) \left(\mathrm{e}^{-i\vec{\phi}\cdot\vec{\psi}} - 1\right)\right\}, \\
\eta_k(\vec{\psi}) &= \Omega_k \exp\left\{\frac{i}{2}\mu\vec{\psi}^2 + pNQ_k \sum_{j=1}^{N} P_j \int \mathrm{d}\vec{\phi}\xi_j(\vec{\phi}) \left(\mathrm{e}^{-i\vec{\phi}\cdot\vec{\psi}} - 1\right)\right\},
\end{aligned}$$

where Θ_j and Ω_k are normalization constants.

In the limit of large mean degree $p \to \infty$, the variational equations are satisfied by the Gaussian ansatz

$$\xi_j(\vec{\phi}) = \frac{1}{(2\pi i\sigma_j)^{n/2}} \exp\left(-\frac{\vec{\phi}^2}{2i\sigma_j}\right), \quad \eta_k(\vec{\psi}) = \frac{1}{(2\pi i\tau_k)^{n/2}} \exp\left(-\frac{\vec{\psi}^2}{2i\tau_k}\right).$$

Putting the Gaussian ansatz into the variational equations, we find

$$\mu - \frac{1}{\sigma_j} - pNP_j \sum_{k=1}^{M} Q_k \tau_k = 0,$$
$$\mu - \frac{1}{\tau_k} - pNQ_k \sum_{j=1}^{N} P_j \sigma_j = 0,$$

which lead to the integral equations

$$Ex^\alpha - \frac{x^\alpha}{s(x)} - (1-\alpha)(1-\beta) \int_0^1 y^{-\beta} t(y) \mathrm{d}y = 0,$$
$$Ey^\beta - \frac{y^\beta}{t(y)} - c(1-\alpha)(1-\beta) \int_0^1 x^{-\alpha} s(x) \mathrm{d}x = 0,$$

where $E = \mu/\sqrt{p}$, $s(x) = \sqrt{p}\sigma_j$ $(x = j/N)$ and $t(y) = \sqrt{p}\tau_k$ $(y = k/M)$.

Let us consider the limit $p \to \infty$ with a scaling variable $E = \mu/\sqrt{p}$. Solving the integral equations, we find the asymptotic spectral density of the adjacency matrix $\mathcal{A}$ as

$$\rho(\mu) \sim \frac{2}{\sqrt{p}(1+c)} \left\{ \frac{c^{1/\beta}(1-\beta)^{1/\beta}}{\beta E^{(2/\beta)+1}} + \frac{c(1-\alpha)^{1/\alpha}}{\alpha E^{(2/\alpha)+1}} \right\}$$

in the tail region $E \to \infty$.

Thus the exponent γ is associated with $\lambda_A = (1/\alpha) + 1$ and $\lambda_B = (1/\beta) + 1$ as $\gamma = 2\min(\lambda_A, \lambda_B) - 1$.

The asymptotic spectral density of the Laplacian matrix $\mathcal{L}$ is similarly derived as

$$\begin{aligned}\rho(\mu) &= \frac{c\{p(1-\alpha)\}^{1/\alpha}}{(1+c)\alpha}\frac{1}{\mu^{(1/\alpha)+1}}\ H\left\{\mu - p(1-\alpha)\right\} \\ &\quad + \frac{\{pc(1-\beta)\}^{1/\beta}}{(1+c)\beta}\frac{1}{\mu^{(1/\beta)+1}}\ H\left\{\mu - pc(1-\beta)\right\}\end{aligned}$$

in the region $\mu = O(p)$ with $p \to \infty$. Here

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

5. Effective medium approximation

We have so far dealt with the spectral density in the limit $p \to \infty$.

The calculation of the spectral density with a finite mean degree p is a much more involved problem.

Here we briefly discuss a simple approximation method (effective medium approximation, EMA) for that problem applied to bipartite scale-free networks.

In EMA, we put the Gaussian ansatz into the definitions of S_0, S_1 and S_2 , and solve the stationary point equations

$$\frac{\partial}{\partial \sigma_j}(S_0 + S_1 + S_2) = 0$$

and

$$\frac{\partial}{\partial \tau_k}(S_0 + S_1 + S_2) = 0.$$

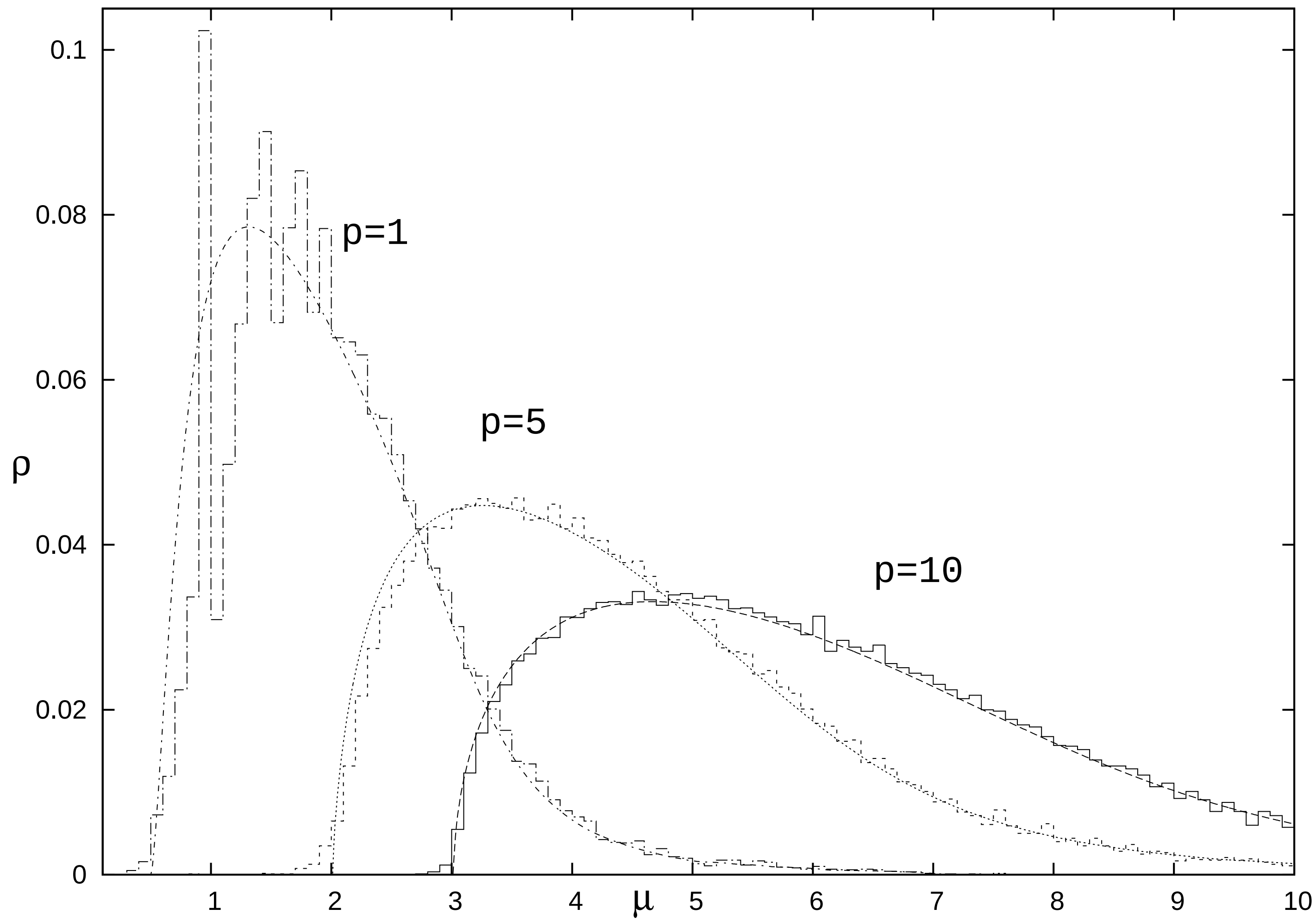
In the case of the adjacency matrix $\mathcal{A}$, the above procedure results in the EMA equations

$$\mu - \frac{1}{\sigma_j} - pNP_j \sum_{k=1}^{M} \frac{Q_k \tau_k}{1 - \sigma_j \tau_k} = 0,$$
$$\mu - \frac{1}{\tau_k} - pNQ_k \sum_{j=1}^{N} \frac{P_j \sigma_j}{1 - \sigma_j \tau_k} = 0.$$

As for scale-free networks with a single species of nodes, TN and Rodgers [1] calculated the $1/p$ expansion of the spectral density by using the corresponding EMA equation.

A similar analytical treatment could also be possible in the bipartite case. Here, however only results of numerical iterations of the EMA equations are shown.

They are compared with the spectral densities of positive eigenvalues calculated by numerical diagonalizations of numerically generated adjacency matrices (averaged over 100 samples).



The EMA spectral densities (curves) and the spectral densities of numerically generated adjacency matrices (histograms) with $p = 1, 5$ and 10. The parameters are $N = 1000$, $M = 200$ and $\alpha = 0.5$, $\beta = 0.2$.

The EMA gives a better fit for a larger p, as expected from the fact that the variational equations are satisfied by the Gaussian ansatz in the limit $p \to \infty$.

When $p = 1$, the agreement significantly breaks down around the origin, although it is still fairly good in the tail region with large μ.

It is thus conjectured that the EMA (with a finite p) gives exact asymptotic results in the limit $\mu \to \infty$.

6. Summary

The generating function is a useful tool in the asymptotic analysis of the connection pattern of a network. It is known that some typical network quantities, such as the average degree, the degree distribution function and the eigenvalue densities of adjacency and Laplacian matrices, can be evaluated by using the generating function.

The asymptotic estimates of the generating functions in the limit of large network size were presented for simple models (the GKK model and a similar bipartite network) of scale-free networks and applied to the analysis of those network quantities.