University of BRISTOL

Universality in Quantum Chaos

Summer school 'Randomness in Physics and Mathematics',
Bielefeld, August 2013

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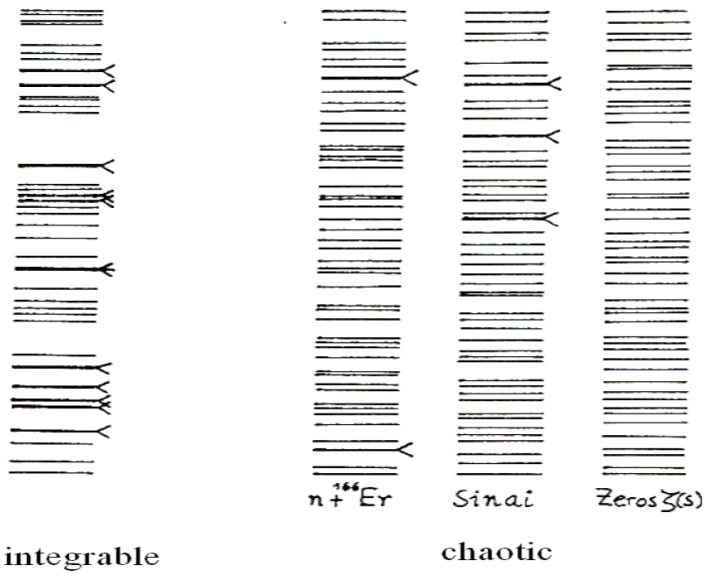
Universality

Many quantum properties of chaotic systems are universal and agree with predictions from random matrix theory,
in particular the statistics of energy levels.

(Bohigas, Giannoni, Schmit 84;
see also Casati, Vals-Gris, Guarneri; Berry, Tabor)



Why?

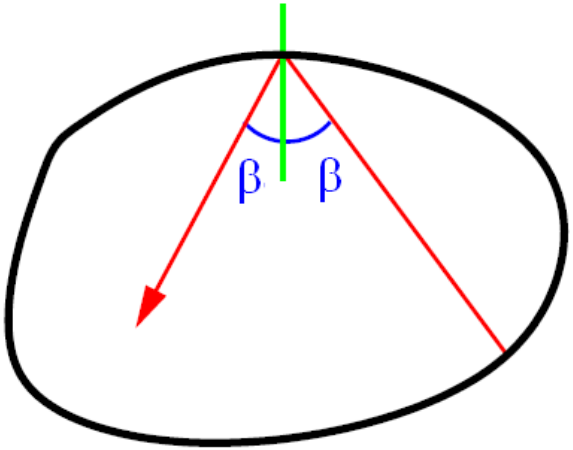
Outline

- Classical chaos
- Quantum chaos
- Semiclassical methods (Gutzwiller trace formula)
- Semiclassical approach to spectral statistics
- Resummation

1 Classical chaos

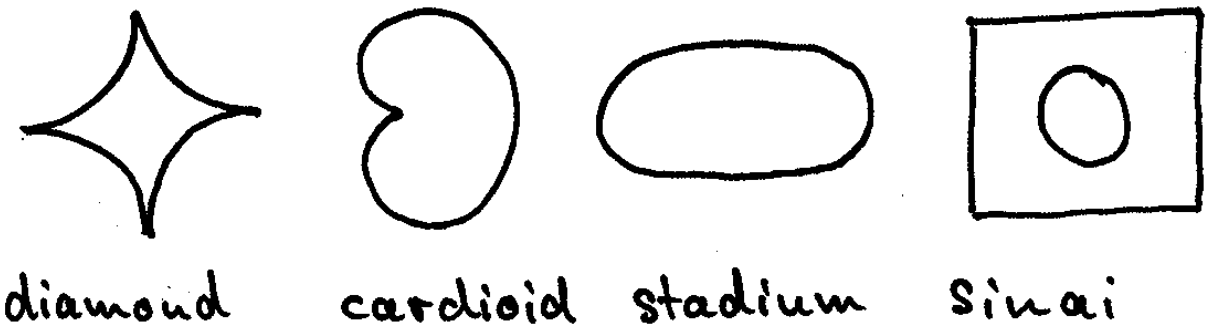
Billiards (classical)

particle moving in domain $B \subset \mathbb{R}^2$



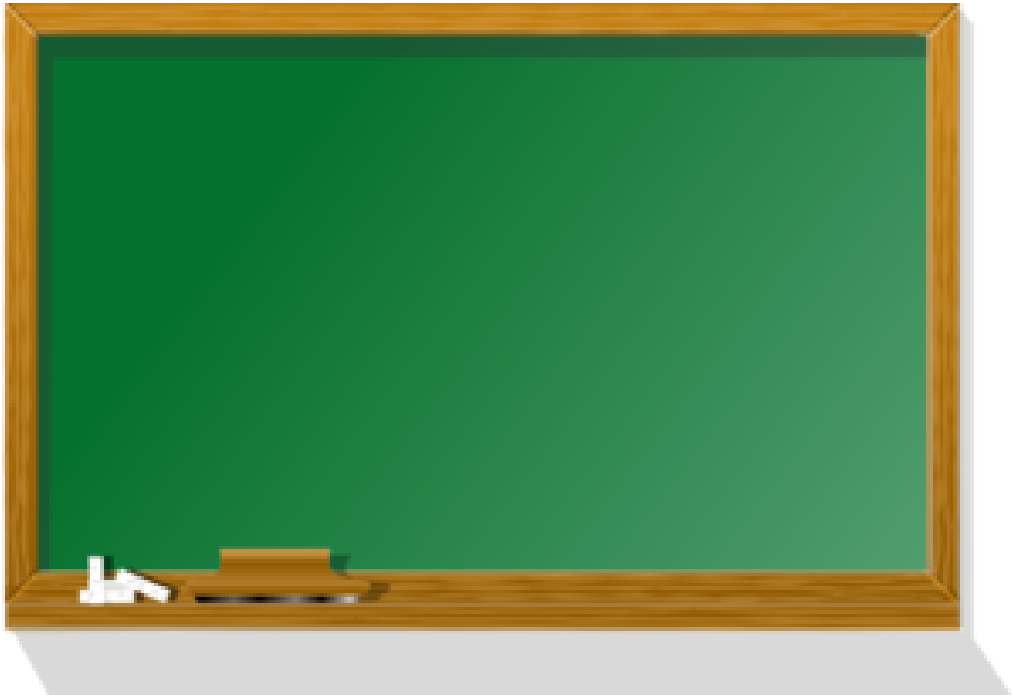
- motion on straight line with constant velocity
- reflection at boundary
  (angle of incidence = angle of reflection)

Billiards with 'irregular' boundary show chaotic behaviour.



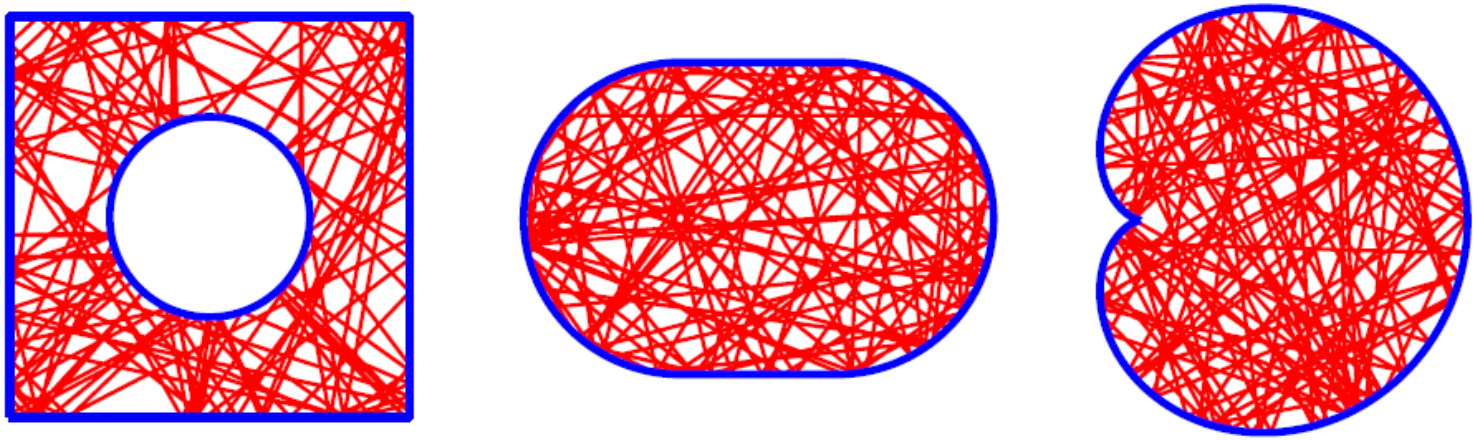
Chaos

(a) sensitive dependence on initial conditions



(b) ergodicity

long trajectories fill the interior almost uniformly
+ all angles are equally likely

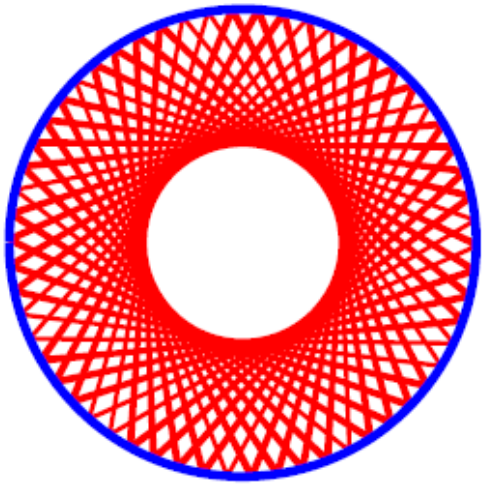


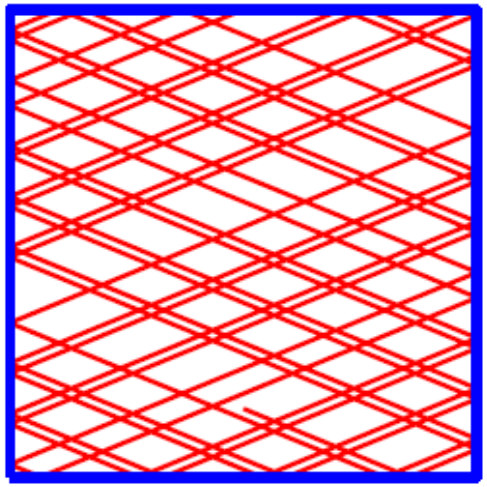
general definition:
energy shell $\Omega = \{(\boldsymbol{q}, \boldsymbol{p}), H(\boldsymbol{q}, \boldsymbol{p}) = E\}$
a system is ergodic if for $\Omega_0 \subset \Omega$ and almost all trajectories

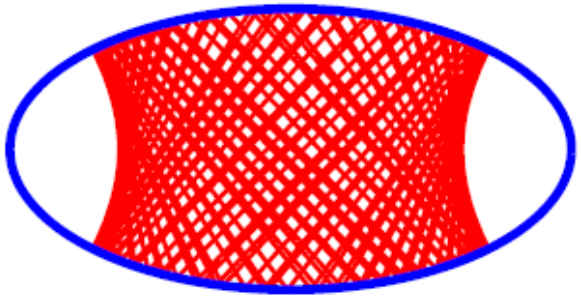
$$\frac{\text{time up to } T \text{ spent in } \Omega_0}{T} \to \frac{\mathrm{vol}(\Omega_0)}{\mathrm{vol}(\Omega)} \qquad (T \to \infty)$$

Ergodicity

billiards that are **not ergodic**:



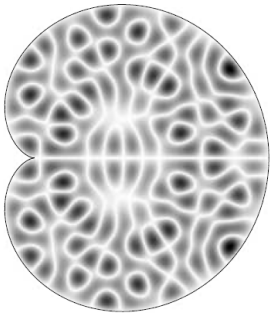




2 Quantum chaos

quantum properties of systems that are classically chaotic

Billiards (quantum)



- **Schrödinger equation** in domain $B \subset \mathbb{R}^2$, no potential

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{q},t) = i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{q},t)$$

Dirichlet boundary conditions:

$$\psi(\boldsymbol{q},t) = 0 \quad \text{for} \quad \boldsymbol{q} \in \partial B$$

- **time independent Schrödinger equation**

$$\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\psi_n(\boldsymbol{q}) &= E_n\psi_n(\boldsymbol{q}) \\ \psi_n(\boldsymbol{q}) &- \text{energy eigenfunctions} \\ E_n &- \text{energy levels} \end{aligned}$$

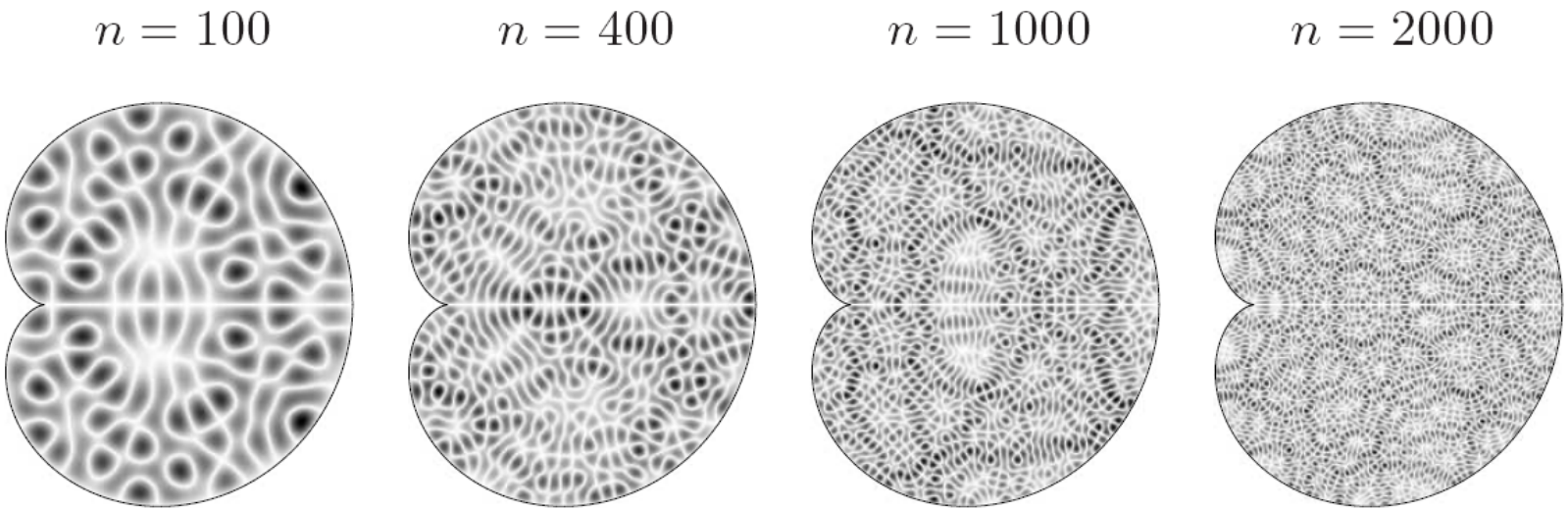
Quantum chaos

(a) **sensitive dependence on initial conditions ?**

NO, because Schrödinger equation is linear

(b) energy eigenfunctions

plot $|\psi_n(\boldsymbol{q})|^2$ for eigenfunctions with increasing energy:



eigenfunctions become more and more equidistributed (**quantum ergodicity**)

Quantum chaos

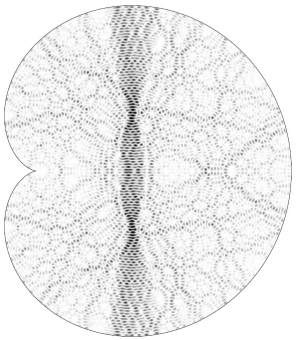
probability that a particle in state n is found in a part B_0 of position space B:

$$\mu_n(B_0) = \int_{B_0} d^2q |\psi_n(\boldsymbol{q})|^2$$

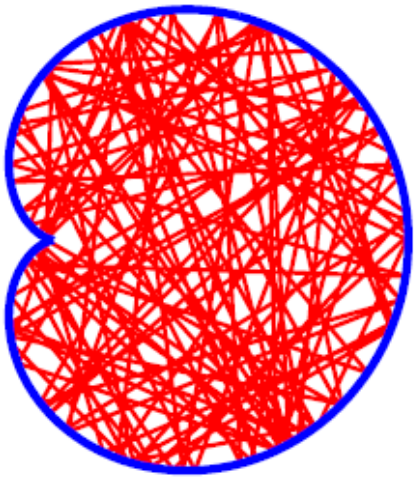
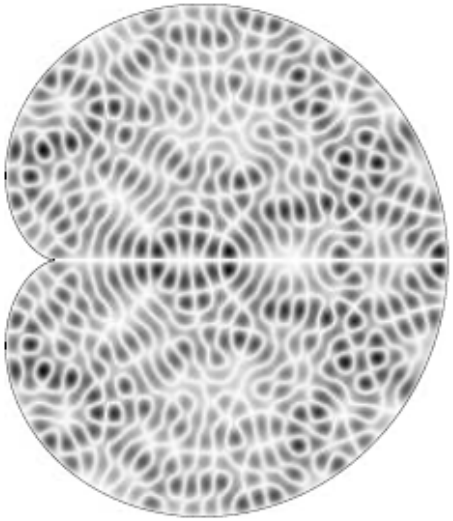
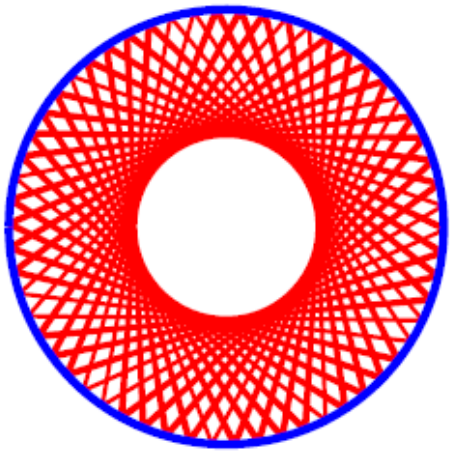
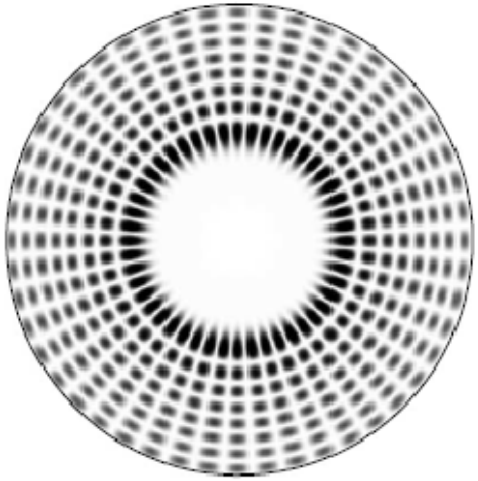
in a quantum ergodic system:

$$\mu_n(B_0) \to \frac{\text{area}(B_0)}{\text{area}(B)}$$

for increasing energies after taking out exceptional wavefunctions ("scars") such as this:

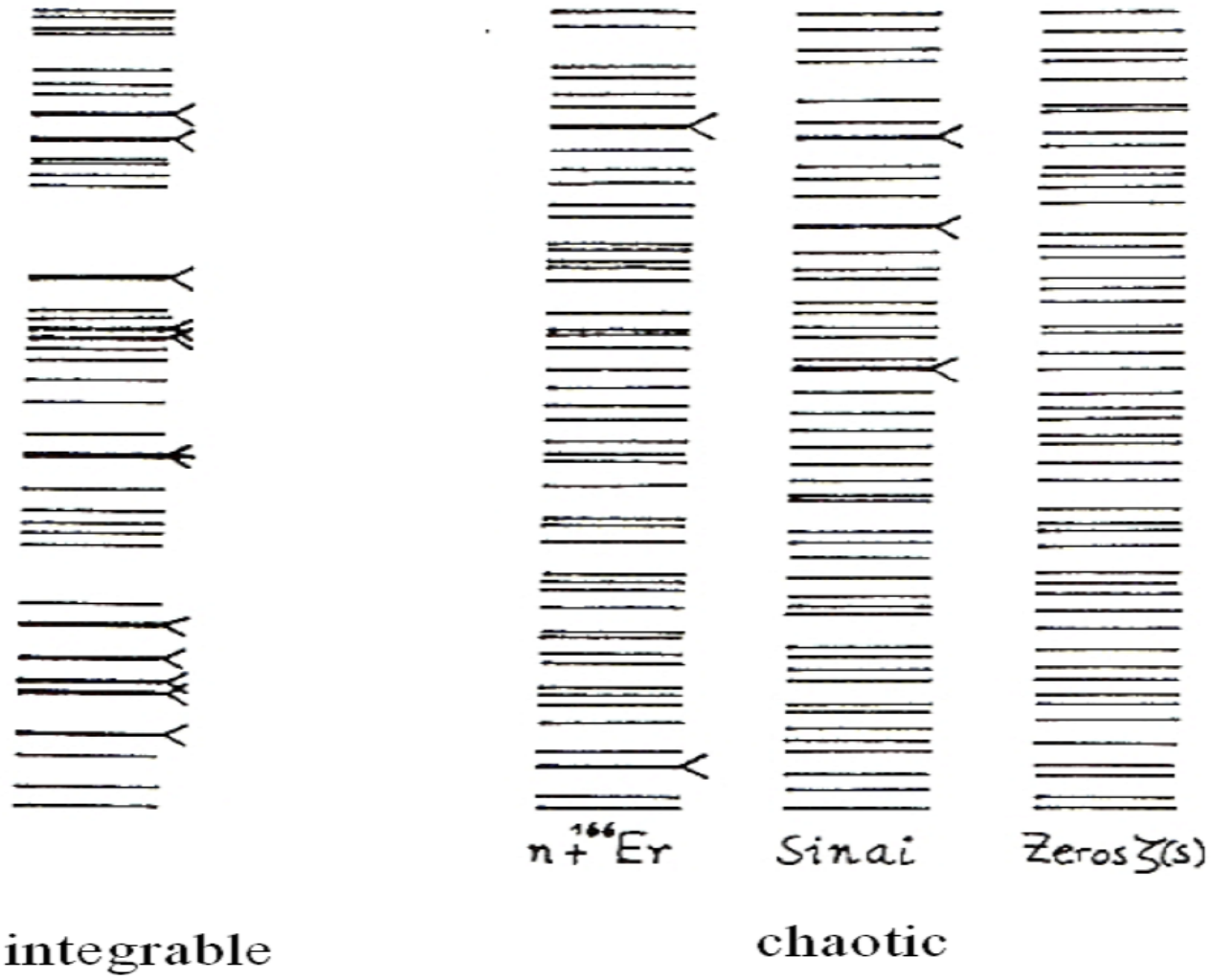


Quantum chaos

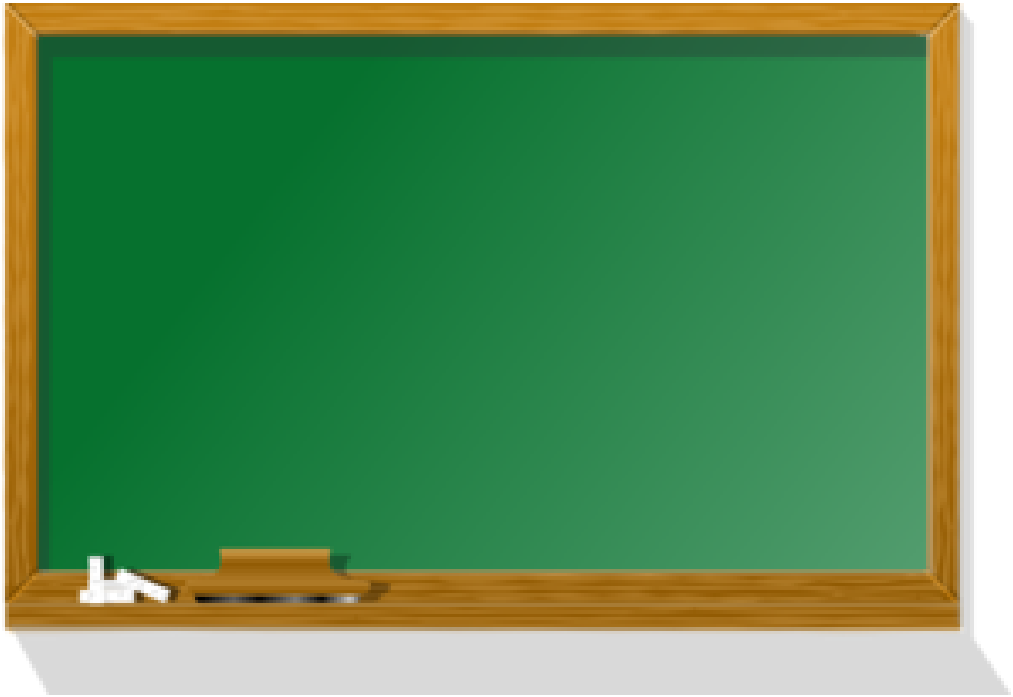
	classical	quantum
chaotic		
integrable		

Quantum chaos

(c) statistics of energy levels

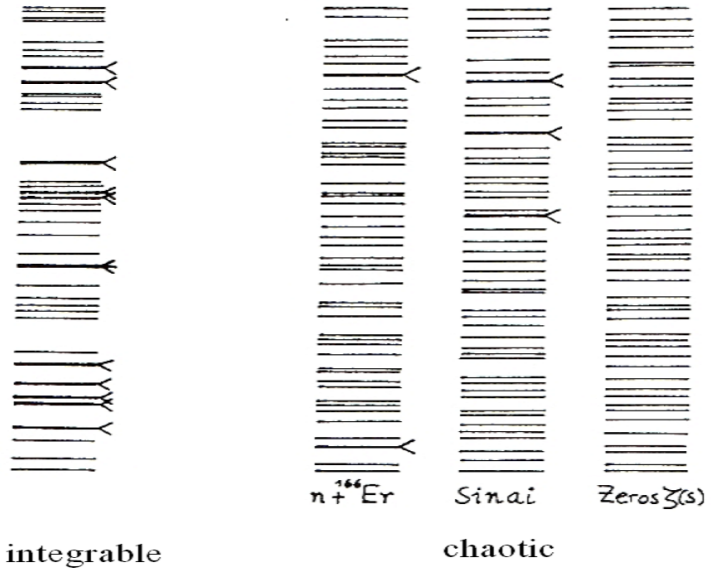


(large) energy levels of a chaotic system **repel** each other!



4 Semiclassical approach to spectral statistics

Two-point correlation function



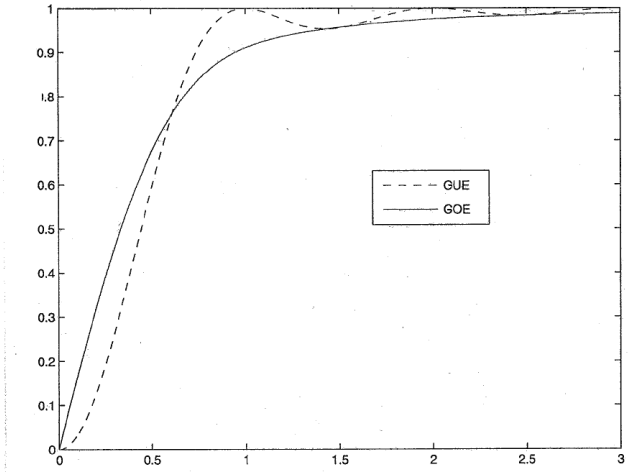
measure likelihood of energy differences:

$$\int_0^\infty dE\, d\left(E+\frac{y}{2}\right) d\left(E-\frac{y}{2}\right) \;=\; \sum_{jk} \delta(y-(E_j-E_k))$$

make everything dimensionless and average over energy and energy differences:

$$R(\epsilon) = \frac{1}{\bar{d}^2}\left\langle d\left(E+\frac{\epsilon}{2\bar{d}}\right) d\left(E-\frac{\epsilon}{2\bar{d}}\right)\right\rangle$$

Random matrix predictions

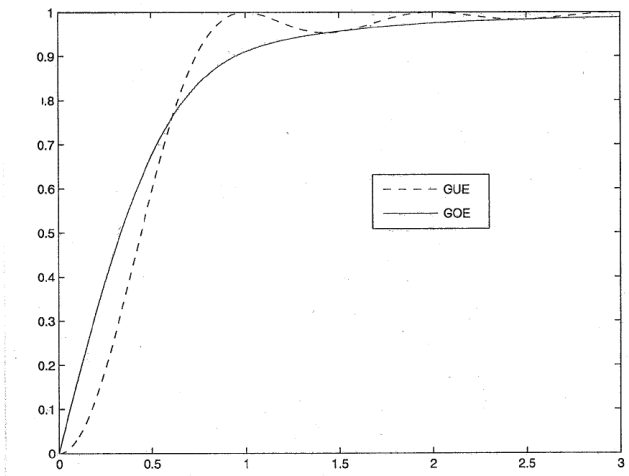


small energy differences are less likely!

- systems without time rev. invariance: Gaussian Unitary Ensemble

$$R_{\mathrm{GUE}}(\epsilon) = 1 - \left(\frac{\sin(\pi\epsilon)}{\pi\epsilon}\right)^2 = \mathrm{Re}\left(1 - \frac{1}{2(\pi\epsilon)^2} + \frac{1}{2(\pi\epsilon)^2} e^{2\pi i\epsilon}\right)$$

Random matrix predictions



- systems with time rev. inv.: Gaussian Orthogonal Ensemble

$$R_{\mathrm{GOE}}(\epsilon) = 1 - \left(\frac{\sin \pi\epsilon}{\pi\epsilon}\right)^2 + \left(\frac{\mathrm{Si}\,\pi\epsilon}{\pi} - \frac{\mathrm{sign}\,\epsilon}{2}\right)\left(\frac{\cos \pi\epsilon}{\pi\epsilon} - \frac{\sin \pi\epsilon}{(\pi\epsilon)^2}\right)$$

$$= \mathrm{Re}\left(\sum_n c_n \left(\frac{1}{\epsilon}\right)^n + \sum_n d_n \left(\frac{1}{\epsilon}\right)^n e^{2\pi i\epsilon}\right)$$

where $\mathrm{Si}(x) = \int_0^x \frac{\sin y}{y} dy$

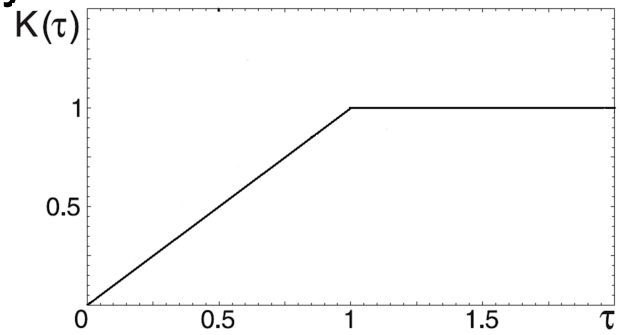
in both cases non-oscillatory terms and oscillatory terms

Spectral form factor

Fourier transform

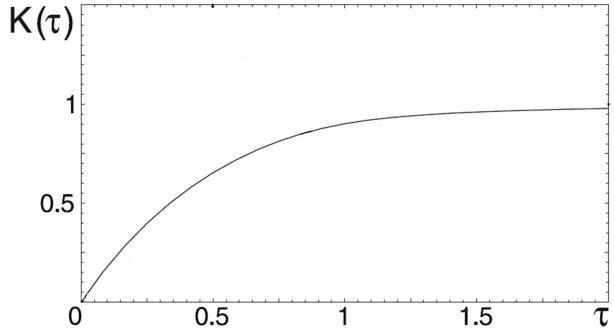
$$K(\tau) = \mathrm{Re} \int_{-\infty}^{\infty} d\epsilon \ \ (R(\epsilon) - 1) e^{2\pi i \epsilon \tau}$$

- systems without time rev. invariance: Gaussian Unitary Ensemble



$$K(\tau) = \begin{cases} \tau & (\tau < 1) \\ 1 & (\tau > 1) \end{cases}$$

- systems with time rev. inv.: Gaussian Orthogonal Ensemble



$$K(\tau) = \begin{cases} 2\tau - \tau \ln(1 + 2\tau) \\ = 2\tau - 2\tau^2 + 2\tau^3 - \frac{8}{3}\tau^4 + \ldots \\ \quad (\tau < 1) \\ 2 - \ln \frac{2\tau+1}{2\tau-1} \quad (\tau > 1) \end{cases}$$

Semiclassical approximation

evaluate

$$R(\epsilon) = \frac{1}{\bar{d}^2} \left\langle d\left(E + \frac{\epsilon}{2\bar{d}}\right) d\left(E - \frac{\epsilon}{2\bar{d}}\right) \right\rangle$$

using

$$\begin{aligned} d(E) &\approx \bar{d} + \frac{1}{\pi\hbar} \mathrm{Re} \sum_p A_p e^{iS_p(E)/\hbar} \\ A_p\left(E \pm \frac{\epsilon}{2\bar{d}}\right) &\approx A_p(E) \\ S_p\left(E \pm \frac{\epsilon}{2\bar{d}}\right) &\approx S_p(E) \pm \frac{T_p(E)\epsilon}{2\bar{d}} \\ T_H &= 2\pi\hbar\bar{d} \qquad \text{(Heisenberg time)} \end{aligned}$$

we get the double sum

$$R(\epsilon) \approx 1 + \frac{1}{T_H^2} \mathrm{Re} \left\langle \sum_{p,p'} A_p A_{p'}^* e^{i(S_p - S_{p'})/\hbar} e^{i\pi\epsilon(T_p + T_{p'})/T_H} \right\rangle$$

$$+ \text{ similar term with } e^{i(S_p + S_{p'})/\hbar}$$

Semiclassical approximation

$$R(\epsilon) \approx 1 + \frac{1}{T_H^2}\mathrm{Re}\left\langle \sum_{p,p'} A_p A_{p'}^* e^{i(S_p - S_{p'})/\hbar} e^{i\pi\epsilon(T_p+T_{p'})/T_H} \right\rangle$$
$$+ \text{ similar term with } e^{i(S_p+S_{p'})/\hbar}$$

energy average $\Rightarrow$ systematic contribution only from first term for small action differences (at most $\sim \hbar$)
constructive interference!

Fourier transform gives:

$$K(\tau) \approx \frac{1}{T_H}\mathrm{Re}\left\langle \sum_{p,p'} A_p A_{p'}^* e^{i(S_p - S_{p'})/\hbar} \delta\left(\tau T_H - \frac{T_p + T_{p'}}{2}\right) \right\rangle$$

relevant orbits have periods of the order

$$T_H = 2\pi\hbar\bar{d} = 2\pi\hbar\frac{\Omega}{(2\pi\hbar)^n} \to \infty \quad (\hbar \to 0)$$

Diagonal approximation

(Berry; Hannay & Ozorio de Almeida)

- for systems without time reversal invariance: take $p' = p$

$$K_{\mathrm{diag}}(\tau) = \frac{1}{T_H} \left\langle \sum_p |A_p|^2 \delta(\tau T_H - T_p) \right\rangle \sim \tau$$

sum over periodic orbits evaluated using ergodicity:

$$\left\langle \sum_p |A_p|^2 \delta(T - T_p) \right\rangle \sim T$$

- time reversal invariant systems: $p' = p$ or time reversed of p

$$K_{\mathrm{diag}}(\tau) = 2\tau$$

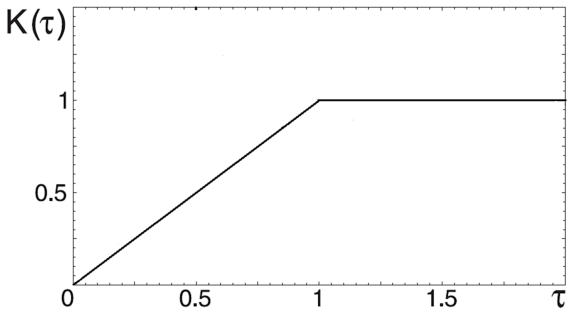
Summary

Gutzwiller trace formula: $d(E) \approx \bar{d} + \frac{1}{\pi\hbar}\mathrm{Re}\sum_p A_p e^{iS_p(E)/\hbar}$
Spectral form factor:

$$K(\tau) \approx \frac{1}{T_H}\mathrm{Re}\left\langle \sum_{p,p'} A_p A_{p'}^* e^{i(S_p - S_{p'})/\hbar}\delta\left(\tau T_H - \frac{T_p + T_{p'}}{2}\right)\right\rangle$$

RMT prediction:

- systems without time rev. invariance: Gaussian Unitary Ensemble



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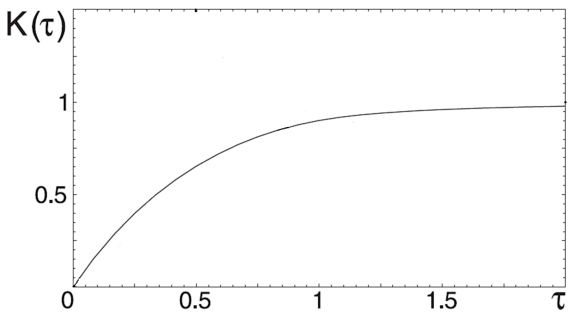

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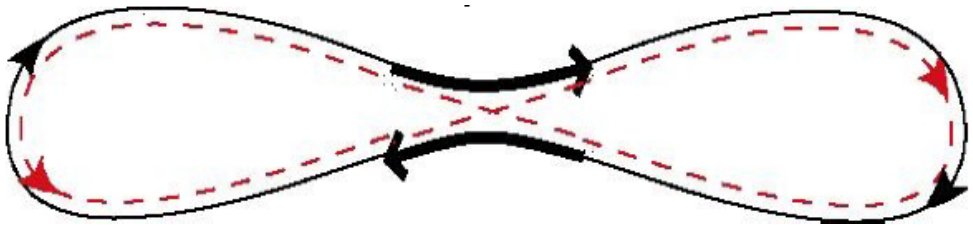
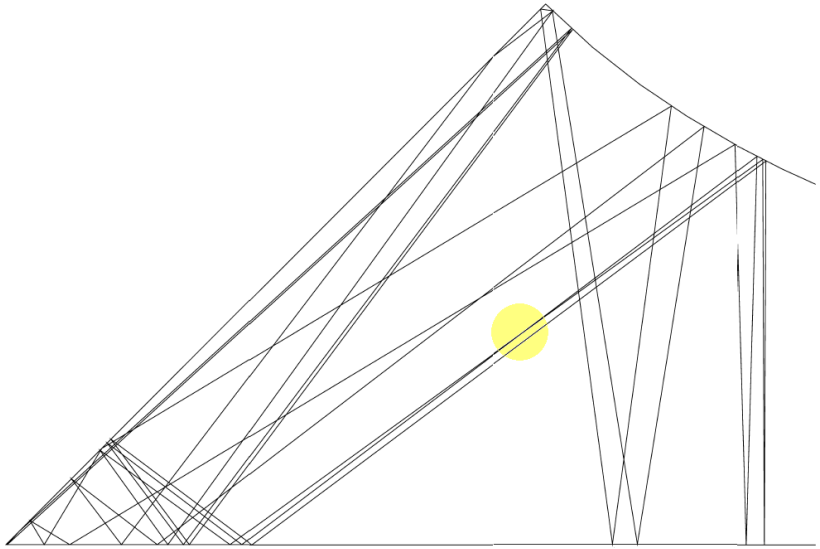
Orbit correlations

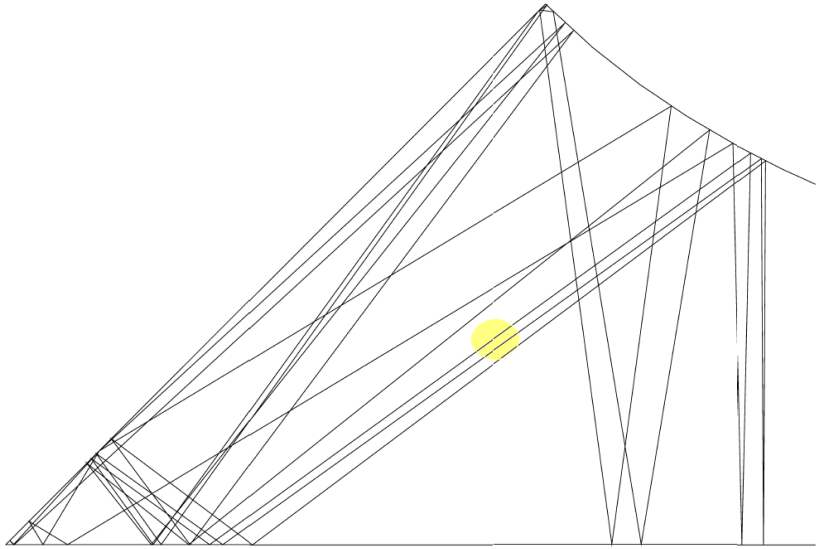
Periodic orbits in chaotic systems come in bunches.

Example (Sieber & Richter 2001):



Realistic picture:





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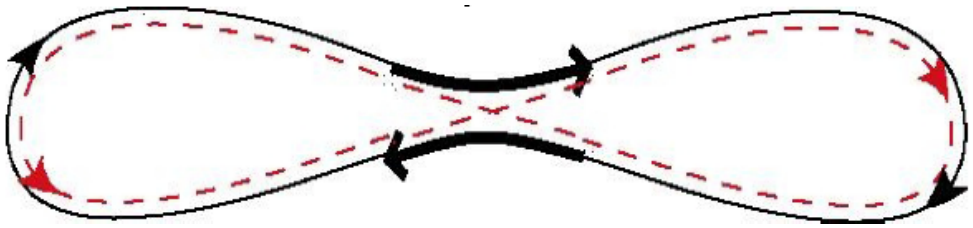


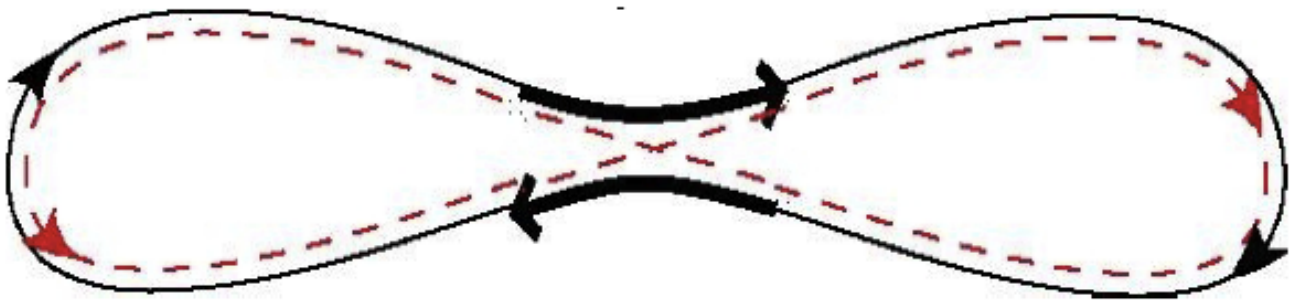
- encounters
  = regions where parts of an orbit come close to each other (up to time reversal)
- can switch connections to get different (but very similar) orbits
- present example requires time reversal invariance

Underlying mechanism

Phase space directions in hyperbolic systems:

- stable direction:
  deviations shrink asymptotically like $e^{-\lambda t}$
  (λ=Lyapunov exponent)
- unstable direction:
  deviations grow for $t \to \infty$ and shrink for $t \to -\infty$ like $e^{\lambda t}$
  $\Rightarrow$ sensitive dependence on initial conditions

Construction of partner orbit:



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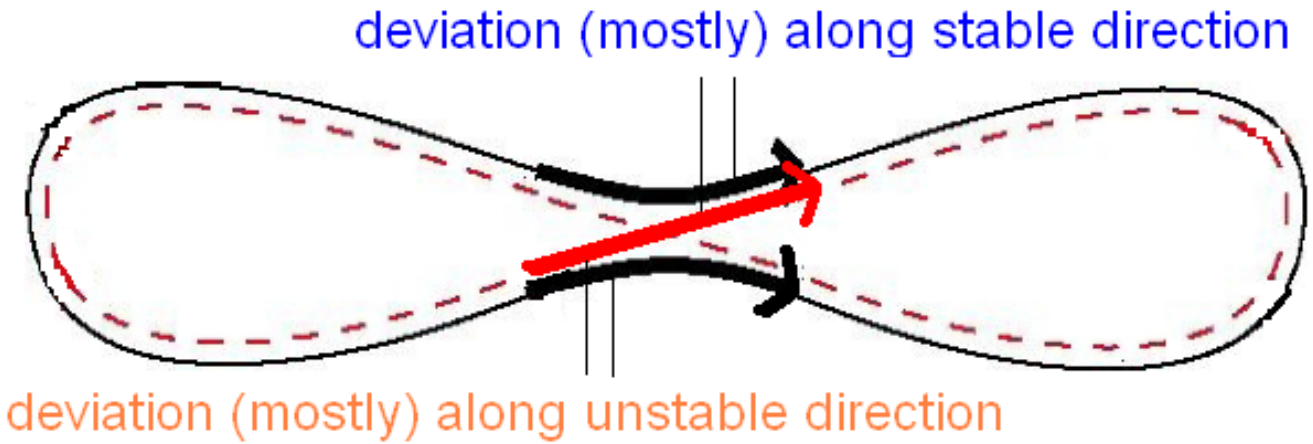

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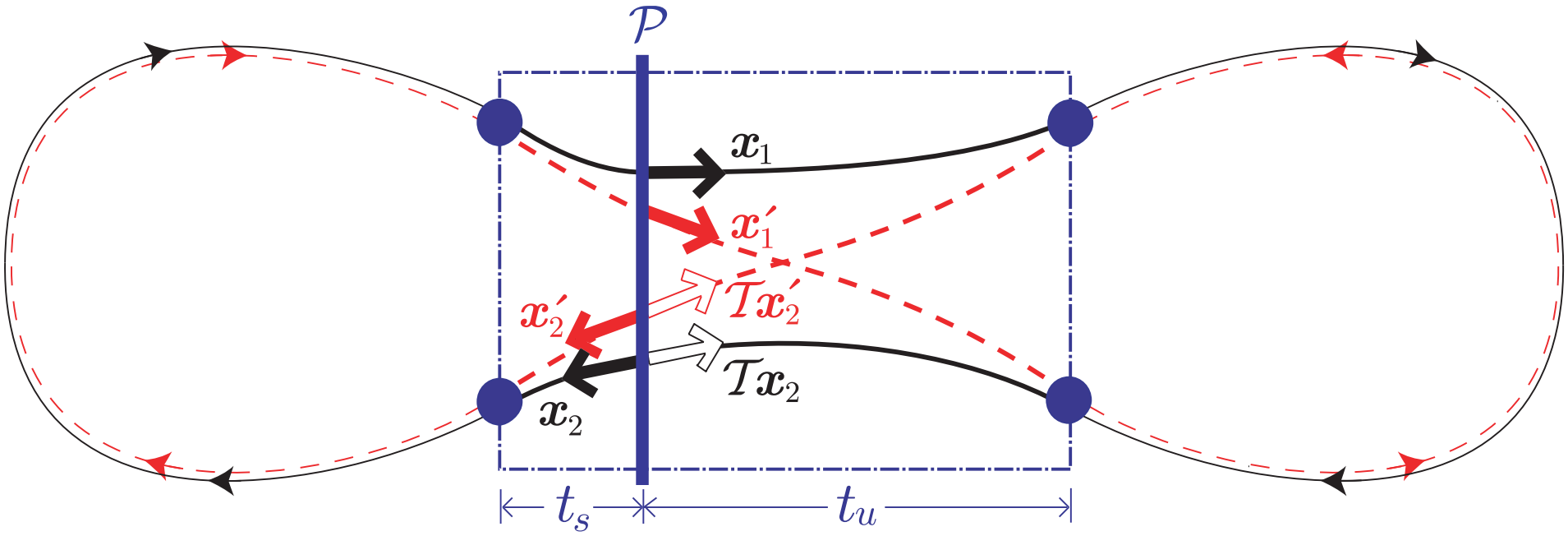
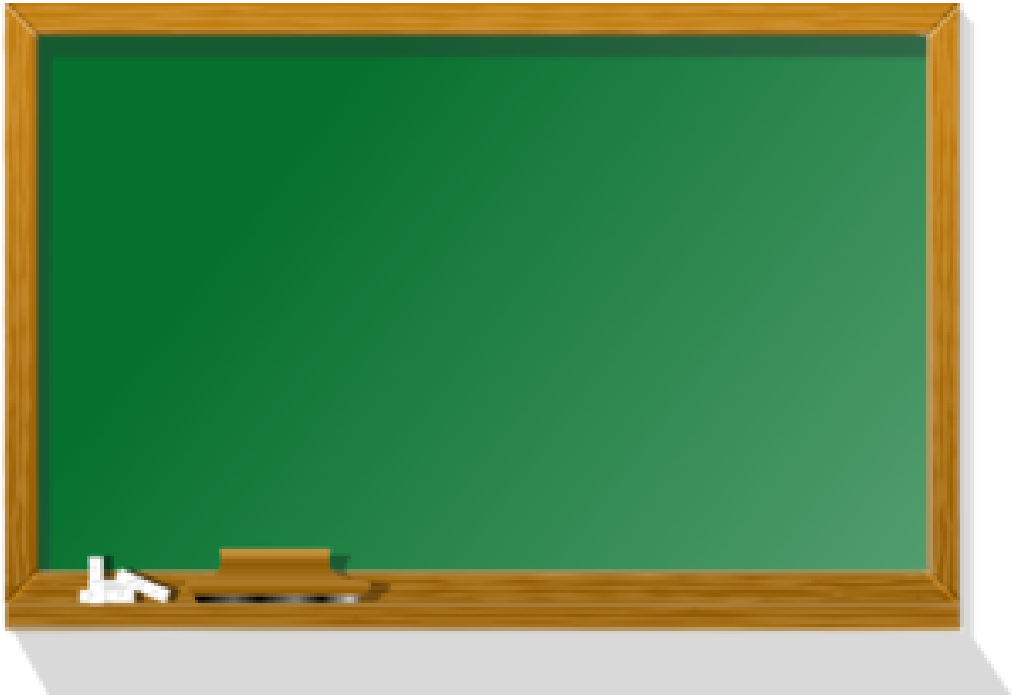
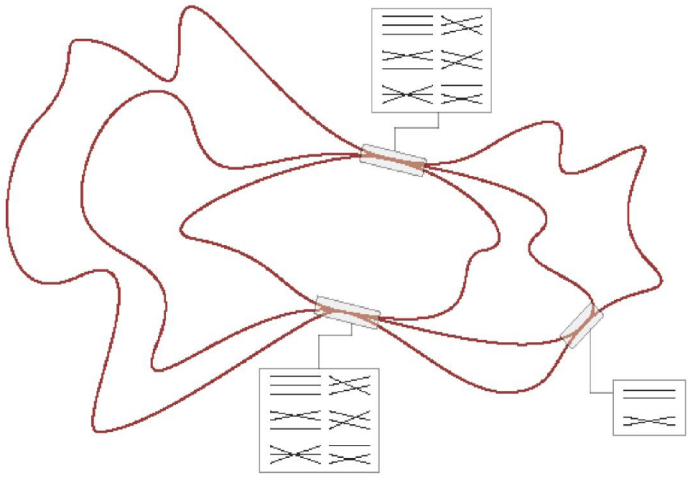


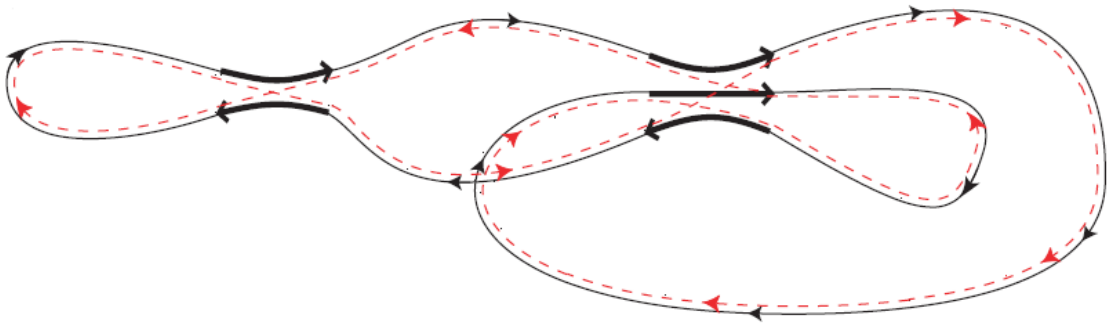


Generalisation

- orbits can differ in arbitrarily many encounters where arbitrarily many stretches come close



- for time reversal invariant systems: stretches may be almost mutually time reversed



Generalisation

- contribution of each “diagram”

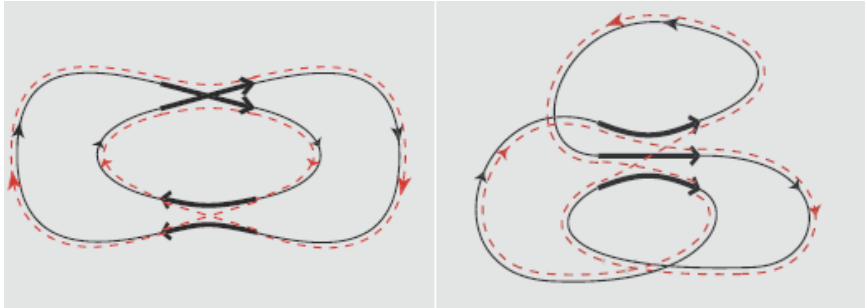
$$\kappa \frac{(-1)^V \prod_l l^{v_l}}{L(L-V-1)!} \tau^{L-V+1}$$

where

$$\begin{aligned}
\kappa &= 2 \text{ with time-rev. invariance}, 1 \text{ without} \\
v_l &= \# \text{ of encounters with } l \text{ stretches} \\
V &= \# \text{ of all encounters} \\
L &= \# \text{ of all stretches}
\end{aligned}$$

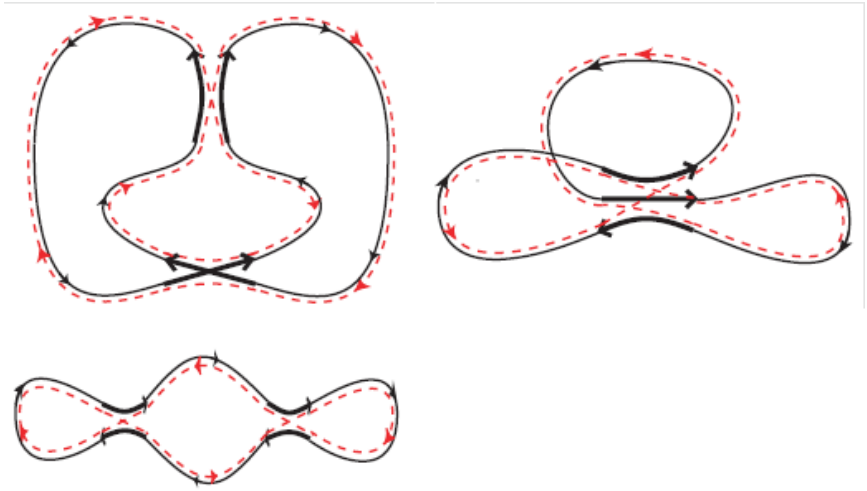
Example: τ^3

- orbit pairs in systems without time reversal invariance



$\Rightarrow$ contributions cancel, agreement with GUE

- additional pairs requiring time reversal invariance



$\Rightarrow$ $2\tau^3$, agreement with GOE

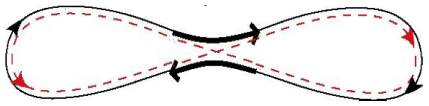
(Heusler, S.M., Braun, Haake 2003)

Summary

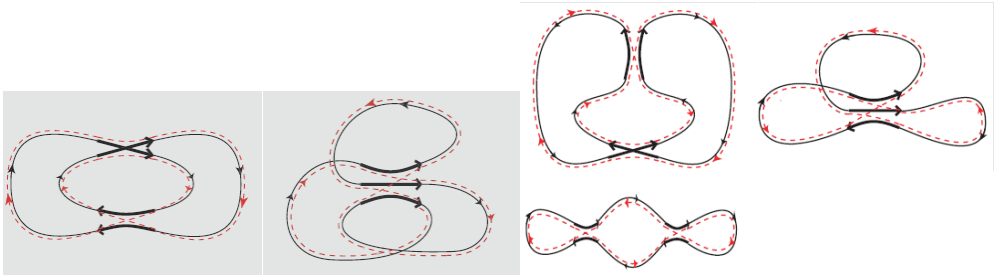
Spectral form factor:

$$K(\tau) \approx \frac{1}{T_H} \mathrm{Re} \left\langle \sum_{p,p'} A_p A_{p'}^* e^{i(S_p - S_{p'})/\hbar} \delta \left(\tau T_H - \frac{T_p + T_{p'}}{2} \right) \right\rangle$$

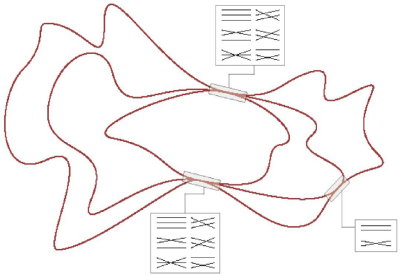
- diagonal approximation: 2τ (τ without time reversal invariance)
- Sieber-Richter pairs: $-2\tau^2$



- $2\tau^3$ from



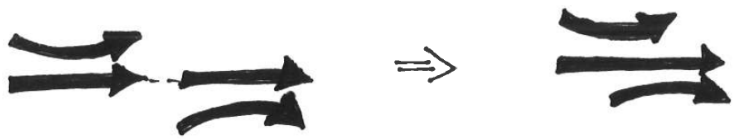
- higher orders: e.g.



All orders

Method I

- relate diagrams to each other by shrinking away links
- example:



  here the order of τ remains the same
  but the sign $(-1)^V$ changes
- this leads to cancellation for systems without time-reversal invariance
- with time-reversal invariance:
  steps where shrinking changes the order are relevant
  get recursion between coefficients of $K(\tau)$
- to implement this, describe diagrams in terms of permutations

All orders

Result:

- systems without time-reversal invariance: agreement with GUE

$$K(\tau) = \tau \qquad (\tau < 1)$$

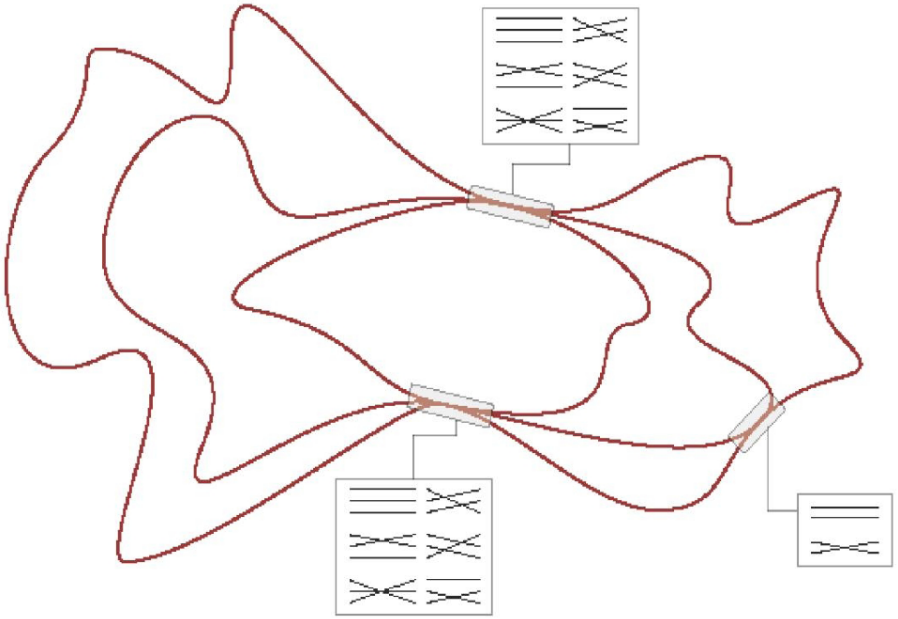
- time-reversal invariant systems: agreement with GOE

$$K(\tau) = 2\tau - \tau \ln(1 + 2\tau) = 2\tau - 2\tau^2 + 2\tau^3 - \frac{8}{3}\tau^4 + \ldots \qquad (\tau < 1)$$

(S.M., Heusler, Braun, Haake, Altland, PRL 2004 + PRE 2005)

All orders in τ

Method II



Compare to nonlinear sigma model (field theoretical implemetation of RMT)

- encounters $\leftrightarrow$ vertices
- links $\leftrightarrow$ propagator lines
- diagrams and their contributions coincide

$\Rightarrow$ Spectral statistics agrees with RMT

5 Resummation

Correlation function

Idea:

to get $\tau > 1$ improve semiclassical approximation
build in that $E_n \in \mathbb{R}$
change setting to make this easier

$\Rightarrow$ go back to **correlation function**

$$
\begin{aligned}
R(\epsilon) &= \left\langle d\left(E+\frac{\epsilon}{2}\right) d\left(E-\frac{\epsilon}{2}\right)\right\rangle \qquad (\text{take } \bar{d}=1) \\
R_{\mathrm{GUE}}(\epsilon) &= \mathrm{Re}\left(1-\frac{1}{2(\pi\epsilon)^2} + \frac{1}{2(\pi\epsilon)^2} e^{2\pi i\epsilon}\right) \\
R_{\mathrm{GOE}}(\epsilon) &= \mathrm{Re}\left(\sum_n c_n \left(\frac{1}{\epsilon}\right)^n + \sum_n d_n \left(\frac{1}{\epsilon}\right)^n e^{2\pi i\epsilon}\right)
\end{aligned}
$$

non-oscillatory terms $\leftrightarrow \tau < 1$ oscillatory terms $\leftrightarrow \tau > 1$

Spectral determinant

access level density from spectral determinant

$$\Delta(E) = \det(E - H) = \prod_j (E - E_j) = \exp \operatorname{tr} \ln (E - H)$$

Motivation:

- used in random-matrix theory
- can now build in that $\Delta(E) \in \mathbb{R}$ for $E \in \mathbb{R}$

Relation to level density:

$$\frac{\partial}{\partial E} \Delta(E)^{-1} = -\operatorname{tr} \frac{1}{E - H} \Delta(E)^{-1}$$

Therefore

$$d(E) = -\frac{1}{\pi} \mathrm{Im} \operatorname{tr} \frac{1}{E - H} = \frac{1}{\pi} \mathrm{Im} \frac{\partial}{\partial E'} \frac{\Delta(E)}{\Delta(E')} \Big|_{E'=E}$$

(where E is taken with a small positive imaginary part)

Generating function

correlation function can thus be accessed through generating function:

$$
\begin{aligned}
Z(\alpha,\beta,\gamma,\delta) &= \left\langle \frac{\Delta(E+\gamma)\Delta(E-\delta)}{\Delta(E+\alpha)\Delta(E-\beta)} \right\rangle \\
R(\epsilon) &\propto \mathrm{Re}\, \frac{\partial^2 Z}{\partial\alpha\partial\beta}\Big|_{\alpha=\beta=\gamma=\delta=\epsilon/2}
\end{aligned}
$$

Conventional semiclassics

$$\mathrm{tr}\,(E-H)^{-1} \sim \underbrace{\boxed{-i\pi\bar{d}}}_{\text{Weyl}} + \underbrace{\boxed{\text{sum over classical periodic orbits}}}_{\text{Gutzwiller}}$$

$$\Rightarrow \Delta\,(E) \sim \exp\left(\int^{E} dE'\,\mathrm{tr}\,(E'-H)^{-1}\right)$$

$$\sim \underbrace{\boxed{e^{-i\pi\bar{d}E}}}_{\text{Weyl}} \times \underbrace{\boxed{\textstyle\sum_A F_A(-1)^{n_A} e^{iS_A(E)/\hbar}}}_{\text{sum over sets of classical periodic orbits } A}$$

”Resummed” semiclassics

But $\Delta(E)$ should be real for real E

$\Rightarrow$ Resummation (Berry & Keating)

$$\left(\sum \text{over orbit sets} > T_H/2\right) = \left(\sum \text{over orbit sets} < T_H/2\right)^*$$

$\Rightarrow$ Riemann-Siegel lookalike

$$\Delta(E) = e^{-i\pi\bar{d}E} \times \sum_{A\ (T_A < T_H/2)} F_A(-1)^{n_A} e^{iS_A(E)/\hbar} + \text{c.c.}$$

Generating function

$$
\begin{aligned}
Z &= \left\langle \frac{\Delta(E+\gamma)\Delta(E-\delta)}{\Delta(E+\alpha)\Delta(E-\beta)} \right\rangle \\
&\sim \boxed{e^{i\pi(\alpha+\beta-\gamma-\delta)}} \\
&\quad \times \left\langle \boxed{\sum F_A F_B^* F_C F_D^* (-1)^{n_C+n_D} e^{i[(S_A(E+\alpha)+S_C(E+\gamma)-S_B(E-\beta)-S_D(E-\delta)]/\hbar}} \right\rangle \\
&\quad + \{\gamma \to -\delta, \delta \to -\gamma\}
\end{aligned}
$$

Weyl factor

sum over orbit sets A, B, C, D ($T_C, T_D < T_H/2$)

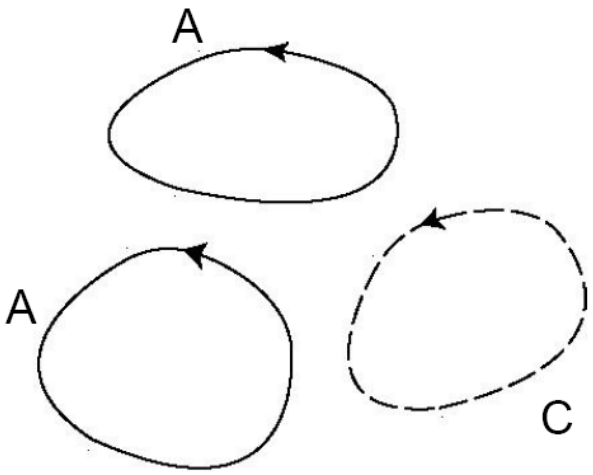
- Weyl factors $\to 1$ or $e^{2\pi i\epsilon}$ as $\alpha, \beta, \gamma, \delta \to \epsilon/2$ ☺
- need **small action differences**

$$\Delta S = S_A(E+\alpha) + S_C(E+\gamma) - S_B(E-\beta) - S_D(E-\delta)$$

$\Rightarrow$ orbit correlations ☺

Contributions

- Diagonal approximation:
  (A, C) contain the same orbits as (B, D)



Contributions

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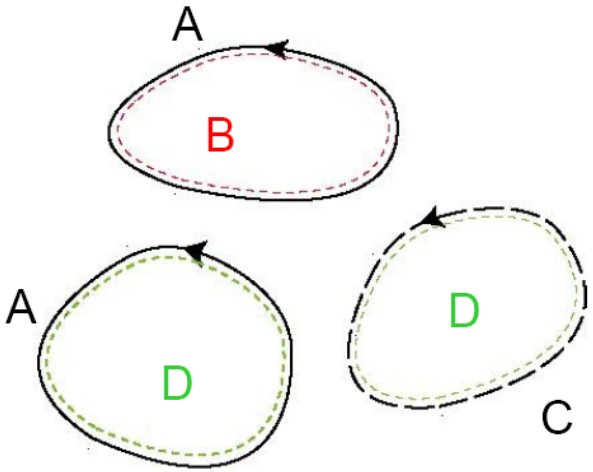
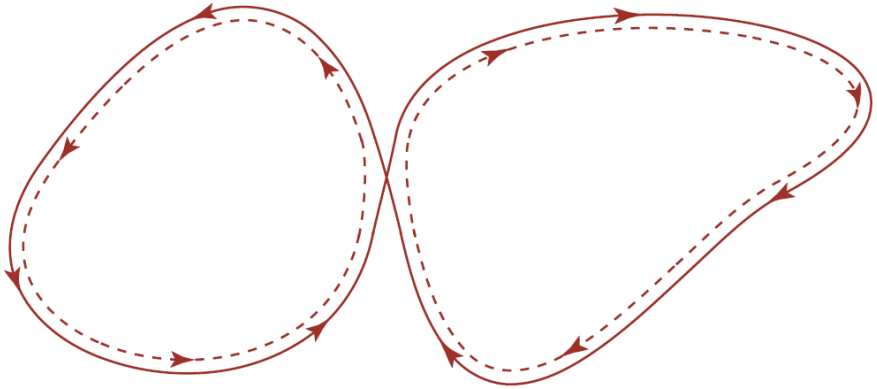


- Encounters:



Full agreement with RMT predictions.

Result

- without time reversal invariance

$$Z = e^{i\pi(\alpha+\gamma-\beta-\delta)} \cdot \frac{(\alpha+\delta)(\gamma+\beta)}{(\alpha+\beta)(\gamma+\delta)} + \{\gamma \to -\delta, \delta \to -\gamma\}$$

- with time reversal invariance

$$Z = e^{i\pi(\alpha+\gamma-\beta-\delta)} \cdot \left(\frac{(\alpha+\delta)(\gamma+\beta)}{(\alpha+\beta)(\gamma+\delta)}\right)^2 + \text{further terms}$$
$$+\{\gamma \to -\delta, \delta \to -\gamma\}$$

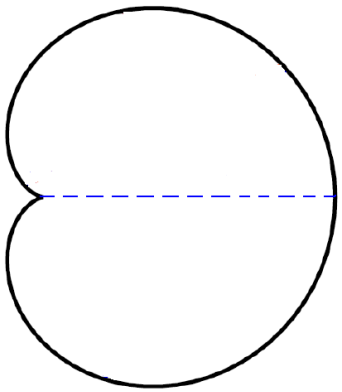
$R(\epsilon) \propto \mathrm{Re}\, \frac{\partial^2 Z}{\partial\alpha\partial\beta}\Big|_{\alpha=\beta=\gamma=\delta=\epsilon/2}$ agrees with GUE and GOE predictions.

Conclusions

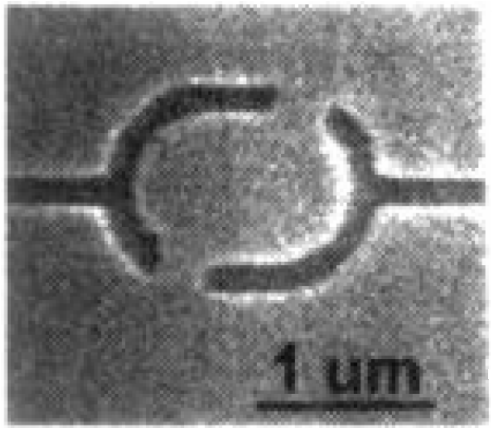
- Classically chaotic systems: sensitive dependence on initial conditions
- Gutzwiller formula: level density written as sum over periodic orbits
- Chaotic systems have universal spectral statistics
- semiclassical explanation involves orbit correlations due to encounters in phase space
- use generating functions and Riemann-Siegel lookalike to understand oscillatory terms in $R(\epsilon)$

Applications

- symmetries



- higher order correlation functions
- transport through chaotic cavities$\Rightarrow$ need open trajectories



Mathematical status

Orbit pairs:

- identical, time-reversed, differing in encounters
  $\Rightarrow$ universal result in agreement with RMT
- other systematic correlations:
  should be related to further symmetries
  universality requires absence of symmetries
  need a general definition of symmetries
- 'random' orbit pairs with small action differences:
  contributions should cancel

Conditions for universality:

- existence of orbit pairs requires hyperbolicity
- universal contribution obtained using
  - ergodicity, mixing
  - semiclassical limit
- no other symmetries

References

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