Universality in Quantum Chaos

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Universality

Many quantum properties of chaotic systems are universal and agree with predictions from random matrix theory, in particular the statistics of energy levels.

(Bohigas, Giannoni, Schmit 84; see also Casati, Vals-Gras, Guarneri; Berry, Tabor)

Why?
Outline

- Classical chaos
- Quantum chaos
- Semiclassical methods (Gutzwiller trace formula)
- Semiclassical approach to spectral statistics
- Resummation
1 Classical chaos
Billiards (classical)

particle moving in domain $B \subset \mathbb{R}^2$

- motion on straight line with constant velocity
- reflection at boundary
  (angle of incidence = angle of reflection)

Billiards with 'irregular' boundary show chaotic behaviour.

- diamond
- cardioid
- stadium
- Sinai
Chaos

(a) sensitive dependence on initial conditions
(b) **ergodicity**

long trajectories fill the interior almost uniformly
+ all angles are equally likely

**general definition:**
energy shell $\Omega = \{(q, p), H(q, p) = E\}$
a system is ergodic if for $\Omega_0 \subset \Omega$ and almost all trajectories

$$\frac{\text{time up to } T \text{ spent in } \Omega_0}{T} \rightarrow \frac{\text{vol}(\Omega_0)}{\text{vol}(\Omega)} \quad (T \rightarrow \infty)$$
Ergodicity

billiards that are **not ergodic**:
2 Quantum chaos

quantum properties of systems that are classically chaotic
**Schrödinger equation** in domain $B \subset \mathbb{R}^2$, no potential

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(q, t) = i\hbar \frac{\partial}{\partial t} \psi(q, t)$$

Dirichlet boundary conditions:

$$\psi(q, t) = 0 \quad \text{for} \quad q \in \partial B$$

**time independent Schrödinger equation**

$$-\frac{\hbar^2}{2m} \nabla^2 \psi_n(q) = E_n \psi_n(q)$$

$\psi_n(q)$ – energy eigenfunctions

$E_n$ – energy levels
(a) sensitive dependence on initial conditions?

NO, because Schrödinger equation is linear
(b) energy eigenfunctions

plot $|\psi_n(q)|^2$ for eigenfunctions with increasing energy:

\[ n = 100 \quad n = 400 \quad n = 1000 \quad n = 2000 \]

eigenfunctions become more and more equidistributed (quantum ergodicity)
Quantum chaos

probability that a particle in state \( n \) is found in a part \( B_0 \) of position space \( B \):

\[
\mu_n(B_0) = \int_{B_0} d^2 q |\psi_n(q)|^2
\]

in a quantum ergodic system:

\[
\mu_n(B_0) \to \frac{\text{area}(B_0)}{\text{area}(B)}
\]

for increasing energies after taking out exceptional wavefunctions ("scars") such as this:
Quantum chaos

- Classical: chaotic and integrable
- Quantum: chaotic and integrable
Quantum chaos

\((c)\) statistics of energy levels

Integrable\hspace{2cm}\text{chaotic}

\((\text{large})\) energy levels of a chaotic system \textbf{repel} each other!
4 Semiclassical approach to spectral statistics
Two-point correlation function

\[
\int_0^\infty dE \ d \left( E + \frac{y}{2} \right) d \left( E - \frac{y}{2} \right) = \sum_{jk} \delta(y - (E_j - E_k))
\]

make everything dimensionless and average over energy and energy differences:

\[
R(\epsilon) = \frac{1}{d^2} \left\langle d \left( E + \frac{\epsilon}{2d} \right) d \left( E - \frac{\epsilon}{2d} \right) \right\rangle
\]
Random matrix predictions

Small energy differences are less likely!

- Systems without time rev. invariance: Gaussian Unitary Ensemble

\[ R_{\text{GUE}}(\epsilon) = 1 - \left( \frac{\sin(\pi \epsilon)}{\pi \epsilon} \right)^2 = \text{Re} \left( 1 - \frac{1}{2(\pi \epsilon)^2} + \frac{1}{2(\pi \epsilon)^2} e^{2\pi i \epsilon} \right) \]
Random matrix predictions

systems with time rev. inv.: Gaussian Orthogonal Ensemble

\[
R_{\text{GOE}}(\epsilon) = 1 - \left( \frac{\sin \pi \epsilon}{\pi \epsilon} \right)^2 + \left( \frac{\text{Si} \frac{\pi \epsilon}{\pi} - \text{sign} \epsilon}{2} \right) \left( \frac{\cos \pi \epsilon}{\pi \epsilon} - \frac{\sin \pi \epsilon}{(\pi \epsilon)^2} \right)
\]

\[
= \text{Re} \left( \sum_n c_n \left( \frac{1}{\epsilon} \right)^n + \sum_n d_n \left( \frac{1}{\epsilon} \right)^n e^{2\pi i \epsilon} \right)
\]

where \( \text{Si}(x) = \int_0^x \frac{\sin y}{y} dy \)

in both cases non-oscillatory terms and oscillatory terms
Spectral form factor

Fourier transform

\[ K(\tau) = \text{Re} \int_{-\infty}^{\infty} d\epsilon \ (R(\epsilon) - 1)e^{2\pi i \epsilon \tau} \]

- systems without time rev. invariance: Gaussian Unitary Ensemble
  \[ K(\tau) = \begin{cases} \tau & (\tau < 1) \\ 1 & (\tau > 1) \end{cases} \]

- systems with time rev. inv.: Gaussian Orthogonal Ensemble
  \[ K(\tau) = \begin{cases} 2\tau - \tau \ln(1 + 2\tau) & (\tau < 1) \\ 2 - \ln \frac{2\tau + 1}{2\tau - 1} & (\tau > 1) \end{cases} \]
Semiclassical approximation

evaluate

\[ R(\epsilon) = \frac{1}{d^2} \left\langle d \left( E + \frac{\epsilon}{2d} \right) d \left( E - \frac{\epsilon}{2d} \right) \right\rangle \]

using

\[ d(E) \approx \bar{d} + \frac{1}{\pi \hbar} \text{Re} \sum_p A_p e^{i S_p(E) / \hbar} \]

\[ A_p \left( E \pm \frac{\epsilon}{2d} \right) \approx A_p(E) \]

\[ S_p \left( E \pm \frac{\epsilon}{2d} \right) \approx S_p(E) \pm \frac{T_p(E)\epsilon}{2d} \]

\[ T_H = 2\pi \hbar \bar{d} \quad (\text{Heisenberg time}) \]

we get the double sum

\[ R(\epsilon) \approx 1 + \frac{1}{T_H^2} \text{Re} \left\langle \sum_{p, p'} A_p A_{p'}^* e^{i(S_p - S_{p'}) / \hbar} e^{i \pi \epsilon (T_p + T_{p'}) / T_H} \right\rangle \]

+ similar term with \( e^{i(S_p + S_{p'}) / \hbar} \)
Semiclassical approximation

\[ R(\epsilon) \approx 1 + \frac{1}{T_H^2} \text{Re} \left\langle \sum_{p,p'} A_p A_p^* \frac{e^{i(S_p - S_{p'})/\hbar}}{\hbar} e^{i\pi\epsilon(T_p + T_{p'})/T_H} \right\rangle \\
+ \text{similar term with } e^{i(S_p + S_{p'})/\hbar} \]

energy average \( \Rightarrow \) systematic contribution only from first term for small action differences (at most \( \sim \hbar \)) constructive interference!

Fourier transform gives:

\[ K(\tau) \approx \frac{1}{T_H} \text{Re} \left\langle \sum_{p,p'} A_p A_p^* \frac{e^{i(S_p - S_{p'})/\hbar}}{\hbar} \delta \left( \tau T_H - \frac{T_p + T_{p'}}{2} \right) \right\rangle \]

relevant orbits have periods of the order

\[ T_H = 2\pi\hbar\tilde{d} = 2\pi\hbar \frac{\Omega}{(2\pi\hbar)^n} \rightarrow \infty \quad (\hbar \rightarrow 0) \]
Diagonal approximation

(Berry; Hannay & Ozorio de Almeida)

- for systems without time reversal invariance: take $p' = p$

$$K_{\text{diag}}(\tau) = \frac{1}{T_H} \left\langle \sum_p |A_p|^2 \delta(\tau T_H - T_p) \right\rangle \sim \tau$$

sum over periodic orbits evaluated using ergodicity:

$$\left\langle \sum_p |A_p|^2 \delta(T - T_p) \right\rangle \sim T$$

- time reversal invariant systems: $p' = p$ or time reversed of $p$

$$K_{\text{diag}}(\tau) = 2\tau$$
Summary

Gutzwiller trace formula: \[ d(E) \approx \bar{d} + \frac{1}{\pi \hbar} \text{Re} \sum_p A_p e^{i S_p(E)/\hbar} \]

Spectral form factor:

\[ K(\tau) \approx \frac{1}{T_H} \text{Re} \left\langle \sum_{p,p'} A_p A_p^* e^{i (S_p - S_{p'})/\hbar} \delta \left( \tau T_H - \frac{T_p + T_{p'}}{2} \right) \right\rangle \]

RMT prediction:

- **systems without time rev. invariance:** Gaussian Unitary Ensemble
  \[ K(\tau) = \begin{cases} \tau & (\tau < 1) \\ 1 & (\tau > 1) \end{cases} \]

- **systems with time rev. inv.:** Gaussian Orthogonal Ensemble
  \[ K(\tau) = \begin{cases} 2\tau - \tau \ln(1 + 2\tau) & (\tau < 1) \\ 2 - \ln \frac{2\tau + 1}{2\tau - 1} & (\tau > 1) \end{cases} \]
Orbit correlations

Periodic orbits in chaotic systems come in bunches.

Example (Sieber & Richter 2001):

Realistic picture:
Periodic orbits in chaotic systems come in bunches.

Example (Sieber & Richter 2001):

- **encounters**
  = regions where parts of an orbit come close to each other (up to time reversal)

- **can switch connections** to get different (but very similar) orbits

- present example requires **time reversal invariance**
Underlying mechanism

Phase space directions in hyperbolic systems:

- **stable direction:**
  deviations shrink asymptotically like $e^{-\lambda t}$
  ($\lambda=$Lyapunov exponent)

- **unstable direction:**
  deviations grow for $t \to \infty$ and shrink for $t \to -\infty$ like $e^{\lambda t}$
  $\Rightarrow$ sensitive dependence on initial conditions

Construction of partner orbit:
Underlying mechanism

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Construction of partner orbit:

![Diagram showing deviation (mostly) along stable and unstable directions]
Underlying mechanism
Generalisation

- orbits can differ in **arbitrarily many encounters where arbitrarily many stretches come close**

- for time reversal invariant systems: stretches may be almost mutually time reversed
Generalisation

- contribution of each “diagram”

\[ \kappa \frac{(-1)^V \prod_l l^{v_l}}{L(L - V - 1)!} \tau^{L - V + 1} \]

where

- \( \kappa = 2 \) with time-rev. invariance, 1 without
- \( v_l = \# \) of encounters with \( l \) stretches
- \( V = \# \) of all encounters
- \( L = \# \) of all stretches
Example: $\tau^3$

- Orbit pairs in systems without time reversal invariance

  $\Rightarrow$ contributions cancel, agreement with GUE

- Additional pairs requiring time reversal invariance

  $\Rightarrow 2\tau^3$, agreement with GOE

(Heusler, S.M., Braun, Haake 2003)
Summary

Spectral form factor:

\[ K(\tau) \approx \frac{1}{T_H} \text{Re} \left\langle \sum_{p,p'} A_p A_{p'}^* e^{i(S_p - S_{p'})/\hbar \delta} \left( \tau T_H - \frac{T_p + T_{p'}}{2} \right) \right\rangle \]

- diagonal approximation: \(2\tau\) (\(\tau\) without time reversal invariance)
- Sieber-Richter pairs: \(-2\tau^2\)
- \(2\tau^3\) from higher orders: e.g.
All orders

Method I

- relate diagrams to each other by **shrinking away links**
- example:

\[ \text{here the order of } \tau \text{ remains the same} \]
\[ \text{but the sign } (-1)^{\nu} \text{ changes} \]

- this leads to **cancellation** for systems **without** time-reversal invariance
- **with** time-reversal invariance:
  - steps where shrinking changes the order are relevant
  - get **recursion** between coefficients of \( K(\tau) \)
- to implement this, describe diagrams in terms of **permutations**
All orders

Result:

- **systems without time-reversal invariance**: agreement with GUE

  \[ K(\tau) = \tau \quad (\tau < 1) \]

- **time-reversal invariant systems**: agreement with GOE

  \[ K(\tau) = 2\tau - \tau \ln(1 + 2\tau) = 2\tau - 2\tau^2 + 2\tau^3 - \frac{8}{3}\tau^4 + \ldots \quad (\tau < 1) \]

All orders in $\tau$

Method II

Compare to nonlinear sigma model (field theoretical implementation of RMT)

- encounters $\leftrightarrow$ vertices
- links $\leftrightarrow$ propagator lines
- diagrams and their contributions coincide

$\Rightarrow$ Spectral statistics agrees with RMT
5 Resummation
Correlation function

Idea:

to get $\tau > 1$ improve semiclassical approximation
build in that $E_n \in \mathbb{R}$
change setting to make this easier

$\Rightarrow$ go back to correlation function

\[
R(\epsilon) = \left\langle d \left( E + \frac{\epsilon}{2} \right) d \left( E - \frac{\epsilon}{2} \right) \right\rangle \quad \text{(take } \tilde{d} = 1)\\
R_{\text{GUE}}(\epsilon) = \text{Re} \left( 1 - \frac{1}{2(\pi \epsilon)^2} + \frac{1}{2(\pi \epsilon)^2} e^{2\pi i \epsilon} \right)\\
R_{\text{GOE}}(\epsilon) = \text{Re} \left( \sum_n c_n \left( \frac{1}{\epsilon} \right)^n + \sum_n d_n \left( \frac{1}{\epsilon} \right)^n e^{2\pi i \epsilon} \right)
\]

non-oscillatory terms $\leftrightarrow \tau < 1$ \quad oscillatory terms $\leftrightarrow \tau > 1$
Spectral determinant

access level density from spectral determinant

\[ \Delta(E) = \det(E - H) = \prod_j (E - E_j) = \exp \text{tr} \ln (E - H) \]

Motivation:
- used in random-matrix theory
- can now build in that \( \Delta(E) \in \mathbb{R} \) for \( E \in \mathbb{R} \)

Relation to level density:

\[
\frac{\partial}{\partial E} \Delta(E)^{-1} = - \text{tr} \left( \frac{1}{E - H} \Delta(E)^{-1} \right)
\]

Therefore

\[ d(E) = - \frac{1}{\pi} \text{Im} \text{tr} \frac{1}{E - H} = \frac{1}{\pi} \text{Im} \frac{\partial}{\partial E'} \frac{\Delta(E)}{\Delta(E')} \bigg|_{E'=E} \]

(where \( E \) is taken with a small positive imaginary part)
correlation function can thus be accessed through generating function:

\[
Z(\alpha, \beta, \gamma, \delta) = \langle \frac{\Delta(E + \gamma)\Delta(E - \delta)}{\Delta(E + \alpha)\Delta(E - \beta)} \rangle
\]

\[
R(\epsilon) \propto \text{Re} \left. \frac{\partial^2 Z}{\partial \alpha \partial \beta} \right|_{\alpha=\beta=\gamma=\delta=\epsilon/2}
\]
Conventional semiclassics

\[ \text{tr} (E - H)^{-1} \sim -i\pi \tilde{d} + \text{sum over classical periodic orbits} \]

Weyl

Gutzwiller

\[ \Rightarrow \Delta (E) \sim \exp \left( \int_{E}^{E'} dE' \text{ tr} (E' - H)^{-1} \right) \]

\[ \sim e^{-i\pi \tilde{d}E} \times \sum_{A} F_{A} (-1)^{n_{A}} e^{iS_{A}(E)/\hbar} \]

Weyl

sum over sets of classical periodic orbits \( A \)
"Resummed" semiclassics

But $\Delta(E)$ should be real for real $E$

⇒ Resummation (Berry & Keating)

$$\left( \sum \text{ over orbit sets } > \frac{T_H}{2} \right) = \left( \sum \text{ over orbit sets } < \frac{T_H}{2} \right)^*$$

⇒ Riemann-Siegel lookalike

$$\Delta(E) = e^{-i\pi \tilde{d}E} \times \sum_{A \ (T_A < T_H/2)} F_A(-1)^{n_A} e^{iS_A(E)/\hbar} + \text{c.c.}$$
Generating function

\[ Z = \left\langle \frac{\Delta(E + \gamma) \Delta(E - \delta)}{\Delta(E + \alpha) \Delta(E - \beta)} \right\rangle \]

\[ \sim e^{i\pi(\alpha + \beta - \gamma - \delta)} \]

Weyl factor

\[ \times \left\langle \sum F_A F_B^* F_C F_D^* (-1)^{n_C + n_D} e^{i[(S_A(E + \alpha) + S_C(E + \gamma) - S_B(E - \beta) - S_D(E - \delta)]/\hbar} \right\rangle \]

sum over orbit sets \( A, B, C, D \) \( (T_C, T_D < T_H/2) \)

\[ + \{ \gamma \rightarrow -\delta, \delta \rightarrow -\gamma \} \]

- Weyl factors \( \rightarrow 1 \) or \( e^{2\pi i \epsilon} \) as \( \alpha, \beta, \gamma, \delta \rightarrow \epsilon/2 \)
- need **small action differences**

\[ \Delta S = S_A(E + \alpha) + S_C(E + \gamma) - S_B(E - \beta) - S_D(E - \delta) \]

\[ \Rightarrow \text{orbit correlations} \]
Contributions

- Diagonal approximation:
  \((A, C)\) contain the same orbits as \((B, D)\)
Contributions

- Diagonal approximation:
  \((A, C)\) contain the same or orbits as \((B, D)\)

- Encounters:

Full agreement with RMT predictions.
**without** time reversal invariance

\[ Z = e^{i \pi (\alpha + \gamma - \beta - \delta)} \cdot \frac{(\alpha + \delta)(\gamma + \beta)}{(\alpha + \beta)(\gamma + \delta)} + \{ \gamma \rightarrow -\delta, \delta \rightarrow -\gamma \} \]

**with** time reversal invariance

\[ Z = e^{i \pi (\alpha + \gamma - \beta - \delta)} \cdot \left( \frac{(\alpha + \delta)(\gamma + \beta)}{(\alpha + \beta)(\gamma + \delta)} \right)^2 + \text{further terms} \]

\[ + \{ \gamma \rightarrow -\delta, \delta \rightarrow -\gamma \} \]

\[ R(\epsilon) \propto \text{Re} \left. \frac{\partial^2 Z}{\partial \alpha \partial \beta} \right|_{\alpha = \beta = \gamma = \delta = \epsilon/2} \text{ agrees with GUE and GOE predictions.} \]
Classically chaotic systems: sensitive dependence on initial conditions

Gutzwiller formula: level density written as sum over periodic orbits

Chaotic systems have universal spectral statistics

Semiclassical explanation involves orbit correlations due to encounters in phase space

Use generating functions and Riemann-Siegel lookalike to understand oscillatory terms in $R(\epsilon)$
Applications

- symmetries

- higher order correlation functions

- transport through chaotic cavities \( \Rightarrow \) need open trajectories
Mathematical status

Orbit pairs:

- identical, time-reversed, differing in encounters
  ⇒ universal result in agreement with RMT
- other systematic correlations:
  should be related to further symmetries
  universality requires absence of symmetries
  need a general definition of symmetries
- ‘random’ orbit pairs with small action differences:
  contributions should cancel

Conditions for universality:

- existence of orbit pairs requires hyperbolicity
- universal contribution obtained using
  - ergodicity, mixing
  - semiclassical limit
- no other symmetries
References

- S. Müller, Quantum Chaos, Undergraduate lecture notes, University of Bristol (2013) [http://www.maths.bris.ac.uk/maxsm/qcnotes.pdf]