Solution of the sign problem in the Potts model at fixed baryon number

Urs Wenger

Albert Einstein Center for Fundamental Physics
University of Bern

Phys.Rev. **D97** (2018) 114503 [1712.07585] in collaboration with
A. Alexandru, G. Bergner, D. Schaich

10 September 2018, SIGN 2018, Bielefeld, Germany

Motivation for canonical formulation of QCD

- Consider the grand-canonical partition function of QCD:

$$Z_{\mathrm{GC}}^{\mathrm{QCD}}(\mu) = \mathrm{Tr}\left[e^{-\mathcal{H}(\mu)/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t(\mu)$$

- The sign problem of QCD is a manifestation of huge cancellations between different states:
  - all states are present for any μ and T
  - some states need to cancel out at different μ and T

- In the canonical formulation:

$$Z_{\mathrm{C}}^{\mathrm{QCD}}(N_Q) = \mathrm{Tr}_{N_Q}\left[e^{-\mathcal{H}(\mu)/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t^{(N_Q)}$$

  - dimension of Fock space tremendously reduced
  - less cancellations necessary
  - e.g. $Z_{\mathrm{C}}^{\mathrm{QCD}}(N_Q) = 0$ for $N_Q \neq 0 \bmod N_c$

Motivation for canonical formulation of QCD

Canonical transfer matrices can be obtained explicitly!

- based on the dimensional reduction of the QCD fermion determinant [Alexandru, Wenger '10; Nagata, Nakamura '10]
- identification of transfer matrices [Steinhauer, Wenger '14]

Motivation for canonical formulation of QCD

Canonical transfer matrices can be obtained explicitly!

- based on the dimensional reduction of the QCD fermion determinant [Alexandru, Wenger '10; Nagata, Nakamura '10]
- identification of transfer matrices [Steinhauer, Wenger '14]

- Motivation
  - QCD in the heavy-dense limit
  - Canonical formulation
  - Absence of the sign problem at strong coupling
- Solution in the 3-state Potts model
  - Canonical formulation
  - Bond formulation and cluster algorithm
  - Solution of the sign problem
  - cf. [Alford, Chandrasekharan, Cox, Wiese 2001]

Dimensional reduction of QCD

- Reduced Wilson fermion determinant is given by

$$\det M_{p,a}(\mu) \propto \prod_t \det Q_t^+ \cdot \det\left[e^{+\mu L_t} \cdot \mathbb{I} \pm \mathcal{T}\right]$$

where $\mathcal{T}$ is a product of transfer matrices given by

$$\mathcal{T} = \prod_t Q_t^+ \cdot \mathcal{U}_t \cdot \left(Q_{t+1}^-\right)^{-1}$$

with

$$Q_t^\pm = B_t P_\pm + P_\mp \,.$$

- Fugacity expansion yields with $N_Q^{\max} = 2 \cdot N_c \cdot L_s^3$

$$\det M_a(\mu) = \sum_{N_Q=-N_Q^{\max}}^{N_Q^{\max}} e^{\mu N_Q/T} \cdot \det M_{N_Q}$$

Heavy-dense limit of grand-canonical QCD

- The heavy-dense approximation in general consists of taking the limit $\kappa \equiv (2m+8)^{-1} \to 0$, $\mu \to \infty$ while keeping $\kappa e^{+\mu}$ fixed.
- Better: just drop the spatial hopping terms, but keep forward and backward hopping in time:
  - system of static quarks and antiquarks
- Reduced Wilson fermion matrix $\Leftrightarrow$ effective action in terms of Polyakov loops P and $P^\dagger$ in $d = 3$:

$$\det M_{p,a}^{HD} = \prod_{\bar{x}} \det\left[\mathbb{I} \pm (2\kappa e^{+\mu})^{L_t} P_{\bar{x}}\right]^2 \det\left[\mathbb{I} \pm (2\kappa e^{-\mu})^{L_t} P_{\bar{x}}^\dagger\right]^2$$

Heavy-dense limit of canonical QCD

- The canonical determinants are given by the trace over the minor matrix $\mathcal{M}$,

$$\det M^{HD}_{N_Q} \propto \mathrm{Tr}\,\mathcal{M}_{N_Q}\left[\left((2\kappa)^{+L_t}\cdot P_+\mathcal{P} + (2\kappa)^{-L_t}\cdot P_-\mathcal{P}\right)\right]$$

where $\mathcal{P}$ denotes the Polyakov loops $\mathcal{P}_{\bar{x},\bar{y}} = \mathbb{I}_{4\times 4}\otimes P_{\bar{x}}\cdot\delta_{\bar{x},\bar{y}}$.

- Under $\mathbb{Z}(N_c)$-transformation by $z_k = e^{2\pi i\cdot k/N_c}\in\mathbb{Z}(N_c)$:

$$\det M^{HD}_{N_Q} \to \det M'^{HD}_{N_Q} = z_k^{-N_Q}\cdot\det\mathcal{M}^{HD}_{N_Q}$$

and summing over z_k yields $\det\mathcal{M}_{N_Q} = 0$ for $N_Q \neq 0 \bmod N_c$

  - reduces cancellations by factor of N_c

Heavy-dense limit of canonical QCD

- Canonical determinant describing no quarks w.r.t. $N_Q^{\max}$:

$$\det M^{HD}_{N_Q^{\max}} = 1 \qquad \Leftrightarrow \qquad \text{quenched case}$$

- Canonical determinant describing a single quark, i.e. $N_Q = 1$:

$$\det M^{HD}_{N_Q^{\max}-1} = \left((2\kappa)^{L_t} + (2\kappa)^{-L_t}\right) \cdot \sum_{\bar{x}} \mathrm{Tr}\, P_{\bar{x}}$$

- For $N_Q = 2$ quarks:

$$\det M^{HD}_{N_Q^{\max}-2}/\Omega \propto 2 \sum_{\bar{x}} \mathrm{Tr}\, P_{\bar{x}} \sum_{\bar{y}} \mathrm{Tr}\, P_{\bar{y}} + \left(4 \sum_{\bar{x}} \mathrm{Tr}\, P_{\bar{x}} \sum_{\bar{y}} \mathrm{Tr}\, P_{\bar{y}} - 3 \sum_{\bar{x}} \left(\mathrm{Tr}\, P_{\bar{x}}\right)^2 + 2\, \mathrm{Tr}\, P^{\dagger}_{\bar{x}}\right)$$

- Both determinants vanish under global $\mathbb{Z}(3)$-transformations.

Heavy-dense limit of canonical QCD

- Canonical determinant for $N_Q = 3$ quarks:

$$\det M^{HD}_{N_Q^{\max}-3}/\Omega = h_3 \cdot \left(4\sum_{\bar{x}} \mathrm{Tr}\, P^\dagger_{\bar{x}} \sum_{\bar{y}} \mathrm{Tr}\, P_{\bar{y}} - 3\sum_{\bar{x}} \mathrm{Tr}\, P_{\bar{x}}\, \mathrm{Tr}\, P^\dagger_{\bar{x}} + 2L_s^3 \right)$$

$$+ h_1 \left(4\sum_{\bar{x}} \mathrm{Tr}\, P^\dagger_{\bar{x}} \sum_{\bar{y}} \mathrm{Tr}\, P_{\bar{y}} + 2\sum_{\bar{x}} (\mathrm{Tr}\, P_{\bar{x}})^2 \sum_{\bar{y}} \mathrm{Tr}\, P_{\bar{y}} \right.$$

$$\left. + 4\sum_{\bar{x}} \mathrm{Tr}\, P_{\bar{x}} \sum_{\bar{y}\neq\bar{x}} \mathrm{Tr}\, P_{\bar{y}} \sum_{\bar{z}} \mathrm{Tr}\, P_{\bar{z}} \right)$$

  - describes the propagation of mesons and baryons

- Invariant under global $\mathbb{Z}(3)$-transformations
- Description in terms of (anti-)quark occupation numbers $n_{\bar{x}}$
- Suffers from a severe sign problem, unless
  - all $P_{\bar{x}}$ align $\iff$ deconfined phase
  - global $\mathbb{Z}(3)$ is promoted to a local one $\iff$ strong coupling

The 3-state Potts model in $d = 3$ dimensions

- We use the 3-state Potts model as a proxy for the effective Polyakov loop action of heavy-dense QCD.

- Grand-canonical partition function of the Potts model:

$$Z_{\mathrm{GC}}(h) = \int \mathcal{D}z \, \exp(-S[z] + h \sum_x z_x)$$

  - Polyakov loops are represented by the Potts spins $z_x \in \mathbb{Z}(3)$
  - standard nearest-neighbour interaction

$$S[z] = -\gamma \sum_{\langle xy \rangle} \delta_{z_x, z_y}$$

  - external 'magnetic' field $h = (2\kappa e^{\mu})^{\beta}$ $\Rightarrow$ breaks $\mathbb{Z}(3)$

- There is a sign problem for $h \neq 0$, i.e. at finite quark density.

The 3-state Potts model in $d = 3$ dimensions

- Canonical partition function for N_Q quarks:

$$Z_C(N_Q) = \sum_{\{n\},|n|=N_Q} \int \mathcal{D}z \exp(-S[z]) \cdot \prod_x f[z_x, n_x]$$

  - local quark occupation number $n_x \le n_x^{\max}$ with $|n| = N_Q$
  - use the simple local fermionic weights

$$f[z, n] = z^n$$

  - equivalent to the grand-canonical partition function for small h i.e. small density:

$$Z_{GC}(h) \simeq \sum_{N_Q=0}^{\infty} e^{\mu N_Q} Z_C(N_Q) \quad \text{for } h = e^{\mu} \ll 1$$

The 3-state Potts model in $d = 3$ dimensions

Canonical partition function

$$Z_{\mathrm{C}}(N_Q) = \sum_{\{n\}} \int \mathcal{D}z \exp(\gamma \sum_{\langle xy \rangle} \delta_{z_x, z_y}) \prod_x z_x^{n_x}$$

- Action is manifestly complex $\Rightarrow$ fermion sign problem!
- Global $\mathbb{Z}(3)$ symmetry ensures $Z_{\mathrm{C}}(N_Q \neq 0 \bmod 3) = 0$:
  - projection onto integer baryon numbers
- In the limit $\gamma \to 0$, the global $\mathbb{Z}(3)$ becomes a local one:
  - projection onto integer baryon numbers on single sites

$$n_x = 0 \bmod 3 \quad (\text{limit}\, \gamma \to 0)$$

  - sign problem is absent

Bond formulation and cluster algorithm

- Introduce bonds to express the action as

$$e^{\gamma \cdot \delta_{z_x, z_y}} = \sum_{b_{xy}=0}^{1} \left(\delta_{z_x, z_y} \delta_{b_{xy}, 1} \left(e^{\gamma} - 1 \right) + \delta_{b_{xy}, 0} \right)$$

- The canonical partition function now becomes

$$Z_{\mathrm{C}}(N_Q) = \sum_{\{n\}} \sum_{\{b\}} \int \mathcal{D}z \prod_{\langle xy \rangle} \left(\delta_{z_x, z_y} \delta_{b_{xy}, 1} \left(e^{\gamma} - 1 \right) + \delta_{b_{xy}, 0} \right) \prod_x z_x^{n_x}$$

  - sum over all bond configurations $\{b\}$

- Define the sum of bond weights over $\{n\}, \{b\}, \mathcal{D}z$ as $\langle\!\langle \cdot \rangle\!\rangle_{N_Q}$:

$$Z_{\mathrm{C}}(N_Q) = \langle\!\langle \prod_x z_x^{n_x} \rangle\!\rangle_{N_Q}$$

Bond formulation and cluster algorithm

- $N_Q = 0$ gives the usual Swendsen-Wang cluster construction:
  - bond b_{xy} is occupied with probability $p(b_{xy} = 1) = (e^{\gamma} - 1)$, if $z_x = z_y$
  - weight of bond configuration is $W(\{b\}) = (e^{\gamma} - 1)^{N_b}$, with $N_b = \sum_{\langle xy \rangle} b_{xy}$

- Summation over $\mathbb{Z}(3)$ spins within connected cluster yields

$$Z_{\mathrm{C}}(N_Q = 0) = \langle\!\langle 1 \rangle\!\rangle_{N_Q=0} = \sum_{\{b\}} (e^{\gamma} - 1)^{N_b} \cdot 3^{N_C}$$

  - total number of clusters N_C
  - cluster algorithm requires Euler-tour trees to achieve dynamic connectivity in $\mathcal{O}(\ln^2 V)$ instead of $\mathcal{O}(V \ln V)$ or $\mathcal{O}(V^2)$

Solution of the sign problem in the canonical formulation

- In the canonical formulation the cluster algorithm solves the sign problem:
  - include the fermionic contribution with an improved estimator
  - similar to idea in the grand canonical formulation [Alford, Chandrasekharan, Cox, Wiese 2001]

- Average $\prod_x z_x^{n_x}$ over the subensemble of the 3^{N_C} configurations related by the $\mathbb{Z}(3)$ transformations:
  - total weight can be factorized into individual cluster weights $W_0(C)$,

$$[\![\prod_x z_x^{n_x}]\!]_{3^{N_C}} = [\![\prod_C \prod_{x \in C} z_x^{n_x}]\!]_{3^{N_C}} = \prod_C [\![\prod_{x \in C} z_x^{n_x}]\!]_3 = \prod_C W_0(C)$$

where

$$W_0(C) = [\![\prod_{x \in C} z_x^{n_x}]\!]_3 = [\![z^{\sum_{x \in C} n_x}]\!]_3 = \begin{cases} 1 & \text{if } \sum_{x \in C} n_x = 0 \bmod 3, \\ 0 & \text{else.} \end{cases}$$

Solution of the sign problem in the canonical formulation

- Hence, the canonical partition function becomes

Sign free canonical partition function

$$Z_C(N_Q) = \langle\!\langle \prod_x z_x^{n_x} \rangle\!\rangle_{N_Q} = \sum_{\{n\}} \sum_{\{b\}} (e^\gamma - 1)^{N_b} \cdot 3^{N_C} \cdot \prod_C \delta_{n_C,0}$$

  - $n_C = \sum_{x \in C} n_x \bmod 3$ denotes the triality of the cluster C
  - $\prod_C \delta_{n_C,0}$ projects on sector of configurations with triality-0 clusters only

- An intuitive, physical picture emerges:
  - only clusters with integer baryon number contribute

    $\Rightarrow$ confinement
  - quarks can move freely within the cluster

    $\Rightarrow$ deconfinement within cluster

Improved estimators for physical quantities

- We can define an improved estimator for the quark-antiquark correlator:

$$\langle z_x z_y^* \rangle_{N_Q} \equiv \langle\!\langle z_x z_y^* \prod_w z_w^{n_w} \rangle\!\rangle_{N_Q} / \langle\!\langle \prod_{w \in C} z_w^{n_w} \rangle\!\rangle_{N_Q}$$

- First calculate weight for cluster C containing source z_y^*:

$$[\![\prod_{w \in C} z_w^{n_w - \delta_{w,y}}]\!]_3 = [\![z^{\sum_{w \in C} n_w - \delta_{w,y}}]\!]_3 = \left\{ \begin{array}{ll} 1 & \text{if } \sum_{w \in C} n_w = 1 \bmod 3 \\ 0 & \text{else} \end{array} \right\} = \delta_{n_C,1}$$

Improved estimators for physical quantities

- We can define an improved estimator for the quark-antiquark correlator:

$$\langle z_x z_y^* \rangle_{N_Q} \equiv \langle\!\langle z_x z_y^* \prod_w z_w^{n_w} \rangle\!\rangle_{N_Q} / \langle\!\langle \prod_{w \in C} z_w^{n_w} \rangle\!\rangle_{N_Q}$$

- ... and the weight for cluster C containing source z_x:

$$[\![\prod_{w \in C} z_w^{n_w + \delta_{w,x}}]\!]_3 = [\![z^{\sum_{w \in C} n_w + \delta_{w,x}}]\!]_3 = \left\{ \begin{array}{ll} 1 & \text{if } \sum_{w \in C} n_w = 2 \bmod 3 \\ 0 & \text{else} \end{array} \right\} = \delta_{n_C, 2}$$

Improved estimators for physical quantities

- We can define an improved estimator for the quark-antiquark correlator:

$$\langle z_x z_y^* \rangle_{N_Q} \equiv \langle\!\langle z_x z_y^* \prod_w z_w^{n_w} \rangle\!\rangle_{N_Q} / \langle\!\langle \prod_{w \in C} z_w^{n_w} \rangle\!\rangle_{N_Q}$$

- ... and the weight for cluster C containing source z_x:

$$[\![\prod_{w \in C} z_w^{n_w + \delta_{w,x}}]\!]_3 = [\![z^{\sum_{w \in C} n_w + \delta_{w,x}}]\!]_3 = \begin{cases} 1 & \text{if } \sum_{w \in C} n_w = 2 \bmod 3 \\ 0 & \text{else} \end{cases} = \delta_{n_C,2}$$

- Calculate the subensemble average including $z_x z_y^*$:

$$[\![z_x z_y^* \prod_w z_w^{n_w}]\!]_{3^{N_C}} = \delta_{C_x,C_y} \cdot [\![z_x z_y^* \prod_w z_w^{n_w}]\!]_{3^{N_C}} + (1 - \delta_{C_x,C_y}) \cdot [\![z_x z_y^* \prod_w z_w^{n_w}]\!]_{3^{N_C}}$$

Improved estimators for physical quantities

- We can define an improved estimator for the quark-antiquark correlator:

$$\langle z_x z_y^* \rangle_{N_Q} \equiv \langle\!\langle z_x z_y^* \prod_w z_w^{n_w} \rangle\!\rangle_{N_Q} / \langle\!\langle \prod_{w \in C} z_w^{n_w} \rangle\!\rangle_{N_Q}$$

- ... and the weight for cluster C containing source z_x:

$$\left[\!\left[\prod_{w \in C} z_w^{n_w + \delta_{w,x}} \right]\!\right]_3 = \left[\!\left[z^{\sum_{w \in C} n_w + \delta_{w,x}} \right]\!\right]_3 = \left\{ \begin{array}{ll} 1 & \text{if } \sum_{w \in C} n_w = 2 \bmod 3 \\ 0 & \text{else} \end{array} \right\} = \delta_{n_C,2}$$

- Calculate the subensemble average including $z_x z_y^*$:

$$\left[\!\left[z_x z_y^* \prod_w z_w^{n_w} \right]\!\right]_{3^{N_C}} = \delta_{C_x,C_y} \cdot \left[\!\left[z_x z_y^* \prod_w z_w^{n_w} \right]\!\right]_{3^{N_C}} + (1 - \delta_{C_x,C_y}) \cdot \left[\!\left[z_x z_y^* \prod_w z_w^{n_w} \right]\!\right]_{3^{N_C}}$$

Improved estimators for physical quantities

- We can define an improved estimator for the quark-antiquark correlator:

$$\langle z_x z_y^* \rangle_{N_Q} \equiv \langle\!\langle z_x z_y^* \prod_{w} z_w^{n_w} \rangle\!\rangle_{N_Q} / \langle\!\langle \prod_{w \in C} z_w^{n_w} \rangle\!\rangle_{N_Q}$$

- ... and the weight for cluster C containing source z_x:

$$[\![\prod_{w \in C} z_w^{n_w + \delta_{w,x}}]\!]_3 = [\![z^{\sum_{w \in C} n_w + \delta_{w,x}}]\!]_3 = \left\{ \begin{array}{ll} 1 & \text{if } \sum_{w \in C} n_w = 2 \bmod 3 \\ 0 & \text{else} \end{array} \right\} = \delta_{n_C,2}$$

- Calculate the subensemble average including $z_x z_y^*$:

$$[\![z_x z_y^* \prod_{w} z_w^{n_w}]\!]_{3^{N_C}} = \delta_{C_x,C_y} \cdot \prod_{C} \delta_{n_C,0} + (1 - \delta_{C_x,C_y}) \cdot [\![z_x z_y^* \prod_{w} z_w^{n_w}]\!]_{3^{N_C}}$$

Improved estimators for physical quantities

- We can define an improved estimator for the quark-antiquark correlator:

$$\langle z_x z_y^* \rangle_{N_Q} \equiv \langle\!\langle z_x z_y^* \prod_w z_w^{n_w} \rangle\!\rangle_{N_Q} / \langle\!\langle \prod_{w \in C} z_w^{n_w} \rangle\!\rangle_{N_Q}$$

- ... and the weight for cluster C containing source z_x:

$$[\![\prod_{w \in C} z_w^{n_w + \delta_{w,x}}]\!]_3 = [\![z^{\sum_{w \in C} n_w + \delta_{w,x}}]\!]_3 = \begin{Bmatrix} 1 & \text{if } \sum_{w \in C} n_w = 2 \bmod 3 \\ 0 & \text{else} \end{Bmatrix} = \delta_{n_C,2}$$

- Calculate the subensemble average including $z_x z_y^*$:

$$[\![z_x z_y^* \prod_w z_w^{n_w}]\!]_{3^{N_C}} = \delta_{C_x,C_y} \cdot \prod_C \delta_{n_C,0} + (1 - \delta_{C_x,C_y}) \cdot [\![z_x z_y^* \prod_w z_w^{n_w}]\!]_{3^{N_C}}$$

Improved estimators for physical quantities

- We can define an improved estimator for the quark-antiquark correlator:

$$\langle z_x z_y^* \rangle_{N_Q} \equiv \langle\!\langle z_x z_y^* \prod_w z_w^{n_w} \rangle\!\rangle_{N_Q} / \langle\!\langle \prod_{w \in C} z_w^{n_w} \rangle\!\rangle_{N_Q}$$

- ... and the weight for cluster C containing source z_x:

$$[\![\prod_{w \in C} z_w^{n_w + \delta_{w,x}}]\!]_3 = [\![z^{\sum_{w \in C} n_w + \delta_{w,x}}]\!]_3 = \left\{ \begin{array}{ll} 1 & \text{if } \sum_{w \in C} n_w = 2 \bmod 3 \\ 0 & \text{else} \end{array} \right\} = \delta_{n_C,2}$$

- Calculate the subensemble average including $z_x z_y^*$:

$$[\![z_x z_y^* \prod_w z_w^{n_w}]\!]_{3^{N_C}} = \delta_{C_x,C_y} \cdot \prod_C \delta_{n_C,0} + (1 - \delta_{C_x,C_y}) \cdot \delta_{n_{C_x},2} \cdot \delta_{n_{C_y},1} \prod_{\substack{C \neq C_x \\ C \neq C_y}} \delta_{n_C,0}$$

Improved estimators for physical quantities

- Similar expressions for $z_x z_y$, $z_x^* z_y^*$, z_x, $z_x^* = z_x^2$, ...
- An interesting quantity is the the quark chemical potential:

$$\mu(\rho) \equiv -\frac{1}{3} \log \frac{Z_C(N_Q+3)}{Z_C(N_Q)}$$

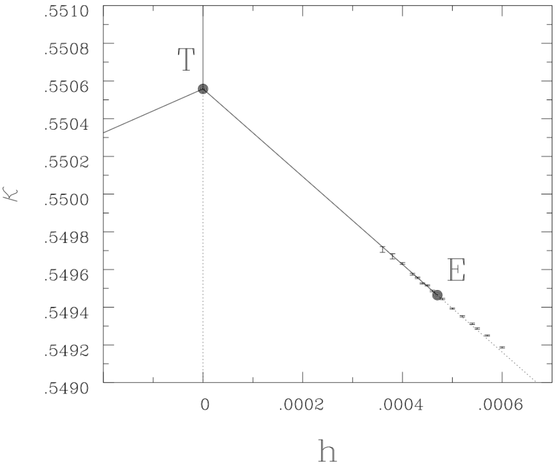
  - quark density $\rho \equiv (N_Q + {}^3\!/\!_2)/V$

- The expectation value of the phase of $Z(N_Q)$:

$$\ln\langle \exp(i\phi)\rangle_{|\cdot|,N_Q} = \frac{\langle\!\langle \prod_x z_x^{n_x} \rangle\!\rangle_{N_Q}}{\langle\!\langle 1 \rangle\!\rangle_{N_Q}} = -3 \sum_{k=0}^{N_Q/3-1} \mu(\frac{3}{2}+3k) - \ln \mathcal{P}(N_Q, V)$$

Physics of the 3-state Potts model

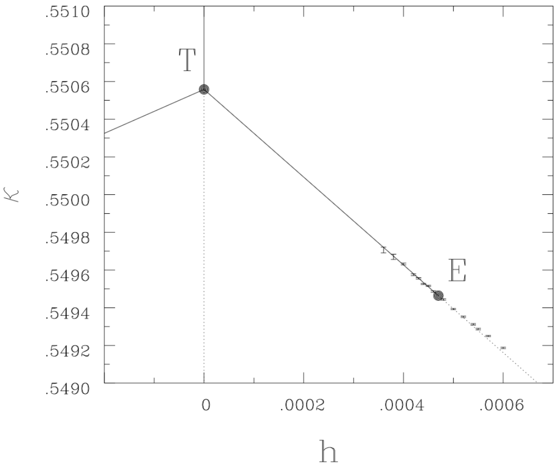
- Phase diagram in the $(e^{\mu}, \gamma) \equiv (h, \kappa)$-plane:
  [Alford, Chandrasekharan, Cox and Wiese 2001]



- deconfinement phase transition at $T = (0, 0.550565(10))$

Physics of the 3-state Potts model

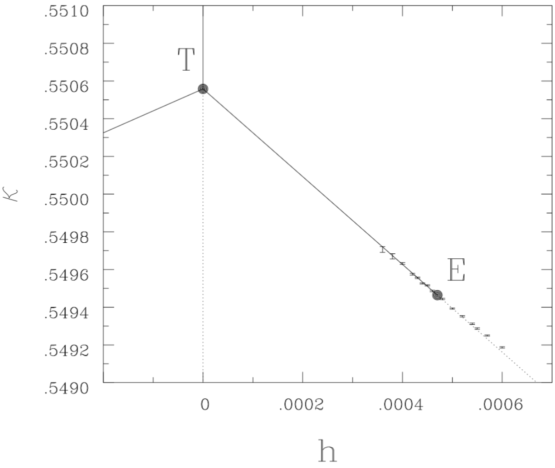
- Phase diagram in the $(e^{\mu}, \gamma) \equiv (h, \kappa)$-plane:
  [Alford, Chandrasekharan, Cox and Wiese 2001]



  - line of first order phase transitions from T to E

Physics of the 3-state Potts model

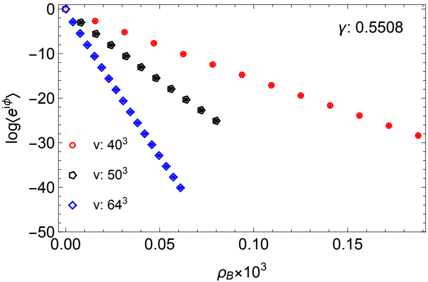
- Phase diagram in the $(e^{\mu}, \gamma) \equiv (h, \kappa)$-plane:
  [Alford, Chandrasekharan, Cox and Wiese 2001]



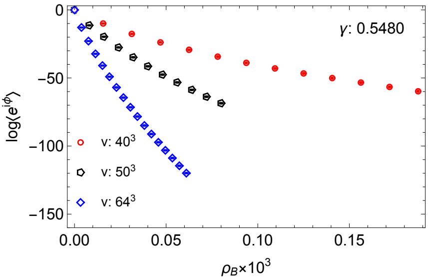
  - critical endpoint $E = (0.000470(2), 0.549463(13))$

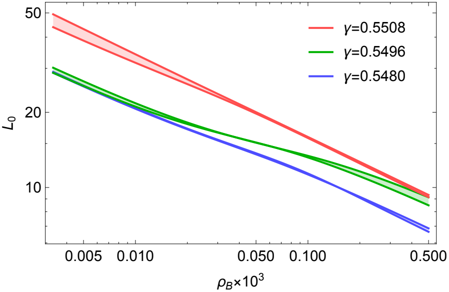
Severity of the sign problem

Deconfined phase:



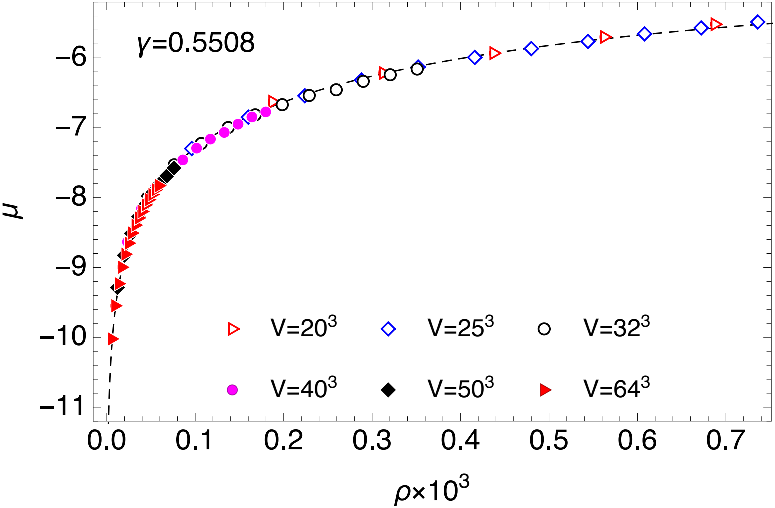
Confined phase:





Physics of the 3-state Potts model

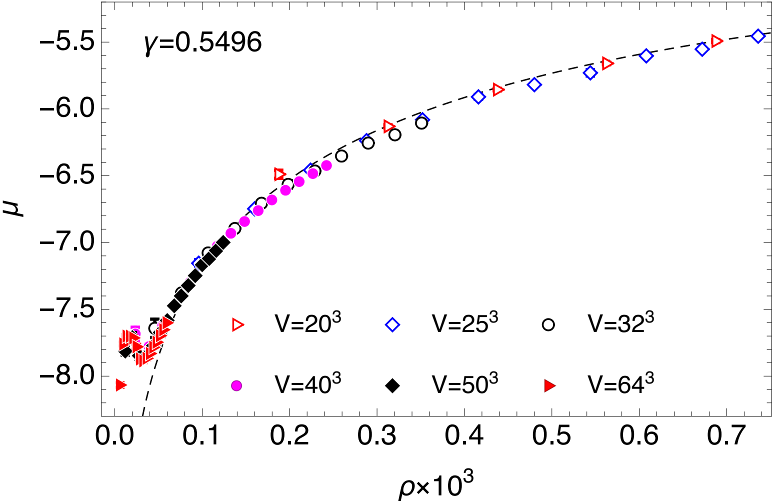
- Canonical simulation results in the deconfined phase:



  - description in terms of a gas of (free) quarks

Physics of the 3-state Potts model

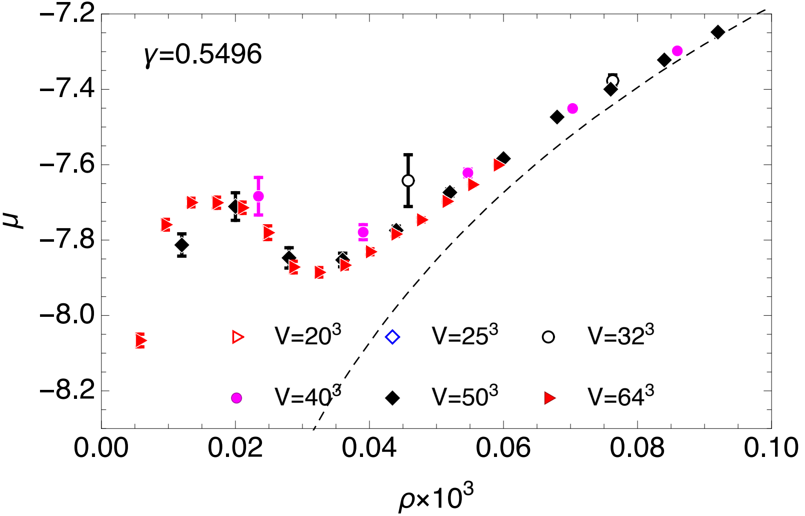
- Results from below the deconfinement transition:



- transition from the confined into the deconfined phase

Physics of the 3-state Potts model

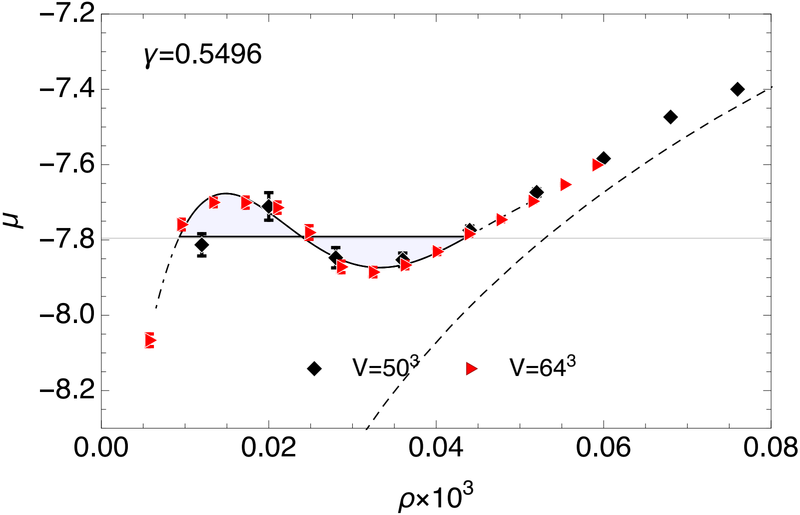
- Results from below the deconfinement transition:



  - typical signature of a 1st order phase transition

Physics of the 3-state Potts model

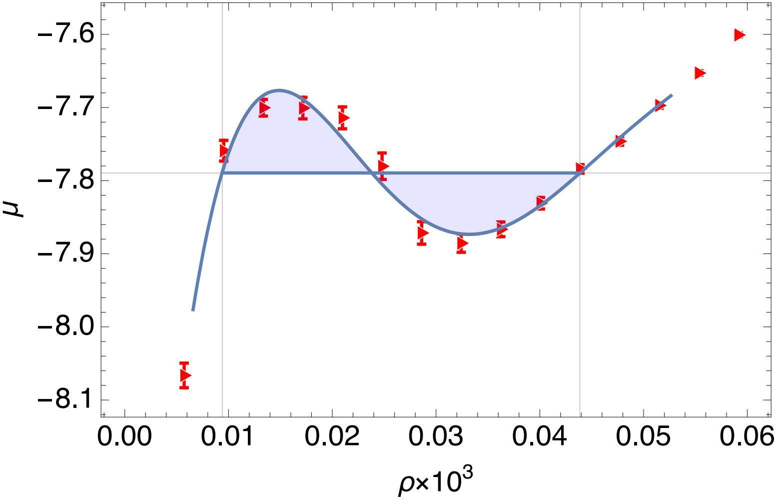
- Results from below the deconfinement transition:



- Maxwell construction yields critical μ_c

Physics of the 3-state Potts model

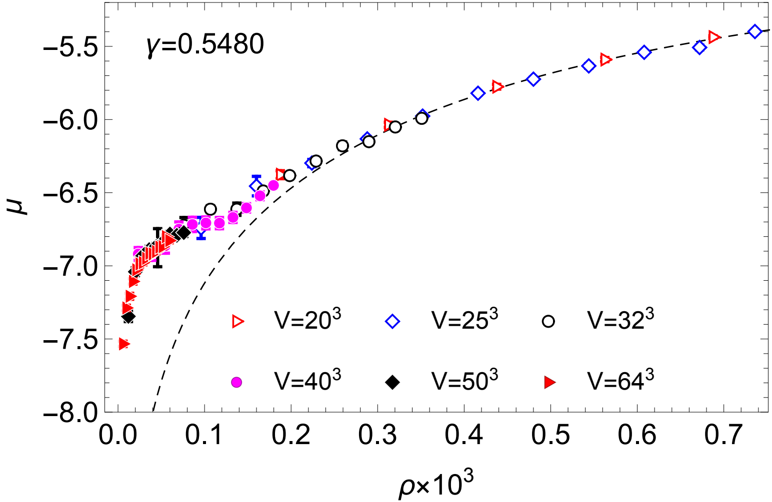
- Results from below the deconfinement transition:



  - Maxwell construction yields critical μ_c

Physics of the 3-state Potts model

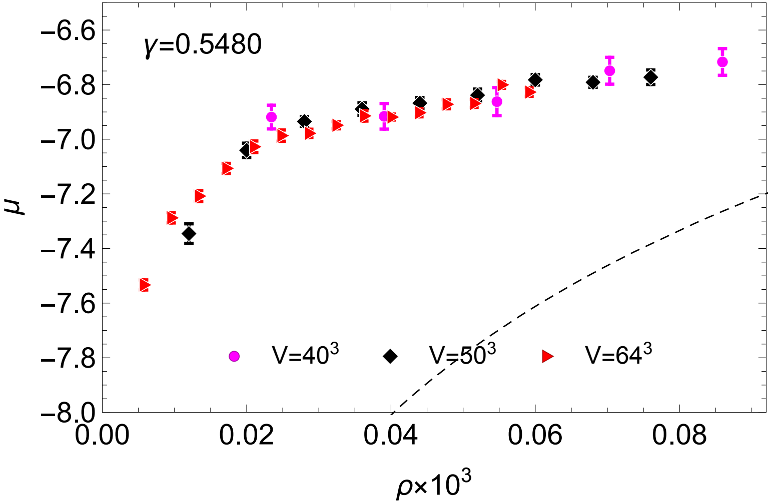
- Results from below the critical endpoint:



  - crossover from the confined into the deconfined phase

Physics of the 3-state Potts model

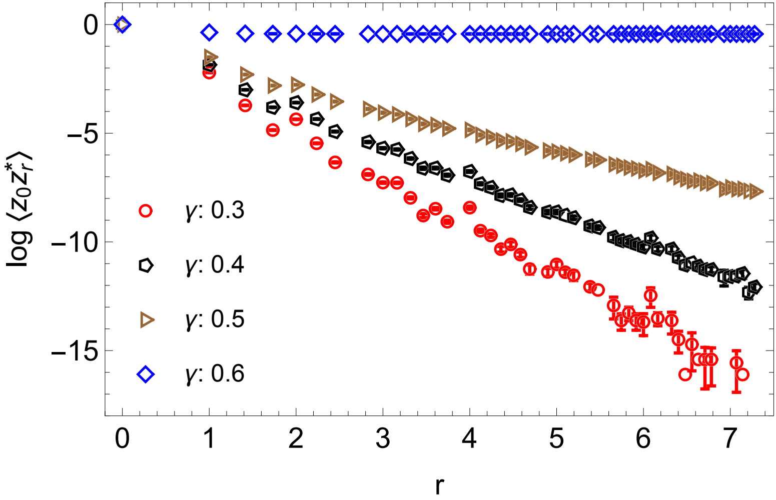
- Results from below the critical endpoint:



- crossover from the confined into the deconfined phase

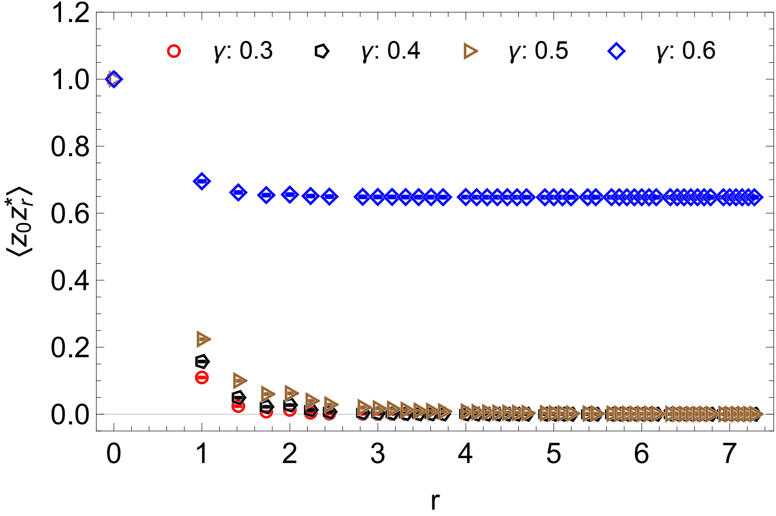
Physics of the 3-state Potts model

- (Anti)Quark-(anti)quark potentials at zero density:



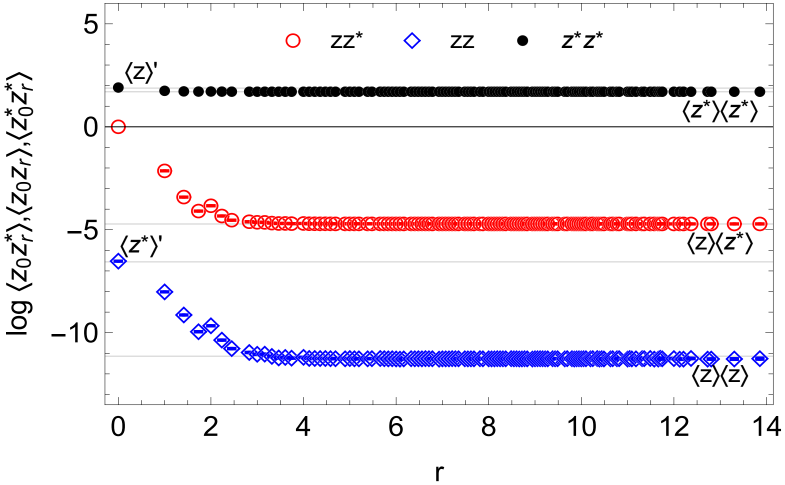
Physics of the 3-state Potts model

- (Anti)Quark-(anti)quark potentials at zero density:



Physics of the 3-state Potts model

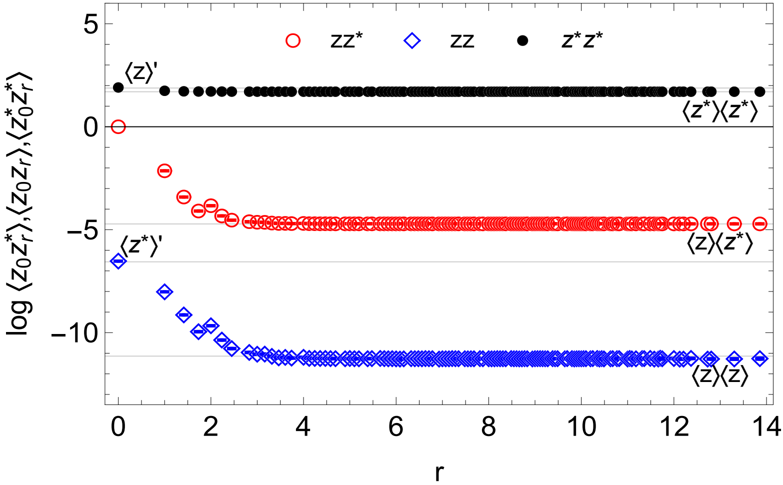
- (Anti)Quark-(anti)quark potentials at low temperature:



- confined phase: $\gamma = 0.3$ for $N_Q = 24$, $V = 16^3$, i.e. $\rho = 5.9 \cdot 10^{-3}$

Physics of the 3-state Potts model

- (Anti)Quark-(anti)quark potentials at low temperature:



- values at $r = 0$ and $r \to \infty$ match $\langle z \rangle$, $\langle z^* \rangle$, $\langle z^* \rangle \langle z^* \rangle, \ldots$

Conclusions

- We solved the fermion sign problem for the $\mathbb{Z}(3)$ Potts model
  - isolate coherent dynamics of the $\mathbb{Z}(3)$ spins in clusters
  - cluster subaverages project on nonzero, positive contributions

    $\Rightarrow$ increase of statistics exponential in N_C

- The solution provides an appealing physical picture:
  - quarks confined in clusters, but move freely within
  - at $\gamma \to 0$ clusters are confined to single sites only
  - deconfinement corresponds to appearance of a percolating cluster

Good algorithms reflect true physics insight!

- Extension to Polyakov loop models could be possible:
  - mechanism at work at $\beta = 0$
  - extend it to $\beta > 0$ $\Rightarrow$ $\mathbb{Z}(N_c)$ clusters for gauge fields