IR Correlation in de Sitter Space: Field Theoretic vs. Stochastic Approaches

In collaboration with Björn Garbrecht & Gerasimos Rigopoulos.
Based on PRD 89, 063506 (2014) [arXiv:1310.0367]
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Kosmologietag Bielefeld

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Outline

De Sitter space

Three types of maximally symmetric spacetime

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$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

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$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] = a^2(t)[-d\eta^2 + dx^2 + dy^2 + dz^2] \\ &= -dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 \end{aligned}$$

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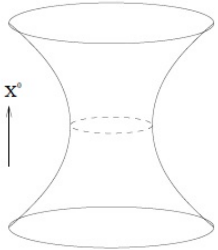
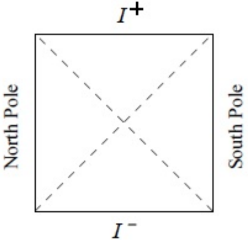
[M. Spradlin, A. Strominger & A. Volovich, arXiv:hep-th/0110007]



Hyperboloid illustrating de Sitter Space



Penrose diagram for de Sitter

Infrared divergence of massless scalar field

Quantum Field Theory

Infrared (IR) divergence ($m = 0$)   Ultraviolet (UV) divergence

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In de Sitter space, soft modes accumulate on superhorizon scales formally resulting in an IR divergence.

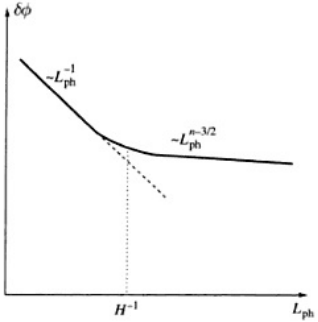

Figure: The fluctuation amplitude $\delta\phi$ as function of physical scale L_{ph} at fix time η.

[V. Mukhanov & S. Winitzki, Introduction to Quantum Effects in Gravity, 2007]

IR divergence and self-regulation

For $m^2 \ll H^2$ (and $\lambda \ll m^4/H^4$), the free propagator for a minimally coupled scalar field with $\lambda\phi^4$ self-interaction can be expanded as

$$i\Delta_0^{fg}(x;x') = \frac{H^2}{4\pi^2}\{-\frac{1}{y^{fg}} - \frac{1}{2}\log(-y^{fg}) + \frac{3H^2}{2m^2} + 1 - \log 2 + O(\frac{m^2}{H^2})\},$$

where y is the de Sitter invariant distance between x and x', and $\{f,g\} = \{+,-\}$ are the closed time path (CTP) indices.

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Self Regulatory: m_{dyn} is generated by loop effects

Quantum Field Theory		Statistical Physics
Dynamical S-D Eqs.[1]	Functional RG[2]	Stochastic process[3]

$$m_{dyn}^2 \propto \sqrt{\lambda}H^2, \quad \text{and} \quad i\Delta \propto \frac{H^2}{\sqrt{\lambda}}$$

[1] B. Garbrecht & G. Rigopoulos, arXiv:1105.0418; A. Riotto & M. S. Sloth, arXiv:0801.1845.
[2] C. P. Burgess, et al., arXiv:0912.1608.
[3] A. A. Starobinsky & J. Yokoyama, arXiv:astro-ph/9407016.

Stochastic Method

Stochastic process

One can divide a scalar field ϕ into long wavelength part $\bar{\phi}$ and short wavelength part respect to a IR cutoff $\epsilon a(t)H$, as following

$$\phi(\mathbf{x},t) = \bar{\phi}(\mathbf{x},t) + \int \frac{d^3k}{(2\pi)^{3/2}} \theta(|k| - \epsilon a(t)H) \left[a_{\mathbf{k}} \phi_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^{\dagger} \phi_{\mathbf{k}}{}^{*}(t) e^{i\mathbf{k}\cdot\mathbf{x}} \right].$$

Here ϵ is a small constant parameter, and $\phi_{\mathbf{k}}(t)$ are free mode functions for ϕ.

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From equation$\Box\phi + \frac{\partial V}{\partial \phi} = 0$, we obtain a Langevin equation for the long wavelength part $\bar{\phi}$

$$\dot{\bar{\phi}}(x) + \frac{1}{3H} \frac{\partial V(\phi)}{\partial \phi} = \xi(x),$$

where $\xi(x)$ is a Gaussian random force with

$$\langle \xi(t) \xi(t') \rangle = \frac{H^3}{4\pi^2} \delta(t - t').$$

Diagrammatic stochastic approach

The expectation value of an operator $\mathcal{O}[\phi]$ is given by

[G. Rigopoulos, arXiv:1305.0229]

$$\langle \mathcal{O}[\phi] \rangle = \int D[\phi,\psi]\,\mathcal{O}[\phi]\,\mathrm{e}^{-\mathrm{i}\int dt\left[\frac{1}{2}(\,\phi\,,\,\psi\,)\begin{pmatrix} 0 & -\partial_t+\frac{m^2}{3H} \\ \partial_t+\frac{m^2}{3H} & -\mathrm{i}\frac{H^3}{4\pi^2}\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}+\frac{\lambda}{3!}\frac{\psi\phi^3}{3H}\right]}.$$

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If the stochastic process is taken to have begun early enough (t and t' are sufficiently large), the correlation functions have the form as following

$$G^R(t,t') = G^A(t',t) = \mathrm{e}^{-\frac{m^2}{3H}(t-t')}\Theta(t-t')\,, \quad F(t,t') \simeq \frac{3H^4}{8\pi^2 m^2}\mathrm{e}^{-\frac{m^2}{3H}|t-t'|}$$

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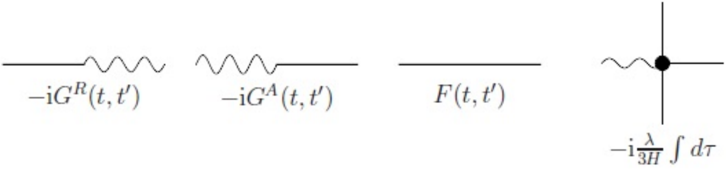

Figure: The elements out of which stochastic diagrams are constructed.

Diagrammatic stochastic approach

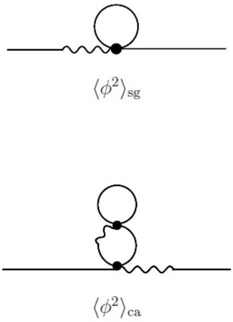

Figure: The stochastic diagrams contributing to $\langle\phi(t)\phi(t')\rangle$ up to order λ^2.

$$\lim_{t\to\infty}\langle\phi(t)^2\rangle = \langle\phi^2\rangle_{\rm sg} + \langle\phi^2\rangle_{\rm ca} + \langle\phi^2\rangle_{\rm sgsg} + \langle\phi^2\rangle_{\rm ss}$$

$$= \frac{3H^4}{8\pi^2m^2} - \lambda\frac{9H^8}{128\pi^4m^6} + \lambda^2\frac{9H^{12}}{256\pi^6m^{10}}\,.$$

Probability distribution function

A different way to obtain this result:

Langvin equation → Fokker-Planck equation → Probability distribution

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At late time limit, the probability distribution function reads

$$\varrho(\phi) = \mathcal{N} \mathrm{e}^{-\frac{8\pi^2}{3H^4} V(\phi)} ,$$

where the normalisation $\mathcal{N}$ is determined by the condition

$$\int_{-\infty}^{\infty} d\phi \varrho(\phi) = 1 .$$

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Expectation value (at equal times) is given by

$$\langle \phi^2 \rangle = \int\limits_{-\infty}^{\infty} d\phi \phi^2 \varrho(\phi) = \langle \phi^2 \rangle = \frac{3H^4}{8\pi^2 m^2} - \frac{9\lambda H^8}{128\pi^4 m^6} + \frac{9\lambda^2 H^{12}}{256\pi^6 m^{10}} + \cdots \,.$$

Field Theoretical Approach

Closed time path formalism

In-out formalism: Designed to address scattering problems.

- Free particles coming in from $t = -\infty$,
- Interactions adiabatically switch on and off,
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- We are not interested here in the calculation of S-matrix elements, but rather in evaluating expectation values of products of fields at a fixed time.
- Conditions are not imposed on the fields at both very early and very late times, but only at very early times.
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In-in formalism (Closed time path (CTP) formalism, Schwinger-Keldysh formalism): a generalisation of equilibrium field theory.

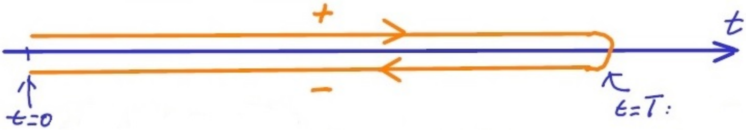

Figure: Closed time path

Closed time path formalism

The basic building block of the Feynman diagrams is the free propagators. They are

$$i\Delta^{T} = i\Delta^{++}(t;t') = \langle \mathrm{T}\phi(t)\phi(t')\rangle, \quad i\Delta^{>} = i\Delta^{+-}(t;t') = \langle \phi(t')\phi(t)\rangle$$

$$i\Delta^{<} = i\Delta^{-+}(t;t') = \langle \phi(t)\phi(t')\rangle, \quad i\Delta^{\overline{T}} = i\Delta^{--}(t;t') = \langle \overline{\mathrm{T}}\phi(t)\phi(t')\rangle$$

which satisfy the Klein-Gordon equation

$$a^4\left(\nabla_x^2 - m^2\right)\mathrm{i}\Delta^{(0)fg}(x;x') = fg\delta^{fg}\,\mathrm{i}\delta^4(x-x')\,,$$

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For $m^2 \ll H^2$ (and $\lambda \ll m^4/H^4$), sizeable IR fluctuations in the field ϕ occur due to the expansion of the Universe. In this limit, we can use the approximation

$$\mathrm{i}\Delta^{(0)fg}(y) = \frac{H^2}{4\pi^2}\left[-\frac{1}{y^{fg}} + \frac{3H^2}{2m^2}\left(-\frac{1}{y^{fg}}\right)^{\frac{1}{3}\frac{m^2}{H^2}} + \mathcal{O}\left(y^{-2}\frac{m^2}{H^2}\right)\right],$$

Schwinger-Dyson equation

While the basic propagators all contain IR-enhanced contributions, within the causal propagators, *i.e.* the retarded and advanced ones

$$\mathrm{i}\Delta^{(0)R,A}(x;x') = \mathrm{i}\Delta^{(0)T}(x;x') - \mathrm{i}\Delta^{(0)<,>}(x;x')\,,$$

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The Schwinger-Dyson equations on the CTP are

$$a^4\left(\nabla_x^2 - m^2\right)\mathrm{i}\Delta^{fg}(x;x') = \delta^{fg}\mathrm{i}\delta^4(x-x') - \mathrm{i}\int d^4w\,\mathrm{i}\Pi^{fh}(x;w)\mathrm{i}\Delta^{hg}(w;x')\,,$$

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Figure: Diagrammatic representation of the Schwinger-Dyson equations that are perturbatively truncated at order λ^2.

Schwinger-Dyson equation

we can take various linear combinations of the Schwinger-Dyson equations. A particularly useful one is

$$
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a^4(-\nabla_x^2 - m^2)\mathrm{i}\Delta^{<,>}(x;x') =& - \mathrm{i}\int d^4w\, \mathrm{i}\Pi^R(x;w)\mathrm{i}\Delta^{<,>}(w,x') \\
& - \mathrm{i}\int d^4w\, \mathrm{i}\Pi^{<,>}(x;w)\mathrm{i}\Delta^A(w,x')\,, \\
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which are known as the Kadanoff-Baym equations.
We can make the ansatz

$$\mathrm{i}\Delta(y) = -\frac{H^2}{4\pi^2 y} + A\log y + C + \cdots\,.$$

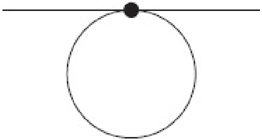
This ansatz is also an approximate solution to the free Klein-Gordon equation when

$$\mathrm{i}\Delta^{(0)} \to \mathrm{i}\Delta, \quad m^2 \to m^2_{\rm dyn} = \frac{3H^4}{8\pi^2 C}$$

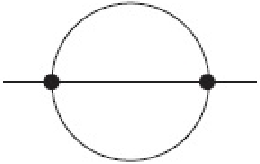
This is why it qualifies as a 'dynamical mass' ansatz

IR fluctuations up to λ^2 order

Up to λ^2 order, the self-energy is contributed by two diagrams as following,



Seagull diagram



Sunset diagram

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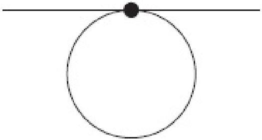

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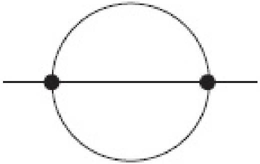

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Then, we obtain

$$C = \langle \phi^2 \rangle = \frac{3H^4}{8\pi^2 m^2} - \frac{9\lambda H^8}{128\pi^4 m^6} + \frac{9\lambda^2 H^{12}}{256\pi^6 m^{10}} + \cdots ,$$

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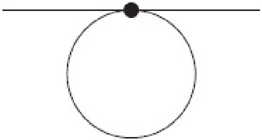

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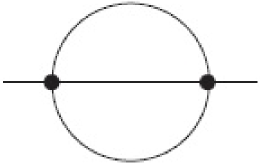

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Note that we may also infer a dynamical mass through

$$m_{\rm dyn}^2 = \frac{3H^4}{8\pi^2 C} = m^2 + \frac{3\lambda H^4}{16\pi^2 m^2} - \frac{15\lambda^2 H^8}{256\pi^4 m^6} + \mathcal{O}(\lambda^3) .$$

Conclusions

- We developed a diagrammatic method for the stochastic approach.
- Up to λ^2 order, the scalar field fluctuations obtained by QFT methods and the fact that these agree with the corresponding quantities derived by the stochastic method.

Thank you!