Random Matrix Theory

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- **What is RMT about?**

  Nuclear Physics, Number Theory, Quantum Chaos, . . .
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  describes spectrum of **QCD** Dirac-operator in precise limit ($N_c = 3$):
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- **What can we compute in RMT?**

  - analytic form of spectral density, distribution of smallest eigenvalue
  - as functions of: $\Sigma = |\langle \bar{q} q \rangle|$, F (if we include μ_{iso}), χSB pattern, gauge field topology = ν zero modes of D (index Theorem)

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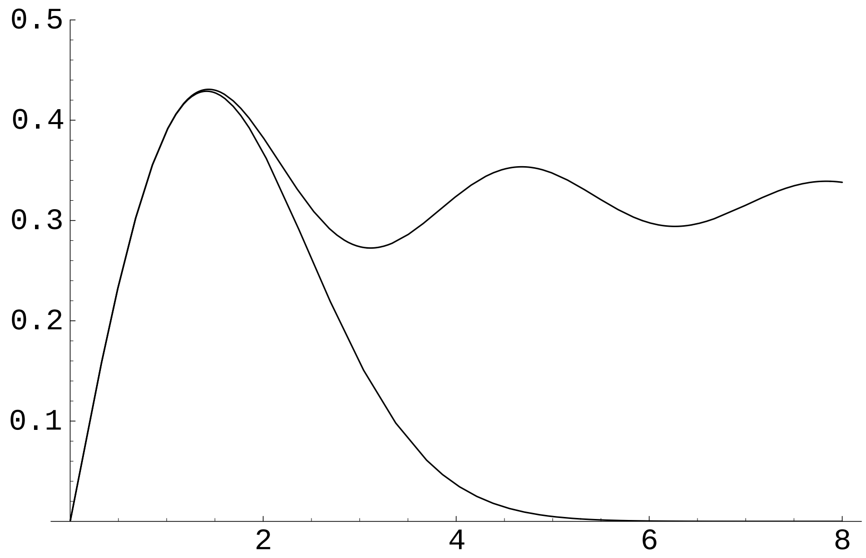
- **What cannot be computed?**

  - actual value of Σ or F: fit density$[\Sigma, F]$ to Lattice data
  - χSB or not: $\beta(N_f)$

Example: Lattice vs spectrum of Dirac eigenvalue density



density $\rho_D(\lambda) \equiv \langle \text{Tr}\delta(D-\lambda)\rangle_{\mathbf{QCD}}$

$\lim \frac{1}{V\Sigma}\rho(x) = \frac{x}{2}[J_0(x^2) + J_1(x^2)]$

1st eigenvalue $p_1(x) = \frac{x}{2}e^{-x^2/4}$

at $\nu = 0$, with $\lambda V\Sigma = x$

spacing $1/V \neq 1/L$ free case

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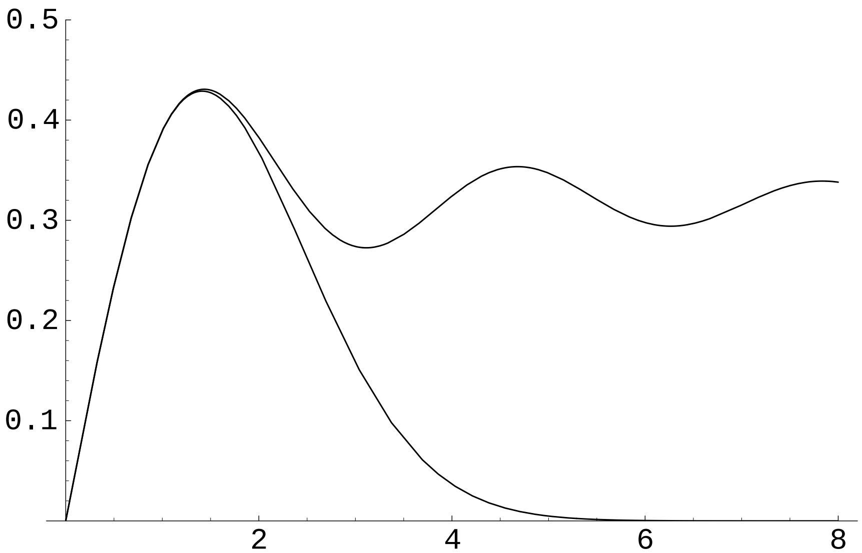

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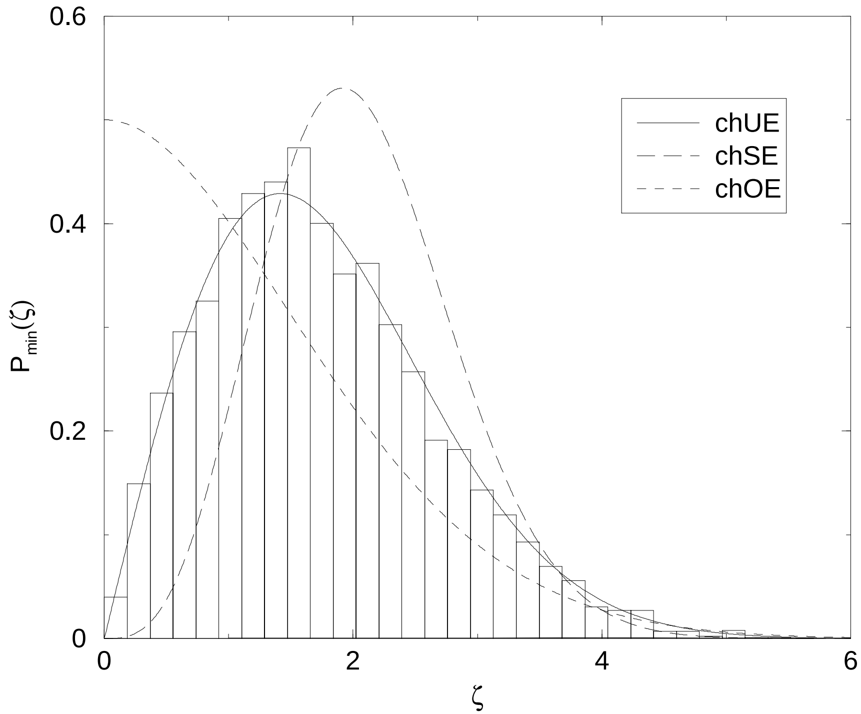

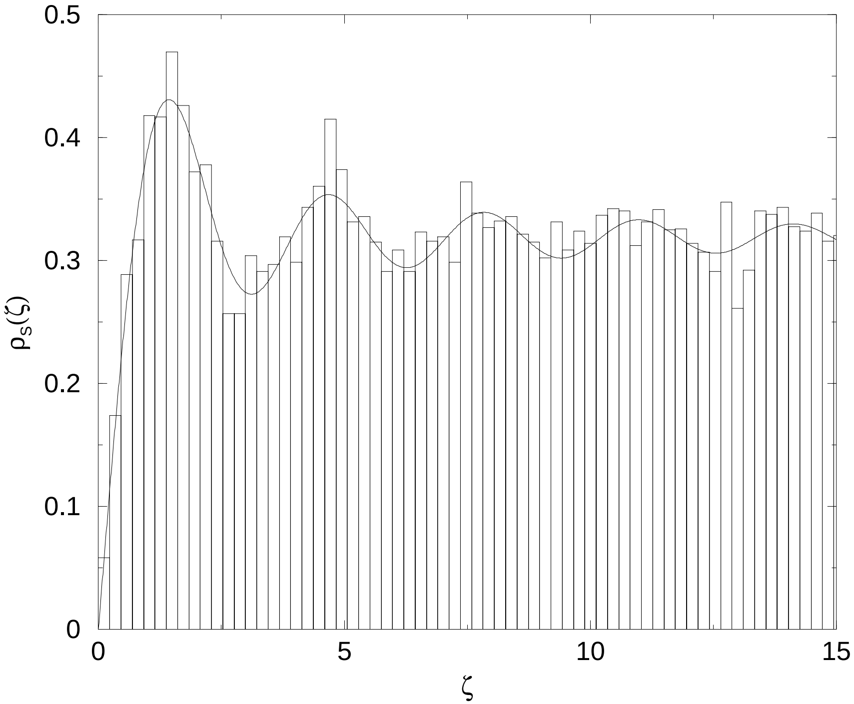

Lattice vs. smallest Dirac eigenvalues

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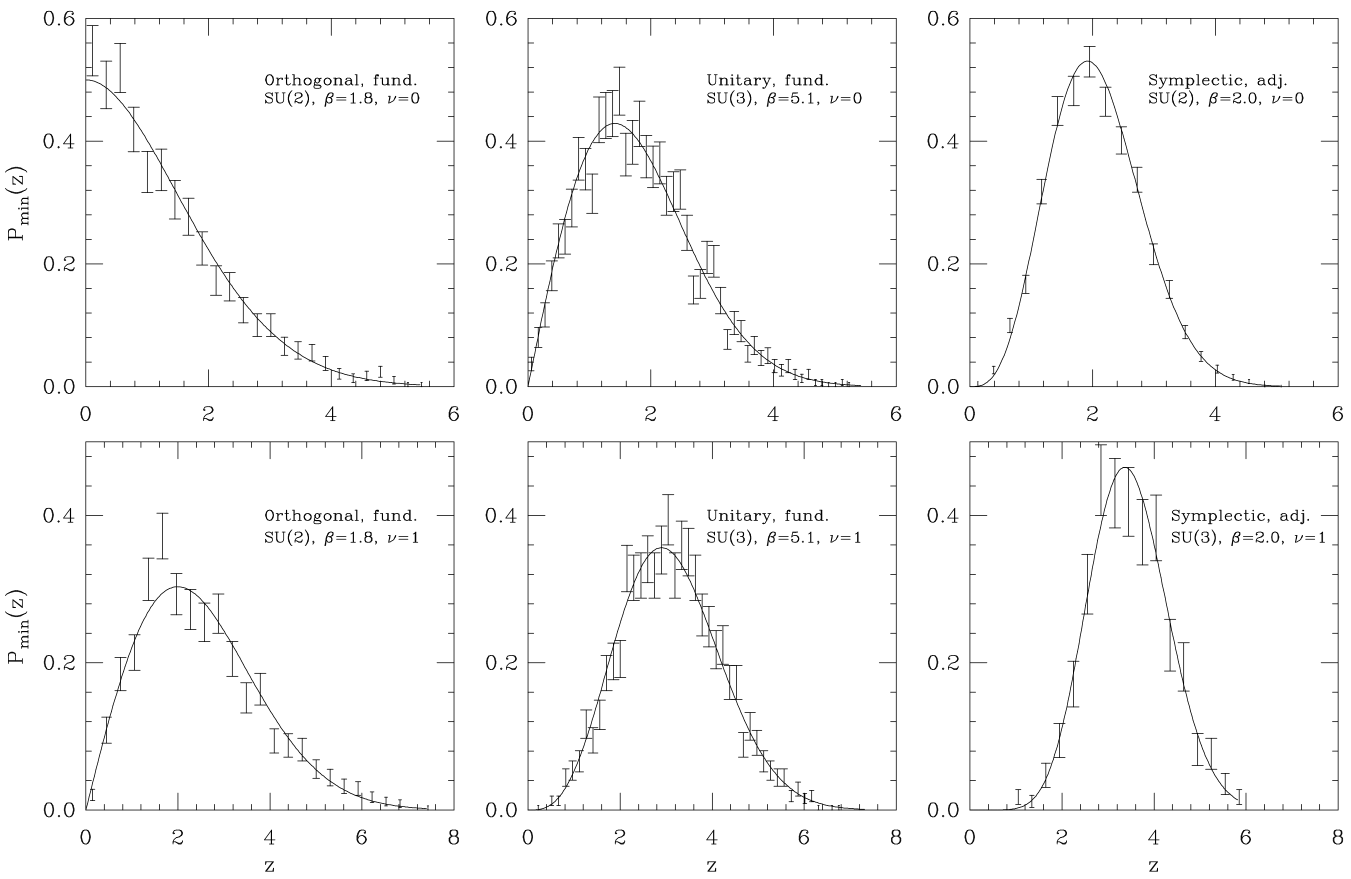
- $\nu = 0$ (top): SU(2) — SU(3) — SU(2) adj.

  & $\nu = 1$ (bottom): different χSB patters



Lecture 1

- the approximation of **QCD** $\longrightarrow \varepsilon\chi$PT
- equivalence $\varepsilon\chi$PT to RMT and limitations
- first results from RMT:

  - flavor-topology duality and

  - smallest eigenvalue distribution

Setup QCD

- full theory (Euclidean) $+\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ topological term

$$\boxed{\mathcal{Z}_{\mathbf{QCD}} \;\equiv\; \int [dA][d\Psi]\;\exp[-\mathrm{Tr}\int \bar{\Psi}(\not{D}+M)\Psi \;+\; FF \;+\; i\Theta F\tilde{F}]}$$

with N_f quarks Ψ of masses M, gauge fields A and field strength F

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- **properties of the Dirac operator:**

  - diagonalise: $\boxed{D\psi_k = i\lambda_k\psi_k}$ in finite V, Euclidean $D^\dagger = -D$

  - index theorem: ν # zero modes $\longleftrightarrow$ winding # of A

  - sum over topology $\mathcal{Z}_{\mathbf{QCD}} \equiv \sum_\nu \mathcal{Z}_\nu \exp[i\Theta\nu]$

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- formally

$$\boxed{\mathcal{Z}_{\mathbf{QCD}} = \sum_\nu \int_0^\infty [d\lambda]\prod_{f=1}^{N_f} m_f^\nu e^{i\nu\Theta/N_f}\prod_k(\lambda_k^2+m_f^2)\;\exp[-\mathrm{Tr}\int F^2[\lambda]]}$$

Start: chiral perturbation theory χPT for QCD

Integration of all non-Goldstone modes:

- in a box $V = L^4$: valid for momenta $1/L \ll \Lambda$ non-Goldstone scale

$$\mathcal{Z}_{\chi PT} \equiv \int_{SU(N_f)} [dU(x)] \ \exp[-\int dx \text{Tr}\ \mathcal{L}(U, \partial U)]$$

- expansion in higher orders $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots$

$$\mathcal{L}_2 = \frac{1}{4} F^2 \partial_\alpha U(x) \partial_\alpha U^\dagger(x) + \frac{1}{2} \Sigma M \left(\mathrm{e}^{i \frac{\Theta}{N_f}} U(x) + \mathrm{e}^{-i \frac{\Theta}{N_f}} U^\dagger(x) \right)$$

- LECs: **Pion decay constant** F & **chiral condensate** Σ

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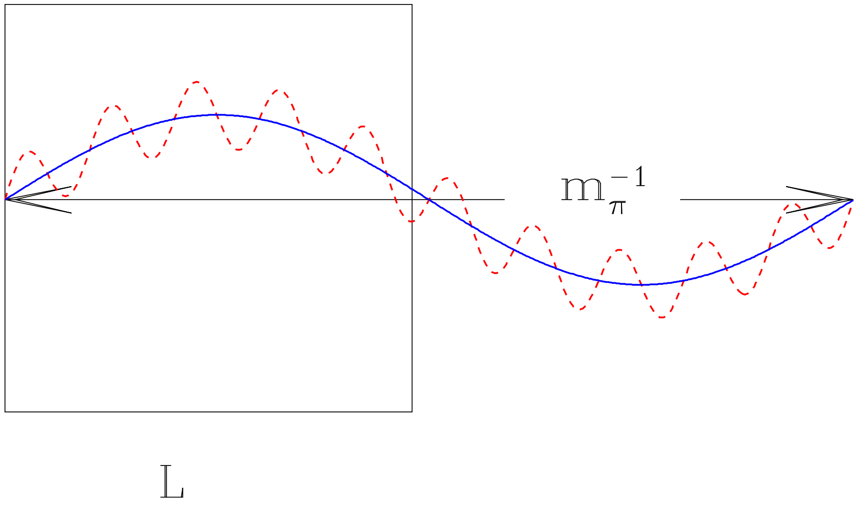
- Fourier trafo: fix topology $\int d\Theta e^{i\nu\Theta} \int_{SU(N_f)} dU = \int_{U(N_f)} dU \det[U]^\nu$

The ε-regime

- ε counting [Gasser, Leutwyler 87]

$$\boxed{m_\pi \sim \frac{1}{L^2} \ll \frac{1}{L} = \varepsilon = V^{-4}} \Rightarrow \quad (m_\pi^2 V)^{-1} = \mathcal{O}(1)$$



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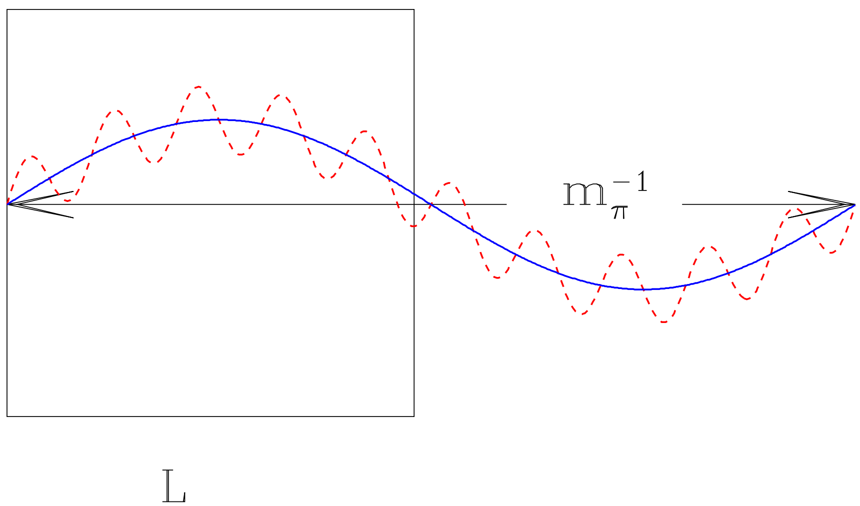

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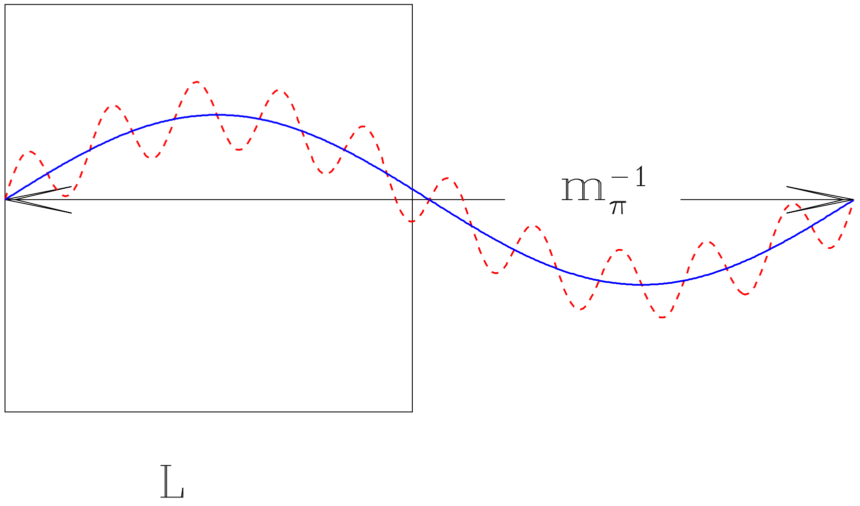

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- standard p regime counting:

$$\boxed{\partial \sim m_\pi \sim 1/L}$$

Partition function of $\varepsilon\chi$PT- zero mode dominance

- fluctuations from integrating quadratic term $\pi_k(p^2 + m_\pi^2)\pi_k$:

  $\boxed{\mathcal{O}\left(\frac{1}{V(p^2+m_\pi^2)}\right)}$ small for $p \neq 0$ NOT for $p = 0$

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- group integral part **equivalent to RMT** to LO: $\mathcal{O}(\varepsilon^0)$

Limitations of RMT and corrections from $\varepsilon\chi$PT

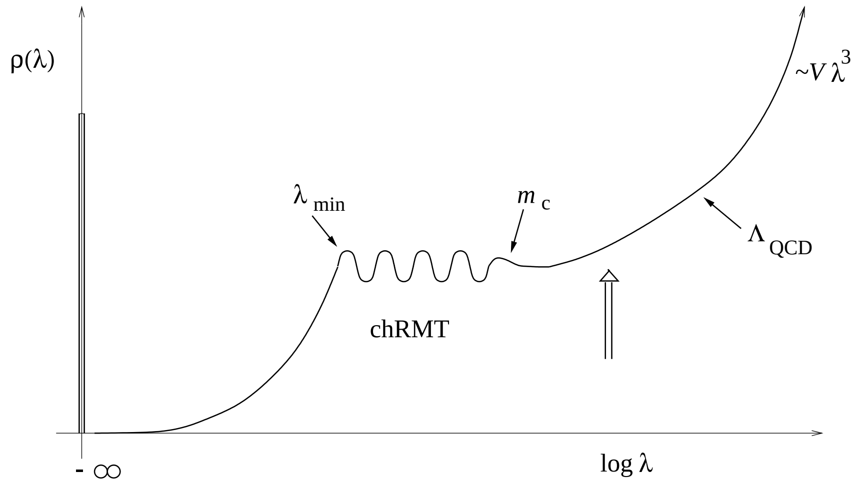

fig: [Verbaarschot]

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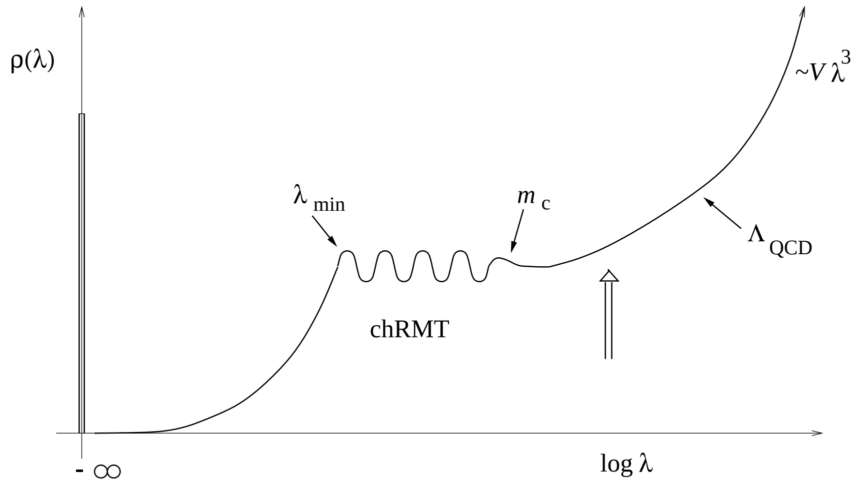

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- **corrections to RMT** regime can be computed: 1-loop $\varepsilon\chi$PT

  in $\mathcal{Z}$ corrections $\Sigma_{eff} = \Sigma\left(1 - \frac{N_f^2-1}{N_f F}\bar{\Delta}(0)\right)$, $F_{eff} = \ldots$ in $1\sqrt{V}$

  [Gasser, Leutwyler 87; G.A., Basile, Lellouch 08]

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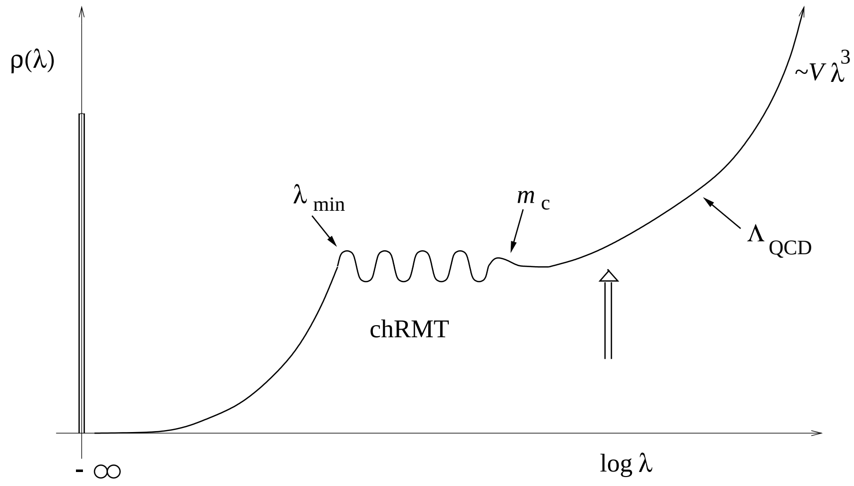

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  $\Rightarrow$ spectral D-correlations from RMT with eff. LEC
- for Green's function $\langle\pi(x)\pi(0)\rangle$: need $\varepsilon\chi$PT [Hansen 90, Damgaard et al. 01]

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- study quark mass dependence of partition function:

  - Taylor expand in m_q formal expression for $\mathcal{Z}_{\mathbf{QCD}}$ in terms of λ_k

$$\mathcal{Z}_\nu(\{m_q\}) = \int_0^\infty [d\lambda] \prod_{f=1}^{N_f} m_f^\nu \prod_k (\lambda_k^2 + m_f^2) \; \exp[-\mathrm{Tr} \int F^2[\lambda]]$$

  - compare to group integral

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- Leutwyler-Smilga sum rules:

$$m^\nu \left(1 + m^2 \left\langle \sum_k \frac{1}{\lambda_k^2} \right\rangle_{\mathbf{QCD}} + \ldots \right) = I_\nu(m) \ \Leftrightarrow$$

$$\boxed{\left\langle \sum_k \frac{1}{V\Sigma\lambda_k^2} \right\rangle_{\mathbf{QCD}} = \frac{1}{4(\nu+1)}}$$

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- computation of spectral 1-point and 2-point density possible by adding extra source quarks (fermionic and bosonic) $\rightarrow$ next lecture

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[Shuryak, Verbaarschot 93]

- block matrix has **same global symmetry** as QCD-Dirac operator D:
  $W_{ij} \in \mathbb{C}$: $N \times (N+\nu)$ Gaussian random variables (chGUE)
  $2N+\nu = V$ eigenvalues: N $\pm\lambda_k$, ν **zero eigenvalues**

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- **equivalence** of other objects than $\mathcal{Z}$?

  \- all Dirac eigenvalue correlations [Damgaard et al 01, Basile, G.A. 07]

The Matrix Model of QCD

• in RMT computation of ALL spectral densities and individual eigenvalue distributions possible and much simpler

$$\mathcal{Z}_{\nu,\ RMT} \equiv \int dW\ \Pi_{f=1}^{N_f} \det \begin{bmatrix} m_f & iW \\ iW^\dagger & m_f \end{bmatrix} e^{-N\Sigma^2 \mathrm{Tr} WW^\dagger}$$

[Shuryak, Verbaarschot 93]

- block matrix has **same global symmetry** as **QCD**-Dirac operator D:

  $W_{ij} \in \mathbb{C}$: $N \times (N+\nu)$ Gaussian random variables (chGUE)

  $2N+\nu = V$ eigenvalues: N $\pm\lambda_k$, ν **zero eigenvalues**

- **universal large** $N \sim V$ **(volume) limit:**

  $$\lim_{N\to\infty} \mathcal{Z}_{\nu,RMT} = \mathcal{Z}_{\nu,\varepsilon\chi\mathrm{PT}}$$

- **equivalence** of other objects than $\mathcal{Z}$?

  - all Dirac eigenvalue correlations [Damgaard et al 01, Basile, G.A. 07]

- different $\varepsilon\chi$PT-χSB, classes: $W_{ij} \in \mathbb{R},\ \mathbb{H}$: RMT still solvable

Large-N limit and equivalence to $\varepsilon\chi$PT

- $\Sigma = 1$ for simplicity:

$$\mathcal{Z}_{\nu,RMT} = \int dW \prod_f^{N_f} \det[WW^\dagger + m_f^2] e^{-N \text{Tr}\, WW^\dagger}$$

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$$\mathcal{Z}_{\nu,RMT} \sim \int dQ \det[Q^\dagger]^\nu \det\left[Q^\dagger Q\right]^N e^{-N \mathrm{Tr}\left(Q^\dagger Q \,-\, M(Q+Q^\dagger)\right)}$$

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- *step 3.* **Saddle Point** $N \rightarrow \infty$:
  parametrise $Q = UR$, at SP: $R \sim 1|$, + scale masses Nm_f

$$\mathcal{Z}_{\nu,RMT} \sim \int dU_{U(N_f)} \det[U^\dagger]^\nu e^{N\text{Tr} M(U+U^\dagger)}$$

  0-dimensional σ-model

RMT eigenvalue representation

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- diagonalise $WW^\dagger$ positive definite matrix : eigenvalues $\lambda_k \geq 0$ (or singular values of W: y_k with $y_k^2 = \lambda_k$ Dirac eigenvalues
- $\boxed{\mathcal{Z}_{\nu,RMT} \sim \int_0^\infty \prod_k^N d\lambda_k \lambda_k^\nu e^{-N\Sigma^2 \lambda_k} \prod_f^{N_f} (\lambda_k + m_f^2) \; |\Delta_N(\lambda)|^2}$
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- Vandermonde determinant
  $\Delta_N(\lambda) = \prod_{k>l} (\lambda_k - \lambda_l) = \exp[2 \sum_{k>l} \log[\lambda_k - \lambda_l]]$
  eigenvalues strongly coupled: log-gas, repulsion of eigenvalues

Results from RMT

- exhibits flavour-topology duality [Verbarschot]:

$$\boxed{\lim_{m\to 0} \mathcal{Z}_{\nu}^{(N_f)}/m^{\nu} \sim \mathcal{Z}_{\nu+1}^{(N_f-1)}}$$

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- define spectral density of Dirac eigenvalues $y = \sqrt{\lambda}$: insert δ

  $\rho(y) = \langle \mathrm{Tr}\delta(D - y)\rangle = N\langle\delta(y_1 - y)\rangle \sim \int dy_2 \ldots dy_N \mathcal{P}_{jpdf}(\{y\})$

  - can be computed exactly for finite-N

Exercise 1st eigenvalue

- distribution of the smallest eigenvalue $p_1(y)$:

  from gap probability = probability that all eigenvalues are $> \lambda$:

$$E(\lambda) = \frac{1}{\mathcal{Z}} \int_\lambda^\infty \prod_{k=1}^{N} d\lambda_k \lambda_k^\nu e^{-N\Sigma^2 \lambda_k} \prod_f^{N_f} (\lambda_k + m_f^2) \, |\Delta_N(\lambda)|^2$$

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- **Exercise:**

  check for $\nu = 0$ that the following holds for Dirac eigenvalues

  1) for $N_f = 0$ derive $p_1(x) = x/2 e^{-x^2/4}$ with $x = 2N\Sigma y$

  2) derive $p_1(x)$ for arbitrary $N_f > 0$

Some literature

- H. Leutwyler, A. Smilga Phys Rev **D46** (1992) 5607-5632
- E. Shuryak, J. Verbaarschot, ”Random matrix theory and spectral sum rules for the Dirac operator in QCD” hep-th/9212088 = Nucl.Phys.A560:306-320,1993
- Jacobus Verbaarschot ”The spectrum of the QCD Dirac operator and chiral random matrix theory: the threefold way”, hep-th/9401059 = Phys.Rev.Lett. 72 (1994) 2531-2533
- G.A., P. H. Damgaard, U. Magnea, S. Nishigaki ”Universality of random matrices in the microscopic limit and the Dirac operator spectrum” hep-th/9609174 = Nucl.Phys.B487:721-738,1997

Lecture 2

- spectral statistics from $\varepsilon\chi$PTand from RMT
- inclusion of chemical potential - why χPTis sensitive to baryonic μ
- further extensions: Wilson χPT