Random Matrix Theory

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Lecture 2

- spectral statistics from $\varepsilon\chi$PTand from RMT
- inclusion of chemical potential - why χPTis sensitive to baryonic μ
- further extensions: Wilson χPT

Relating Chiral Symmetry & Dirac Spectrum

- **Banks Casher Relation** $\boxed{\rho_D(\lambda = 0) = \frac{1}{\pi} V\Sigma}$ **macroscopic density**

$$\Sigma(m) \equiv \frac{1}{V}\partial_m \log \mathcal{Z}_{\mathbf{QCD}} = \left\langle \sum_k \frac{2m}{V(\lambda_k^2 + m^2)} \right\rangle = \int d\lambda \rho_D(\lambda) \frac{2m}{(\lambda^2 + m^2)}$$

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- **microscopic scaling:** $\boxed{\lim_{V\to\infty} \frac{1}{\Sigma V} \rho_D(\lambda \Sigma V)}$ **zoom into origin**

Resolvent - spectrum relation

- **Resolvent definition**

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- alternative: Replicas $\lim_{n \to 0} \frac{1}{n} \partial_m \det[D + m]^n$

How to compute densities from $\varepsilon\chi$PT- $\mu = 0$

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- **generating functions** for all k-point densities:

  $U(N_F + k|k)$**–supergroup integral of $\varepsilon\chi$PT**, take derivatives & discontinuities

  vs RMT $\boxed{\rho_k(y_1, \ldots, y_k) \sim \int dy_{k+1} \ldots dy_N \mathcal{P}_{jpdf}}$ (or + source too)

  generating functions agree, so also all individual eigenvalues do!

  [F. Basile, G.A. 07]

  proof: Superbosonisation Theorem also [Littelmann,Sommers,Zirnbauer]

  $$\boxed{\int d\Psi f(\Sigma_k \Psi_k \times \Psi_k) \sim \int_{U(N_F+k|k)} dU_0 \,\text{sdet}[U_0]^N f(U_0)}$$

$\varepsilon\chi$PT - and chemical potential

$$\mathcal{Z}_{\varepsilon\chi PT} = \int dU_0 \det[U_0]^\nu \exp\left[-\frac{1}{4}\mu^2 F^2 V \text{Tr}[U_0, B][U_0^\dagger, B] - \frac{1}{2}\Sigma V \text{Tr} M(U_0 + U_0^\dagger)\right]$$

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- μ_B has $B_0 =$id: $\varepsilon\chi$PTis independent of μ (BUT: density ρ_D is NOT!)

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- **example** $N_f = 2$ **& isospin** $B = \sigma_3$ (F back!)

  $\mathcal{Z}_\nu = \int_0^1 dt t \exp[-2t^2\mu^2F^2V] I_\nu(V\Sigma m_1 t) I_\nu(V\Sigma m_2 t)$

  check $\mu \to 0$: $[I_\nu(\hat{m}_1)\hat{m}_2 I_{\nu+1}(\hat{m}_2) - I_\nu(\hat{m}_2)\hat{m}_1 I_{\nu+1}(\hat{m}_1)]/(m_1^2 - m_2^2)$

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- $\mu \to i\mu$ **trivial for $\mathcal{Z}$, but NOT for spectrum of $\not{D} + \gamma_0\mu$!**

Why is the spectral density derived from $\varepsilon\chi$PT μ-dependent?

- density resolvent relation for $\mu \neq 0$: $\rho(\lambda) \sim \partial_\lambda^* G(\lambda)$
- other rep. of 2-dim delta:

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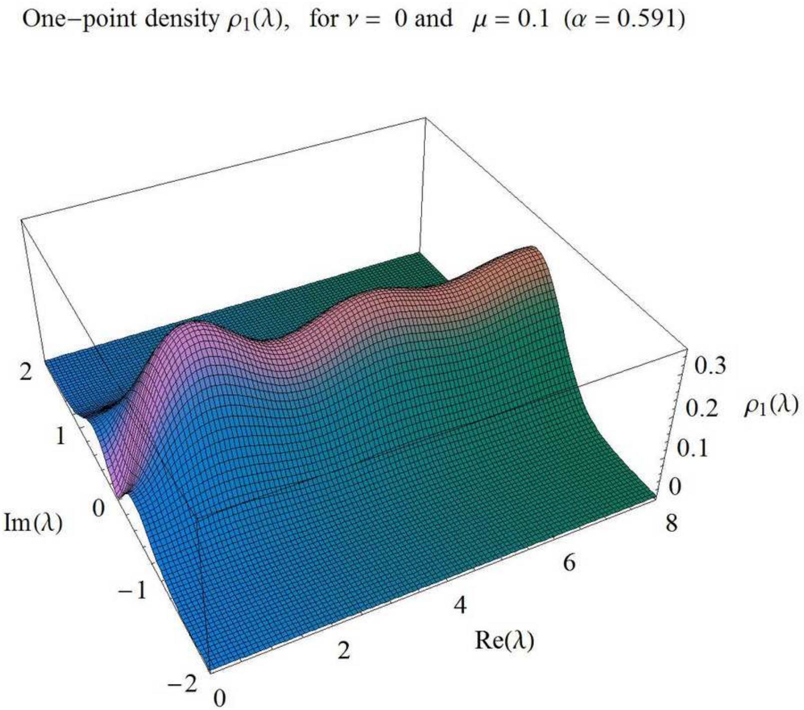
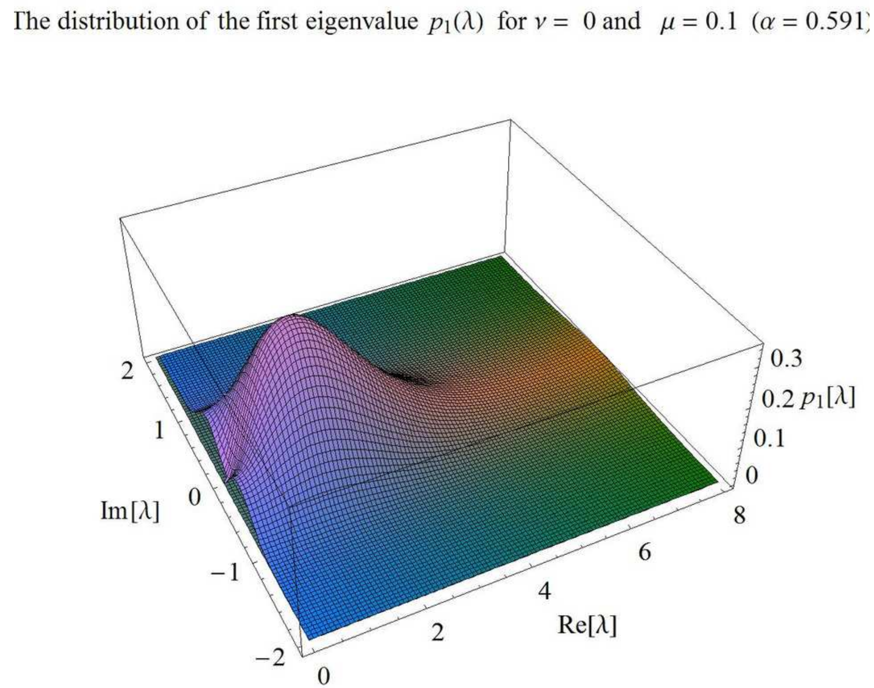
- **example quenched density:** identify $N\mu^2 = VF^2\mu^2_{Lat} \equiv \alpha^2$

$$\boxed{\rho(\xi) = \frac{1}{\alpha^2}|\xi|^2 K_\nu\left(\frac{|\xi|^2}{2\alpha^2}\right) \mathrm{e}^{+\frac{1}{4\alpha^2}(\xi^2+\xi^{*\,2})} \int_0^1 dt\, t\, \mathrm{e}^{-t^2\alpha^2} J_\nu(\xi) J_\nu(t\xi^*)}$$

Quenched density in the complex plane





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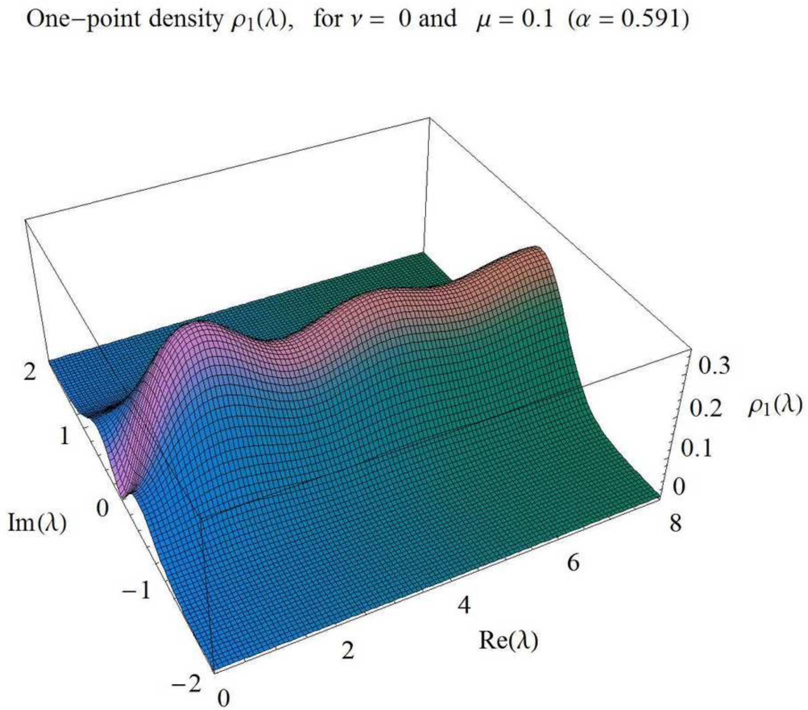

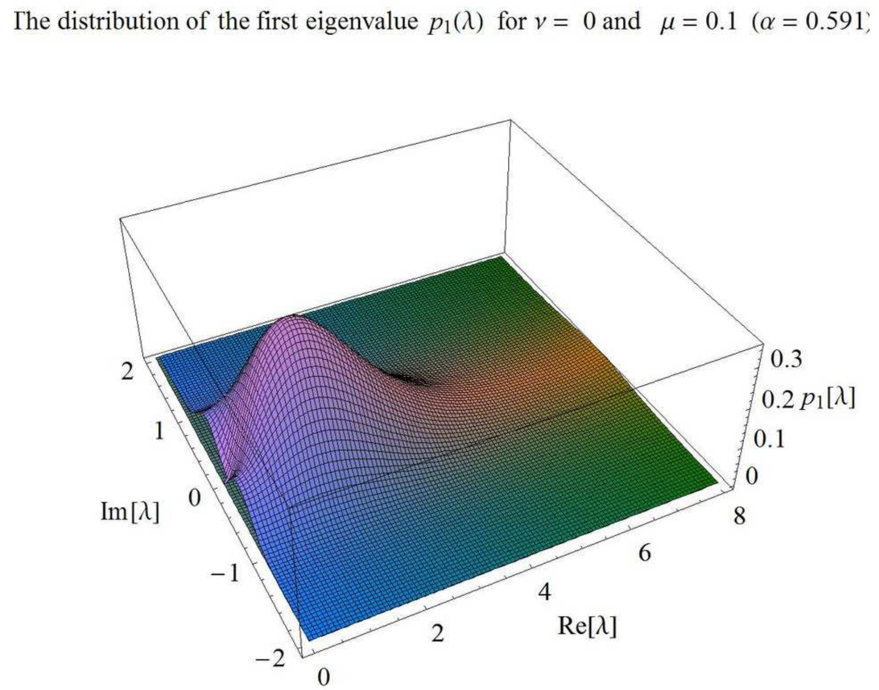

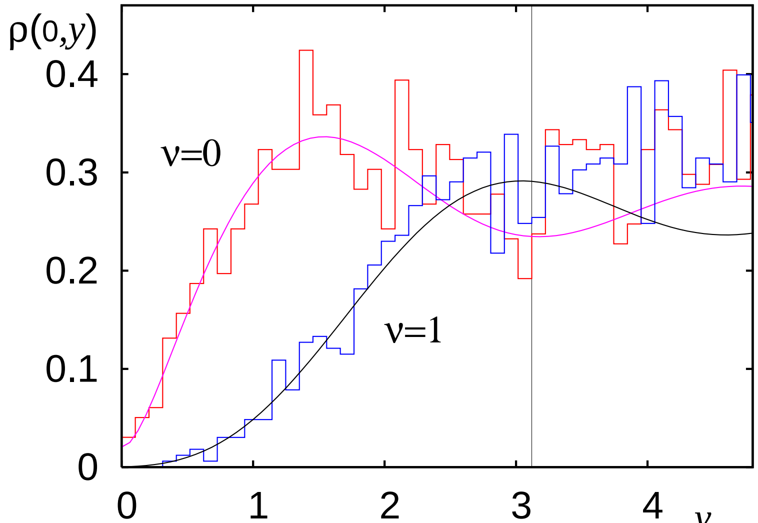

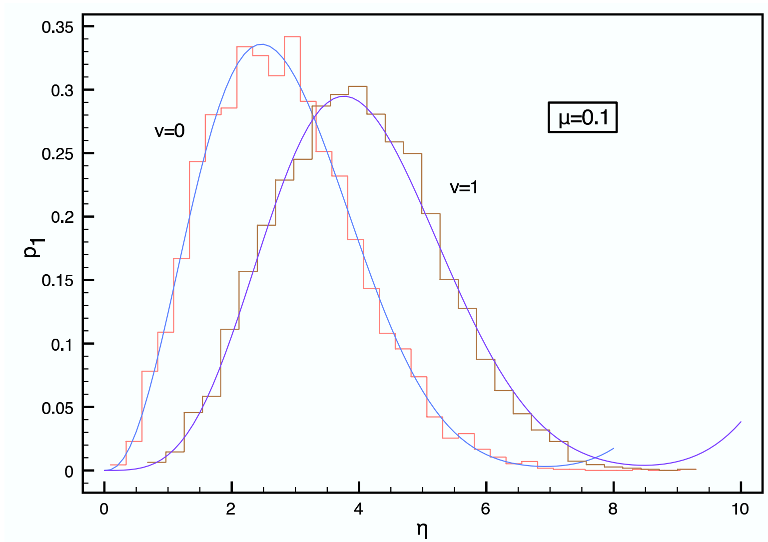

- density (left) [Bloch, Wettig 06], "smallest" eigenvalue (right) [+ G.A., Shifrin 07]

Solution of RMT - with or without μ

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$$\boxed{\Delta_N(\lambda) = \prod_{k>l} |\lambda_k - \lambda_j| = \begin{vmatrix} 1 & \lambda_1 & \dots \\ 1 & \lambda_2 & \dots \\ \dots & & \end{vmatrix} = \begin{vmatrix} P_0(\lambda_1) & P_1(\lambda_1) & \dots \\ P_0(\lambda_2) & P_1(\lambda_2) & \dots \\ \dots & & \end{vmatrix}}$$

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  - for $\mu = 0$ weight $w(\lambda) = \lambda^\nu e^{-\lambda}$ Laguerre polynomials
  - for $\mu \neq 0$ Laguerre polynomials on $\mathbb{C}$ (weight exp $\cdot$ K-Bessel)

  e.g.[Bender, G.A. 10]

- **the partition function**

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- **microscopic limit**

  large-N Laguerre : $L_{N+j}^{(\nu)}(-m_f^2) \rightarrow I_\nu(\hat{m}_f)$ gives again $\mathcal{Z}_{\varepsilon\chi\text{PT}}$

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- **gap probabilities** $E(\lambda) = \int_\lambda^\infty \mathcal{P}_{jpdf}$

  \- equals partition function:

  $E(\lambda) = e^{-N^2\Sigma^2\lambda} \mathcal{Z}_{\nu=0}^{(N_f+\nu)}(\{m'\})$

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- **first eigenvalue distribution**

  $p_1(y) = -\partial_y E(y^2)$

  <u>examples ($N_f = 0$):</u>

  $\nu = 0$ $\boxed{p_1(y) = y/2e^{-y^2/4}}$, $\nu = 1$ $\boxed{p_1(y) = y/2e^{-y^2/4} I_2(y)}$ etc.

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- **Integration Theorem Mehta:**

$$\int dz_N w(z_N)\det_{k\times k}[K_N(z_i,z_J^*)] = (N-k+1)\det_{(k-1)\times(k-1)}[K_N(z_i,z_J^*)]$$

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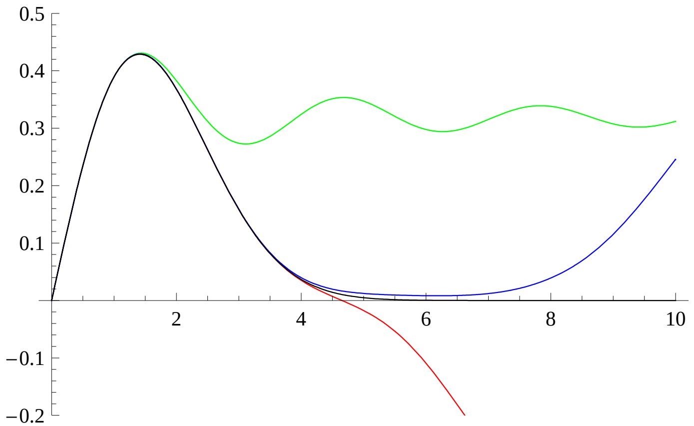

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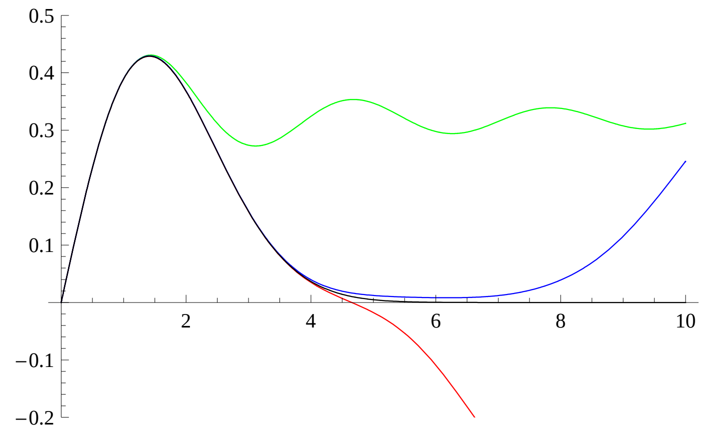

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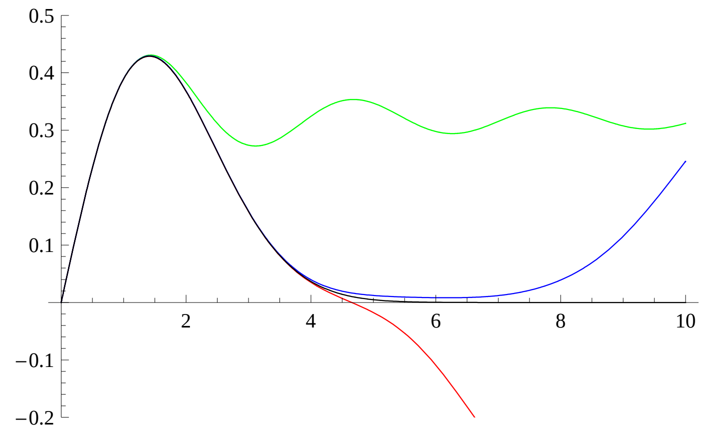

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- for **complex eigenvalues** a more direct computation of $p_1(r)$ is difficult also from RMT

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◦ NEW: corrections from finite lattice spacing

The effect of lattice spacing: Wilson χPT

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- same strategy: ε-regime (with our without RMT approach)

  - compute density and smallest eigenvalue LEC's

  - determine LEC from fit to Lattice data

Wilson $\varepsilon\chi$PT & Wilson RMT

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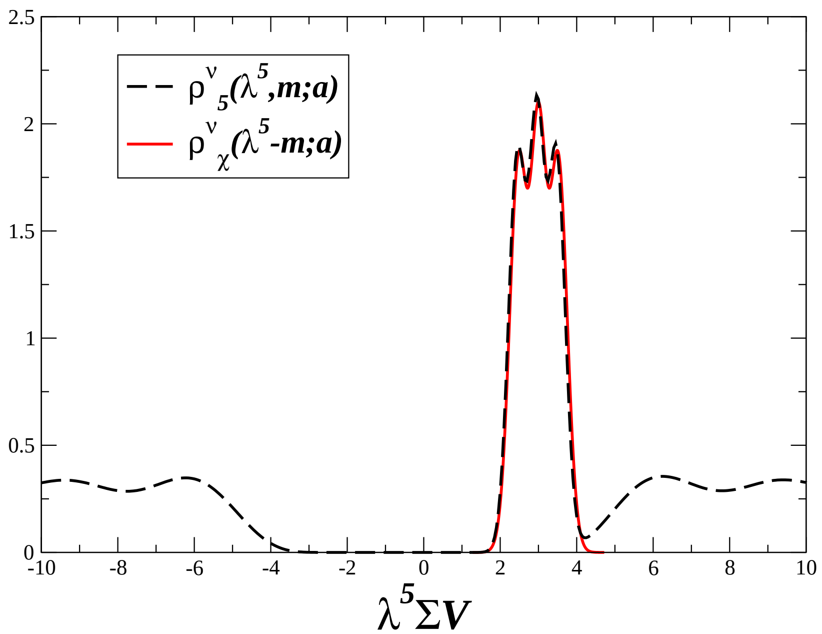
- problems at $a \neq 0$:

  - D_W non-Hermitian

  - concept on **topology no longer well defined,** $\exists$ several related densities of D_5, real modes of D_W, chirality density

  - ν **zero modes get smeared out** and decouple from rest of spectrum

First Wilson results



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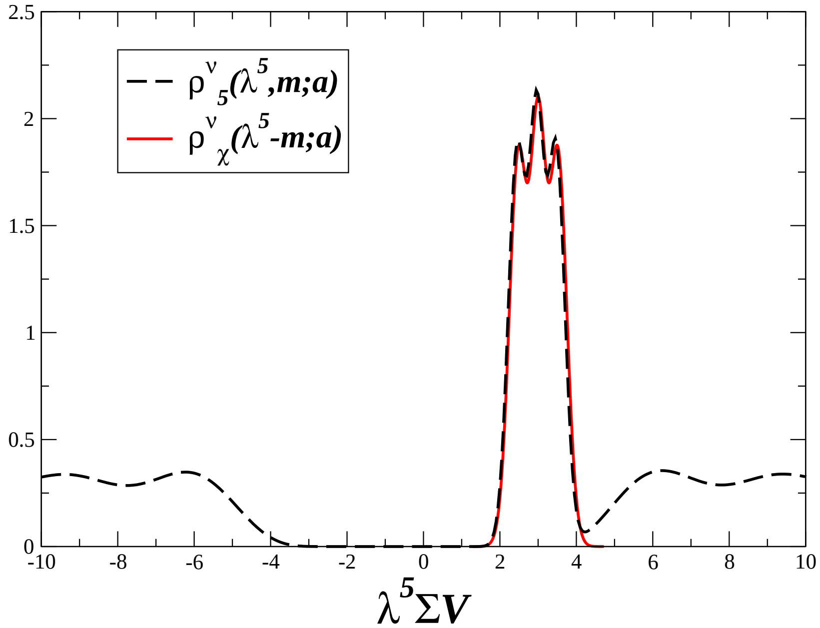

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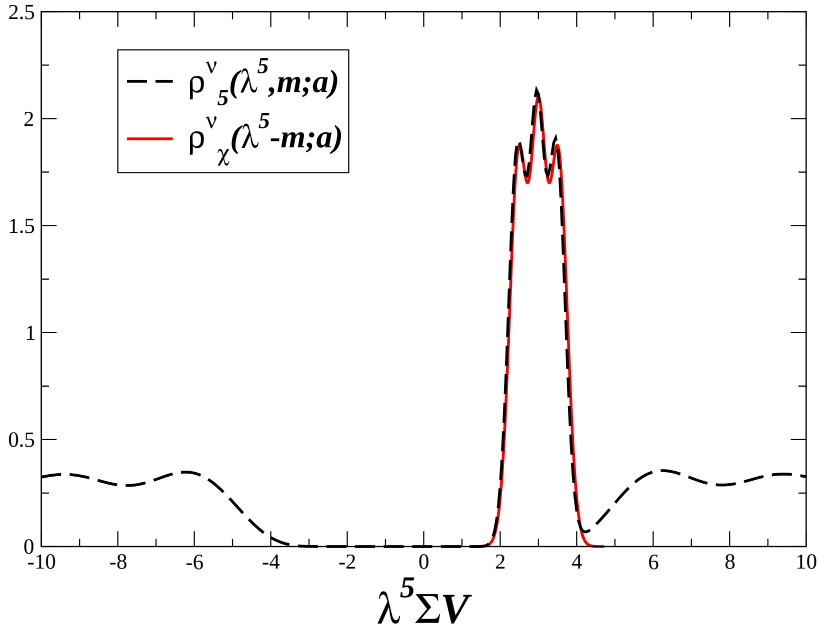

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- meson correlators in Wilson $\varepsilon\chi$PT: [Shindler, Necco, Bär]

More literature

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